Editorial

The Scientific Committee is pleased to present this volume which contains the proceedings of the international conference *Advances in Group Theory and Applications 2009*, held in Porto Cesareo in June 2009.

The committee is especially grateful to all of the contributors. It wishes to record its gratitude to the main speakers for their cooperative interaction and the editorial assistance they provided. Thanks are also due to all those who presented such a wide and varied range of research topics in the short talk programme. We acknowledge the important role played by all the participants, who by their questions and suggestions helped make the event a memorable success.

In this collection the reader will find a comprehensive compendium of background material on which further research investigations can be based. It contains a stimulating account of new results and a challenging range of open questions. It is particularly gratifying to note that a number of the advances concern problems posed at the previous conference held in Otranto in 2007.

It would have been impossible to organize a meeting of this size and duration without special financial sponsorship. We acknowledge the generous support of the *Dipartimento di Matematica E. De Giorgi* of the University of Salento, the *Dipartimento di Matematica e Applicazioni R. Caccioppoli* of the University of Napoli, the *Dipartimento di Matematica e Informatica* of the University of Salerno and the *Dipartimento di Matematica* of the Second University of Napoli. The conference was also supported by the G.N.S.A.G.A. of the *Istituto Nazionale di Alta Matematica* and by the project *Teoria dei gruppi e applicazioni* of the *Ministero dell'Università e della Ricerca Scientifica*. Finally, we are also grateful to *Monte del Paschi di Siena* whose valued assistance enabled the printing of this volume.

For their expert handling of the organizational, managerial and administrative programme we are indebted to Maria Maddalena Miccoli, Alessio Russo, Salvatore Siciliano, Ernesto Spinelli, Giovanni Vincenzi and thank them most sincerely.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 1–7. doi:10.1285/i15900932v30sn1p1

Soluble Products of Finite Groups

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Abstract. If G is the soluble product of its subgroups A and B, then a natural question is how the structure of G is influenced by the structure of A and B. In particular, can the derived length be bounded in terms of invariants of A and B. There are many results in this direction, but very little is known about best possible bounds. We survey some of these results.

Keywords: soluble group, product of groups, derived length.

MSC 2000 classification: 20D40

Suppose A and B are subgroups of a group G. We say that A permutes with B if the product of A and B, $AB = \{ab \ : \ a \in A, \ b \in B\}$, is a subgroup of G. If $AB = G$, we say G is the product of A and B and we call A and B factors of G. A natural question to ask is whether properties of $G = AB$ can be deduced from properties of A and B. There is an extensive literature on this question. Many properties have been considered- see for example the book of Amberg, Franciosi and de Giovanni [2]- and further restrictions on the products have also been considered. I want to concentrate on one particular property and will only consider finite groups, although some of the results do not need finiteness.

Suppose that G is soluble. Then A and B are certainly soluble but A and B soluble is not enough to ensure that G is soluble and so we may ask the following questions:

What further conditions on A and B will ensure that $G = AB$ is soluble?

If G is soluble, can we bound the derived length $d(G)$ of G in terms of invariants of A and B?

If $d(G)$ is bounded, can we find the best possible bound?

For the first question there is an extensive literature, both for restrictions on the factors and on the type of product. Producing bounds is harder and there is very little known about best possible bounds. I want to concentrate here on the second and third questions.

Here the story usually starts with the theorem of Ito:

Theorem 1. (Ito [12]) *Let $G = AB$ with A and B abelian. Then G is metabelian.*

This actually does not need G finite and it is easy to construct groups which are nonabelian and so this bound is best possible.

The next step is due to Hall and Higman who proved

Theorem 2. ((Hall and Higman [9]) *Suppose that $G = AB$ is soluble and A, B are nilpotent of coprime order. Then $d(G) \leq c(A) + c(B)$, where $c(A)$ denotes the nilpotency class of A.*

(This result is often attributed to Pennington [20], who obtains it as a corollary to her main theorem. It first appears as a special case of Theorem 1.2.4 of [9].)

Much of the work since this result has concentrated on products of nilpotent groups.

Shortly after, Wielandt proved if $G = AB$ and A, B are nilpotent of coprime order then G is soluble. Then Kegel removed the restriction on coprimeness of A and B.

Theorem 3. (Wielandt [22]) *If $G = AB$ and A, B are nilpotent of coprime order then G is soluble.*

Theorem 4. (Kegel [16] *If $G = AB$ and A, B are nilpotent then G is soluble.*

There are many papers in the decade before Kegel's theorem that may be regarded as precursors of Kegel's result, proving solubility under further restrictions on the structure of the factors. Kegel's proof does not give a bound on the derived length of the product. For many years it was conjectured that the sum of the nilpotency classes would be an upper bound for the derived length and there are a number of partial results.

One of the first was

Theorem 5. (Pennington [20]) *If $G = AB$ and A, B are nilpotent then $d(G) \leq c(A) + c(B) + d(F(G))$.*

Perhaps more important from the viewpoint of finding a bound was the following result

Theorem 6. (Pennington [20], Amberg [1]) *If $G = AB$ with A, B nilpotent, then $F(G) = (F(G) \cap A)(F(G) \cap B)$.*

These two results tell us that it is enough to consider the case of G nilpotent (and hence a p-group) to find a bound (p will denote a prime throughout this paper).

When G is nilpotent, another approach was started by Kazarin in 1982, who showed that the derived length could be bounded in terms of the orders of the derived groups of A and B. Several authors have improved on his bounds in special cases:

Theorem 7. *Suppose that $G = AB$ is a p-group with $|A'| = p^m$ and $|B'| = p^n$ with $m \geq n$:*
i) (Kazarin [14]) $d(G) \leq 2(m+n)+1$;
ii) (Morigi [19]) $d(G) \leq m+2n+2$; and if B is abelian, $d(G) \leq m+2$;
iii) (McCann [18]) if A has class at most 2 and B is extraspecial, $d(G) \leq m+3$;
iv) (Mann [17]) if B is abelian, $d(G) \leq 2log_2(m+2)+3$.

Note that these results combined with the results of Pennington and Amberg above show that the derived length of the product is bounded (in terms of invariants of the factors). The bound that they give is generally very large. To find bounds in terms of the nilpotency classes of A and B seems more difficult and there are fewer results here. Of course the aim here was to prove Kegel's conjecture. Since soluble groups of derived length at most d form a formation, a minimal counterexample to Kegel's conjecture must have a unique minimal normal subgroup and so in particular the Fitting subgroup must be a p-group for some prime p. It seems natural to consider products in which the Fitting subgroup is a p-group and Stonehewer and I began to look at such groups in the late 90's.

Theorem 8. (Cossey and Stonehewer [5]) *Suppose $G = AB$, with A, B nilpotent, $F(G)$ a p-group. Suppose also $F(G)/\Phi(F(G))$ contains no central chief factors. Then $d(G) \leq c(A) + c(B) + 1$.*

In analysing the structure of a minimal example of a group of derived length $c(A)+c(B)+1$ in the above theorem, Stonehewer and I produced an example of a group of derived length 4 which was the product of an abelian and a metabelian group. We were then able to find a few further examples . In particular we proved:

Theorem 9. (Cossey and Stonehewer [4]) *There are examples of groups $G = AB$ with A nilpotent of class m, B nilpotent of class n and G of derived length $d > m+n$ for the following triples (m, n, d): $(1, 2, 4)$, $(2, 2, 5)$ and $(2, 2, 6)$.*

Although the most surprising thing about these examples is the length of time it took to find them, it seems difficult to extend the construction to give more examples and it seems likely that a linear bound is the correct one.

Another invariant of nilpotent groups that would seem relevant to the derived length of G is the derived length of the factors and this was considered by Kazarin for factors of coprime order. When A and B have small class, the bound of Hall and Higman is best possible, but for larger classes it seems too large. Kazarin showed that it can be replaced by a bound involving the derived lengths of the factors and perhaps not surprisingly a better bound can be found if the product has odd order.

Theorem 10. (Kazarin [15]) *Suppose $G = AB$, A , B nilpotent of coprime order. Then*

$$d(G) \leq 2d(A)d(B) + d(A) + d(B)$$

and if G has odd order then

$$d(G) \leq d(A)d(B) + max\{d(A), d(B)\}.$$

If A is abelian, Kazarin's bound is $3d(B) + 1$ and $2d(B)$ if G has odd order. Are these best possible? If B is metabelian then Kazarin's bounds are 7 and 4 respectively. Wang and I showed that these bounds could be improved.

Theorem 11. (Cossey and Wang [6]) *Suppose $G = AB$ with A abelian and B nilpotent and metabelian. Suppose also that A and B have coprime orders. Then $d(G) \leq 4$ and if G has odd order $d(G) \leq 3$ and these bounds are best possible.*

We actually showed that examples of derived length 4 can be classified and the odd order result follows from the classification.

Recently Jabara has used a different invariant to bound the derived length of products of p-groups. Let $\mathcal{A}_1$ be the class of finite abelian p-groups. Then $G \in \mathcal{A}_n$ if and only if each chief series of G contains a nontrivial abelian term K such that $G/K \in \mathcal{A}_{n-1}$.

Theorem 12. (Jabara [13]) *Suppose $G = AB$ is a p-group, with A abelian, $B \in \mathcal{A}_n$. Then $d(G) \leq 2n$.*

Another result which gives a bound for a product with restrictions on the type of product has been given by Dixon and Stonehewer.

Theorem 13. (Dixon and Stonehewer [7]) *Suppose $G = AB$, with A, B nilpotent quasinormal subgroups of G. Suppose also that $A \cap B = 1$. Then $d(G) \leq max\{2, d(A), d(B)\}$.*

This bound is also clearly best possible. Dixon and Stonehewer ([8]) have also observed that the result holds with A and B soluble. Cossey and Ezquerro [3] have shown that the requirement that A and B be quasinormal can be replaced by G being the totally permutable product of A and B when G has odd order. (G is the totally permutable product of A and B if every subgroup of A permutes with every subgroup of B.)

If $G = AB$ is soluble, even when bounds can be found for $d(G)$, best possible bounds have only been found for special cases, either for small values of some invariant or strong restrictions on the type of product. Perhaps the simplest case where the answer is not known is $G = AB$, where A and B are normal in G.

Of course, if A and B are soluble then G is soluble - this is an easy undergraduate problem, as is the bound $d(G) \leq d(A) + d(B)$. So we may ask if this bound is best possible. Perhaps surprisingly this seems to be a difficult question.

If A, B are abelian, Ito's Theorem gives a bound of 2 and it is easy to find examples where this bound is reached. We can also find examples where A is abelian, $d(B)$ is arbitrary and G has derived length $1 + d(B)$. An easy set of examples is given by the following.

Let $T_n(p)$ be the group of upper unitriangular $n \times n$ matrices over $GF(p)$. Weir [21] showed in 1955- see also Huppert [10] III.16.6- that $T_n(p)$ can be written as the product of $n-1$ abelian normal subgroups $A_1, ..., A_{n-1}$ where A_j consists of the matrices of the form

$$\begin{pmatrix} I_j & S \\ 0 & I_{n-j} \end{pmatrix}.$$

Now given an integer d we can find an n such that $T_n(p)$ has derived length at least $d+1$. For some $i < n$ we will have $N = \langle A_1, ..., A_i \rangle$ of derived length d and $\langle A_1, ..., A_{i+1} \rangle = A_{i+1}N$ of derived length $d+1$.

When A and B are both nonabelian, I know of no examples where the bound is attained. It appears to be a difficult problem, even when A and B are both metabelian. Note that all these examples are groups of prime power order. We might ask if this is an essential feature of such a product in the following sense. If $G = AB$ is the product of normal subgroups and P is a Sylow p-subgroup of G, p a prime, then P is the normal product of $P \cap A$ and $P \cap B$. If $d(G) = d(A) + d(B)$ is it true that for some prime p, $d(G) = d(P) = d(P \cap A) + d(P \cap B)$. This also seems a difficult question in general. It is true if both A and B are abelian but for A metabelian and B abelian it is not true and an example is given below.

Let p be an odd prime, C be a group of order p and $X = \langle x, y \rangle$ be the nonabelian group of order p^3 and exponent p. Put $[x, y] = z$. Put $H = C\mathrm{wr}\mathrm{X}$ and denote the base group of H by Y. Then Y is an elementary abelian group. We let α be the automorphism of H which fixes X and inverts every element of Y and put $K = H\langle \alpha \rangle$. As a $\langle y, z \rangle$ module, $Y = Y_1 \times ... \times Y_p$ where each Y_i is isomorphic to the regular module and so has a unique minimal submodule

M_i. We may assume that $Y_i = Y_1^{x^{i-1}}$. By the dual of [11] Theorem VII.15.5 Y_i/M_i has a submodule N_i/M_i of order p^2 on which $\langle y, z\rangle$ acts trivially and $[N_i, z] \neq 1$. Let M be the product of the M_i's and N the product of the N_i's. Then M and N are normal subgroups of K. We let $G = NX\langle\alpha\rangle$, $A = N\langle x, z, \alpha\rangle$ and $B = M\langle y, z\rangle$. Modulo M, B is normalised by x and centralised by N, z and α and so B is normal in G. Also modulo M, A centralises N and α and normalises $\langle x, z\rangle$ and so B is normal in G. We then have $G' = \langle z\rangle N$. Since z does not centralise N, $G'' \neq 1$ and so G has derived length 3. The Sylow p-subgroup of G is just NX and is the normal product of $N\langle x, z\rangle$ (which is not abelian) and B. Since $(NX)' = [NX, NX] = [N, X]X' = [N, X]\langle z\rangle$ and $[N, X] \leq M \leq B$ we have $(NX)'' = 1$.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 9–20. doi:10.1285/i15900932v30sn1p9

Groups and set theoretic solutions of the Yang-Baxter equation

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Abstract. We discuss some recent results on the link between group theory and set theoretic involutive non-degenerate solutions of the Yang-Baxter equation. Some problems are included.

Keywords: Yang Baxter equation, set theoretic solution, multipermutation solution, permutation group, group of I-type.

MSC 2000 classification: 81R50, 20B25, 20F38, 20B35, 20F16, 20F29

1 Introduction

In a paper on statistical mechanics by Yang [21], the quantum Yang-Baxter equation appeared. It turned out to be one of the basic equations in mathematical physics and it lies at the foundation of the theory of quantum groups. One of the important unsolved problems is to discover all the solutions R of the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where V is a vector space, $R : V \otimes V \to V \otimes V$ is a linear map and R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ that acts as R on the (i, j) tensor factor (in this order) and as the identity on the remaining factor. In recent years, many solutions have been found and the related algebraic structures have been intensively studied (see for example [17]). Drinfeld, in [6], posed the question of finding the simplest solutions, that is, the solutions R that are induced by a linear extension of a mapping $\mathcal{R} : X \times X \to X \times X$, where X is a basis for V. In this case, one says that $\mathcal{R}$ is a set theoretic solution of the quantum Yang-Baxter equation.

Let $\tau : X^2 \to X^2$ be the map defined by $\tau(x, y) = (y, x)$. Observe that $\mathcal{R}$ is a set theoretical solution of the quantum Yang-Baxter equation if and only if the mapping $r = \tau \circ \mathcal{R}$ is a solution of the braided equation (or a solution of the Yang-Baxter equation, in the terminology used for example in [10, 12])

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}.$$

Set theoretic solutions $\mathcal{R} : X^2 \to X^2$ of the quantum Yang-Baxter equation (with X a finite set) that are (left) non-degenerate and such that $r = \tau \circ \mathcal{R}$ is involutive (i.e., r^2 is the identity map on X^2) have received recently a lot of attention by Etingof, Schedler and Soloviev [9], Gateva-Ivanova and Van den Bergh [10, 13], Lu, Yan and Zhu [18], Rump [19, 20], Jespers and Okniński [15, 16] and others. (The set theoretical solutions $\mathcal{R}$ such that r is involutive are called unitary in [19], and in [9] one then says that (X, r) is a symmetric set.)

Recall that a bijective map

$$\begin{array}{cccc} r\colon & X \times X & \longrightarrow & X \times X \\ & (x,y) & \mapsto & (f_x(y), g_y(x)) \end{array}$$

is said to be left (respectively, right) non-degenerate if each map f_x (respectively, g_x) is bijective.

Gateva-Ivanova and Van den Bergh in [13], and Etingof, Schedler and Soloviev in [9], gave a beautiful group theoretical interpretation of involutive non-degenerate solutions of the braided equation. In order to state this, we need to introduce some notation. Let FaM_n be the free abelian monoid of rank n with basis $u_1, \ldots, u_n$. A monoid S generated by a set $X = \{x_1, \ldots, x_n\}$ is said to be of left I-type if there exists a bijection (called a left I-structure)

$$v\colon \mathrm{FaM}_n \longrightarrow S$$

such that

$$v(1) = 1 \quad \text{and} \quad \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\},$$

for all $a \in \mathrm{FaM}_n$. In [13] it is shown that these monoids S have a presentation

$$S = \langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \rangle,$$

with $\binom{n}{2}$ defining relations so that every word $x_i x_j$, with $1 \leq i, j \leq n$, appears at most once in one of the relations. Such a presentation induces a bijective map $r\colon X \times X \longrightarrow X \times X$ defined by

$$r(x_i, x_j) = \begin{cases} (x_k, x_l), & \text{if } x_i x_j = x_k x_l \text{ is a defining relation for } S; \\ (x_i, x_j), & \text{otherwise.} \end{cases}$$

Furthermore, r is an involutive right non-degenerate solution of the braided equation. Conversely, for every involutive right non-degenerate solution of the braided equation $r\colon X\times X\longrightarrow X\times X$ and every bijection $v:\{u_1,\dots,u_n\}\to X$ there is a unique left I-structure $v:\mathrm{FaM}_n\to S$ extending v, where S is the semigroup given by the following presentation

$$S=\langle X\mid ab=cd,\ \text{if } r(a,b)=(c,d)\rangle$$

([16, Theorem 8.1.4.]).

In [15] Jespers and Okniński proved that a monoid S is of left I-type if and only if it is of right I-type; one calls them simply monoids of I-type. Hence, it follows that an involutive solution of the braided equation is right non-degenerate if and only if it is left non-degenerate (see [15, Corollary 2.3] and [16, Corollary 8.2.4]).

Jespers and Okniński in [15] also obtained an alternative description of monoids of I-type. Namely, it is shown that a monoid is of I-type if and only if it is isomorphic to a submonoid S of the semi-direct product $\mathrm{FaM}_n\rtimes Sym_n$, with the natural action of Sym_n on FaM_n (that is, $\sigma(u_i)=u_{\sigma(i)}$ for $\sigma\in Sym_n$), so that the projection onto the first component is a bijective map, that is

$$S=\{(a,\phi(a))\mid a\in\mathrm{FaM}_n\},$$

for some map $\phi\colon\mathrm{FaM}_n\to Sym_n$. It then follows that S has a (two-sided) group of quotients (one needs to invert the central element $((u_1\cdots u_n)^k,1)$; where k is the order of the permutation $\phi(u_1\cdots u_n)$). Of course the group of quotients $\mathcal{G}=S^{-1}S$ of the monoid of I-type S is defined by the same generators and relations as S. These groups have been investigated by Etingof, Guralnick, Schedler and Soloviev in [8, 9], where they are called structural groups. They are simply called groups of I-type.

The group $\mathcal{G}$ can also be described as follows. The map ϕ extends uniquely to a map $\phi\colon\mathrm{Fa}_n\to Sym_n$, where Fa_n is the free abelian group of rank n, and the group $\mathcal{G}$ is isomorphic to a subgroup of the semi-direct product $\mathrm{Fa}_n\rtimes Sym_n$ so that the projection onto the first component is a bijective map, that is

$$\mathcal{G}=\{(a,\phi(a))\mid a\in\mathrm{Fa}_n\}. \tag{1}$$

Note that if we put $f_{u_i}=\phi(u_i)$ then $S=\langle (u_i,f_{u_i})\mid 1\le i\le n\rangle$ and one can easily obtain the associated involutive non-degenerate solution $r:X^2\to X^2$ defining the monoid of I-type. Indeed, if we set $X=\{u_1,\dots,u_n\}$, then $r(u_i,u_j)=(f_{u_i}(u_j),f^{-1}_{f_{u_i}(u_j)}(u_j))$. Obviously, $\phi(\mathrm{Fa}_n)=\langle\phi(a)\mid a\in\mathrm{FaM}_n\rangle=\langle f_{u_i}\mid 1\le i\le n\rangle$ (we will denote this group also as G_r). Note that, because of Proposition 2.2 in [9], if $(x,g)\mapsto(f_x(y),g_y(x))$ is an involutive non-degenerate solution of the braided equation then $T^{-1}{g_x}^{-1}T=f_x$, where $T:X\to X$ is the bijective map defined by $T(y)={g_y}^{-1}(y)$. Hence $\langle f_x : x\in X\rangle$ is isomorphic with $\langle g_x : x\in X\rangle$.

2 Groups of I-type

In order to describe all involutive non-degenerate solutions of the braided equation (equivalently the non-degenerate unitary set theoretic solutions of the quantum Yang-Baxter equation) one needs to solve the following problem.

Problem 1: Characterize the groups of I-type.

An important first step in this direction is to classify the finite groups that are of the type $\phi(\mathrm{Fa}_n)$ for some group of I-type $\mathcal{G}$, as in (1) (equivalently the groups of the form $\langle f_x : x \in X\rangle$, for $(x, y) \in X^2 \mapsto (f_x(y), g_y(x))$ a non-degenerate involutive solution of the braided equation). As in [4], a finite group with this property is called an *involutive Yang-Baxter* (IYB, for short) group. So to tackle the above problem we will need to solve the following two problems.

Problem 1a: Classify involutive Yang-Baxter groups.

Problem 1b: Describe all groups of I-type that have a fixed associated IYB group G.

In [4] these problems are being investigated and in this section we report on the main results of that paper.

Recall that Etingof, Schedler and Soloviev in [9, Theorem 2.15] proved that any group of I-type is solvable. As a consequence, every IYB group is solvable.

Verifying that a finite group is IYB seems to be a non-trivial task. Hence it is useful the give several equivalent properties that guarantee this property. For this we first recall some terminology of [4]. For a finite set X we denote by Sym_X the symmetric group on X. An *involutive Yang-Baxter map* (IYB map, for short) on a finite set X is a map $\lambda : X \to Sym_X$ satisfying

$$\lambda(x)\lambda(\lambda(x)^{-1}(y)) = \lambda(y)\lambda(\lambda(y)^{-1}(x)) \quad (x, y \in X). \tag{2}$$

The justification for this terminology is based on the fact that each IYB map yields an involutive non-degenerate set theoretical solution of the Yang-Baxter equation and conversely. Indeed, let $r : X^2 \to X^2$ be a bijective map. As before, denote $r(x, y) = (f_x(y), g_y(x))$. From the proof of [3, Theorem 4.1], it follows that $r : X^2 \to X^2$ is an involutive non-degenerate set theoretical solution of the Yang-Baxter equation if and only if $f_x \in Sym_X$ for all $x \in X$ and the map $\lambda \colon X \to Sym_X$ defined by $\lambda(x) = f_x$, for all $x \in X$, is an IYB map.

Theorem 1. *[4] The following conditions are equivalent for a finite group G.*

(1) *G is an IYB group, that is, there is a map $\phi : \mathrm{Fa}_n \to Sym_n$ such that $\{(a, \phi(a)) : a \in \mathrm{Fa}_n\}$ is a subgroup of $\mathrm{Fa}_n \rtimes Sym_n$ and G is isomorphic to $\phi(\mathrm{Fa}_n)$.*

(2) *There is an abelian group A, an action of G on A and a group homomorphism $\rho : G \to A \rtimes G$ such that $\pi_G \rho = \mathrm{id}_G$ and $\pi_A \rho : G \to A$ is bijective, where π_G and π_A are the natural projections on G and A respectively.*

(3) *There is an abelian group A, an action of G on A and a bijective 1-cocycle $G \to A$.*

(4) *There exists an IYB map $\lambda\colon A \cup X \to Sym_{A\cup X}$ satisfying the following conditions:*

a. *$\lambda(A)$ is a subgroup of $Sym_{A\cup X}$ isomorphic to G,*

b. *$A \cap X = \emptyset$,*

c. *$\lambda(x) = \mathrm{id}_{A\cup X}$ for all $x \in X$,*

d. *$\lambda(a)(b) \in A$ for all $a, b \in A$ and*

e. *$\lambda|_A$ is injective.*

(5) *$G \cong \lambda(X)$ for some IYB map $\lambda : X \to Sym_X$ whose image is a subgroup of Sym_X.*

(6) *$G \cong \langle\lambda(X)\rangle$ for some IYB map $\lambda : X \to Sym_X$.*

(7) *There exist a group homomorphism $\mu : G \to Sym_G$ satisfying*

$$x\mu(x)^{-1}(y) = y\mu(y)^{-1}(x), \tag{3}$$

for all $x, y \in G$.

(8) *There exist a generating subset Z of G and a group homomorphism $\mu : G \to Sym_Z$ satisfying (3) for all $x, y \in Z$.*

One obtains some constructions of IYB-groups from a give IYB-groups.

Corollary 1. *[4]*

(1) *If G is an IYB group then its Hall subgroups are also IYB.*

(2) *The class of IYB groups is closed under direct products.*

Theorem 2. *[4] Let G be a finite group such that $G = AH$, where A is an abelian normal subgroup of G and H is an IYB subgroup of G. Suppose that there is a bijective 1-cocycle $\pi : H \to B$, with respect to an action of H on the abelian group B such that $H \cap A$ acts trivially on B. Then G is an IYB group.*

In particular, every semi-direct product $A \rtimes H$ of a finite abelian group A by an IYB group H is IYB.

Theorem 3. *[4] Let N and H be IYB groups and let $\pi_N : N \to A$ be a bijective 1-cocycle with respect to an action of N on an abelian group A. If $\gamma : H \to \mathrm{Aut}\,(N)$ and $\delta : H \to \mathrm{Aut}\,(A)$ are actions of H on N and A respectively such that $\delta(h)\pi_N = \pi_N\gamma(h)$ for every $h \in H$, then the semi-direct product $N \rtimes H$, with respect to the action γ, is an IYB group.*

Corollary 2. *[4]*

(1) Let G be an IYB group and H an IYB subgroup of Sym_n. Then the wreath product $G \wr H$ of G and H is an IYB group.

(2) Any finite solvable group is isomorphic to a subgroup of an IYB group.

(3) Let n be a positive integer. Then the Sylow subgroups of Sym_n are IYB groups.

(4) Any finite nilpotent group is isomorphic to a subgroup of an IYB nilpotent group.

The next result yields many examples of IYB groups.

Theorem 4. *[4] Let G be a finite group having a normal sequence*

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

satisfying the following conditions:

(i) for every $1 \leq i \leq n$, $G_i = G_{i-1}A_i$ for some abelian subgroup A_i;

(ii) $(G_{i-1} \cap (A_i \cdots A_n), G_{i-1}) = 1$;

(iii) A_i is normalized by A_j for every $i \leq j$.

Then G is an IYB group.

Corollary 3. *[4]*

(1) Let G be a finite group. If $G = NA$, where N and A are two abelian subgroups of G and N is normal in G, then G is an IYB group. In particular, every abelian-by-cyclic finite group is IYB.

(2) Every finite nilpotent group of class 2 is IYB.

It is unclear whether the class of IYB groups is closed for taking subgroups. As a consequence, it is unknown whether the class of IYB groups contains all finite solvable groups. Hence, our next problem.

Problem 2: Are all finite solvable groups involutive Yang-Baxter groups, i.e. does the class of IYB groups coincide with that of all solvable finite groups.

The results in [4] also indicate that there is no obvious inductive process to prove that solvable finite groups are IYB. Indeed for such a process to exist one would like to be able to lift the IYB structure from subgroups H or quotient groups $\overline{G}$ of a given group G to G. However, in [4], examples are given that show that not every IYB homomorphism of a quotient of G can be lifted to an IYB homomorphism of G.

Concerning Problem 1b: If $r(x_1, x_2) = (f_{x_1}(x_2), g_{x_2}(x_1))$ is an involutive non-degenerate solution on a finite set X of the braided equation then it is easy to produce, in an obvious manner, infinitely many solutions with the same associated IYB group, namely for every set Y let $r_Y : (X \cup Y)^2 \to (X \cup Y)^2$ be given by $r_Y((x_1, y_1), (x_2, y_2)) = ((f_{x_1}(x_2), y_1), (g_{x_2}(x_1), y_2))$. In [4] an alternative way of obtaining another involutive non-degenerate solution on $X \times X$ of the braided equation with the same associated IYB group is given. It follows that, in a non-obvious fashion, infinitely many set theoretic solutions of the Yang-Baxter equation are obtained for the same IYB group.

3 Groups of I-type and poly-$\mathbb{Z}$ groups

There are two other approaches, both originating from the work of Etingof, Schedler and Soloviev, [9], that could lead to successfully classify all possible set theoretic solutions of the Yang-Baxter equation. The idea is to show that every solution can be built in a recursive way from certain solutions of smaller cardinality.

Let (X, r) be a set theoretical solution on the finite set X. The first alternative approach is based on the retract relation $\sim$ on the set X, introduced in [9], and defined by

$$x_i \sim x_j \quad \text{if } \sigma_i = \sigma_j$$

(here we denote by σ_i the permutation f_{x_i}; by γ_j or γ_{x_j} we denote the map g_{x_j}). There is a natural induced solution

$$Ret(X, r) = (X/\!\sim, \tilde{r}),$$

and it is called the retraction of X. A solution (X, r) is called a *multipermutation solution of level* m if m is the smallest nonnegative integer such that the solution $Ret^m(X, r)$ has cardinality 1. Here one defines

$$Ret^k(X, r) = Ret(Ret^{k-1}(X, r))$$

for $k > 1$. If such an m exists then one also says that the solution is *retractable*. In this case, the group $G(X, r)$ is a poly-$\mathbb{Z}$ group (see [16, Proposition 8.2.12]). Recall that a group is called poly-$\mathbb{Z}$ (or, poly-infinite cyclic) if it has a finite subnormal series with factors that are infinite cyclic groups.

The second alternative approach is based on the notion of generalized twisted union. In order to state the definition, first notice that there is a natural action of the associated involutive Yang-Baxter group G_r on X defined by $\sigma(x_i) = x_{\sigma(i)}$. A set theoretic involutive non-degenerate solution (X, r) is called a *generalized twisted union* of solutions (Y, r_Y) and (Z, r_Z) if X is a disjoint union of two G_r-invariant non-empty subsets Y, Z such that for all $z, z' \in Z, y, y' \in Y$ we have

$$\sigma_{\gamma_y(z)|Y} = \sigma_{\gamma_{y'}(z)|Y} \tag{4}$$

$$\gamma_{\sigma_z(y)|Z} = \gamma_{\sigma_{z'}(y)|Z}. \tag{5}$$

Here, to simplify notation, we write σ_x for σ_i if $x = x_i$, and similarly for all γ_i. If, moreover, (X, r) is a square free solution (that is, every defining relation is without words of the form x_i^2), then conditions (4) and (5) are equivalent to

$$\sigma_{\sigma_y(z)|Y} = \sigma_{z|Y} \tag{6}$$

$$\sigma_{\sigma_z(y)|Z} = \sigma_{y|Z}, \tag{7}$$

(see [10, Proposition 8.3]). Let $G_{r,Y}$ be the subgroup of G_r generated by the set $\{\sigma_y \mid y \in Y\}$ and let $G_{r,Z}$ be the subgroup of G_r defined in a similar way. Then (6) and (7) amount to saying that the elements of the same $G_{r,Y}$-orbit on Z determine the same permutation of Y and the elements of the same $G_{r,Z}$-orbit on Y determine the same permutation of Z. The simplest example (called a twisted union in [9]) motivating this definition is obtained by choosing any permutations $\sigma_1, \sigma_2 \in Sym_n$, $n = |X|$, such that $\sigma_i(Y) = Y$ for $i = 1, 2$, and $\sigma_y = \sigma_1$ for every $y \in Y$ and $\sigma_z = \sigma_2$ for every $z \in Z$. An important step supporting this approach was made by Rump [19], who showed that the number of G_r-orbits on X always exceeds 1 if (X, r) is a non-degenerate involutive square free solution with $|X| > 1$.

The following conjectures were formulated by Gateva-Ivanova in [10].

(GI 1) Every set theoretic involutive non-degenerate square free solution (X, r) of cardinality $n \geq 2$ is a multipermutation solution of level $m < n$.

(GI 2) Every multipermutation square free solution of level m and of cardinality $n \geq 2$ is a generalized twisted union of multipermutation solutions of level less than m.

Notice that the square free assumption in (GI 1) is essential. Indeed in [16, Example 8.2.13] an example is given of a set theoretic involutive non-degenerate solution of cardinality 4 that is not a multipermutation solution.

In a recent paper [5] Cedó, Jespers and Okniński investigated these conjectures and obtained the following results. Recall that a set theoretic solution (X, r) is said to be trivial if $r(x_i, x_j) = (x_j, x_i)$ for every i, j. This is equivalent to saying that σ_i is the identity map for every i.

Theorem 5. *[5] Assume that (X, r) is a set theoretic involutive non-degenerate square free solution with abelian associated IYB-group G_r. If r is not trivial then there exist $i, j \in \{1, \dots, n\}$ such that $\sigma_i = \sigma_j$, $i \neq j$ and x_i, x_j in one G_r-orbit.*

An application is an affirmative answer for (GI 1) in case the IYB-group is abelian. Actually a stronger statement is proved. For this the notion of strong retractability of (X, r) was introduced. Let ρ denote the refining of the relation $\sim$ on X by requesting additionally that the elements are in the same G_r-orbit on X. Then, let $Ret_\rho(X, r) = (X/\rho, \bar{r})$ denote the induced solution. One says in [5] that (X, r) is strongly retractable if there exists $m \geq 1$ such that applying m times the operator Ret_ρ we get a trivial solution.

Note that the IYB group corresponding to the solution $(X/\rho, \bar{r})$ also is abelian if G_r is abelian.

Corollary 4. *[5] Assume that (X, r) is a set theoretic involutive non-degenerate square free solution with abelian associated IYB-group G_r. Then (X, r) is strongly retractable.*

The conjecture can also be confirmed in some cases that are not covered by the previous result.

Theorem 6. *[5] Let (X, r) be a set theoretic involutive non-degenerate square free solution with associated IYB group G_r, such that its generators σ_i, $i = 1, \dots, n$, are cyclic permutations. Then, (X, r) is strongly retractable. Moreover, if $|X| > 1$ then (X, r) is a generalized twisted union.*

The following result shows that (GI 2) is not true in general.

Theorem 7. *[5] There exists a multipermutation square free solution of level 3 (on 24 generators and with 3 orbits) that it is not a generalized twisted union. Furthermore, the associated IYB group is abelian.*

In a recent paper [2] Cameron and Gateva-Ivanova introduced the notion of strong twisted union. This notion is weaker than that of a generalized twisted union and is defined as follows. A set theoretic involutive non-degenerate solution (X, r) with associated IYB-group G_r is called a *strong twisted union* of solutions if X is a disjoint union $X = X_1 \cup \cdots \cup X_m$ of G_r-invariant non-empty subsets X_i so that $(X_i \cup X_j, r_{X_i \cup X_j})$ is a generalized twisted union. A new conjecture is then stated.

(GI 2a) Every involutive non-degenerate square-free multipermutation solution of level m is a strong twisted union of multipermutation solutions of levels less than m.

This conjecture is confirmed by Cameron and Gateva-Ivanova in the following cases.

Theorem 8. *[2] The statement (GI 2a) holds in the following cases:*

(1) the associated IYB-group is abelian,

(2) the solution is retractable of multipermutation level not exceeding 3.

We finish this section with posing two problems.

Problem 3: Prove (GI 1) for arbitrary IYB-groups.

Problem 4: Classify the multipermutation square free solutions of level m and of cardinality $n \geq 2$ that are a generalized twisted union of multipermutation solutions of level less than m.

4 Algebras of groups of I-type

If $\mathcal{G} = \{(a, \phi(a)) \mid a \in \mathrm{Fa}_n\}$ is a group of I-type then the IYB group $G = \phi(\mathrm{Fa}_n)$ naturally acts on the quotient group $A = \mathrm{Fa}_n/K$, where $K = \{a \in \mathrm{Fa}_n \mid \phi(a) = 1\}$ and we obtain an associated bijective 1-cocycle $G \to A$ with respect to this action. By a result of Etingof and Gelaki [7], this bijective 1-cocycle yields a non-degenerate 2-cocycle on the semi-direct product $H = A \rtimes G$. This has been generalized by Ben David and Ginosar [1] to more general extensions H of A by G with a bijective 1-cocycle from G to A. This construction of Etingof and Gelaki and of Ben David and Ginosar gives rise to a group of central type in the sense of [1], i.e. a finite group H with a 2-cocycle $c \in Z^2(H, \mathbb{C}^*)$ such that the twisted group algebra $\mathbb{C}^c H$ is isomorphic to a full matrix algebra over the complex numbers, or equivalently $H = K/Z(K)$ for a finite group K with an irreducible character of degree $\sqrt{[K : Z(K)]}$. This provides a nice connection between IYB groups and groups of central type that should be investigated.

The above links groups of I-type (hence solutions of the Yang-Baxter equation) and ring theory. Another important link with ring theory is that the semigroup algebra FS of a monoid of I-type S over an arbitrary field F shares many properties with the polynomial algebra in finitely many commuting variables. For example, in [13], it is shown that FS is a domain that satisfies a polynomial identity and that it is a maximal order in its classical ring of quotients. In particular, the group of I-type SS^{-1} is finitely generated abelian-by-finite and torsion free, i.e., it is a Bieberbach group ([13, Theorem 1.7], see also [15, Corollary 8.27]). The homological properties for FS were the main reasons for studying monoids of I-type in [13] and it was inspired by earlier work of Tate and Van den Bergh on Sklyanin algebras.

Clearly the group algebra $F[S^{-1}S]$ is a central localization of FS and hence shares many properties with FS: it is a domain that satisfies a polynomial identity and it is a maximal order in its classical ring of quotients. Group algebras are a fundamental topic of research, as for example, they are a natural link between group theory and ring theory. Clearly the elements of the form fg, with $0 \neq f \in F$ and $g \in G$ are invertible in FG (these are called trivial units). In case G is torsion-free group, then there is a famous conjecture due to Kaplansky: are all units in FG trivial?

Problem 5: Determine the group of invertible elements in the group algebra FG, for a group G of I-type; i.e. verify Kaplansky's conjecture for such group algebras.

Note that if the group of I-type is poly-infinite cyclic then one obtains immediately that the group algebra FG is a domain that has only trivial units, i.e. all units are trivial. So, in case conjecture (GI 1) has a positive answer then Problem 5 only should be investigated for groups of I-type that are not square free. As mentioned earlier, an example of this type is the following ([16, Example 8.2.14]) $G = \langle x_1, x_2, x_3, x_4 \mid x_1x_2 = x_3x_3,\ x_2x_1 = x_4x_4,\ x_1x_3 = x_2x_4,\ x_1x_4 = x_4x_2,\ x_2x_3 = x_3x_1,\ x_3x_2 = x_4x_1 \rangle$. This group is not poly-infinite cyclic as it contains the subgroup $\langle a, b \mid a^{-1}b^2a = b^{-2},\ b^{-1}a^2b = a^{-2} \rangle$ of which it is well known that it is not poly-infinite cyclic.

The problem of Kaplansky has been open for many decades and it appears to be notoriously difficult. Now, groups G of I-type are such that their group algebra FG behaves in many ways as commutative polynomial algebras. Hence, in this spirit, one would hope that such groups are ideal candidates for which the Kaplansky problem can be solved. The nice combinatorial nature of G should be of great help.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 21–36. doi:10.1285/i15900932v30sn1p21

On some infinite dimensional linear groups

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Abstract. Let F be a field and A an (infinite dimensional) vector space over F. A group G of linear transformations of A is said to be *finitary linear* if for each element $g \in G$ the centralizer $C_A(g)$ has finite codimension over F. Finitary linear groups are natural analogs of FC-groups (i.e. groups with finite conjugacy classes). In this paper we consider linear analogs of groups with boundedly finite conjugacy classes, and also some generalizations corresponding to groups with Chernikov conjugacy classes.

Keywords: Finitary linear group, FC-group, artinian-finitary module.

MSC 2000 classification: 20F24, 20G15

Let F be a field and A a vector space over F. Denote by $GL(F, A)$ the group of all F-automorphisms of A. The subgroups of $GL(F, A)$ are called the *linear groups*. Linear groups play a very important role in algebra and other branches of mathematics. If $dim_F(A)$ (the dimension of A over F) is finite, say n, then a subgroup G of $GL(F, A)$ is a *finite dimensional* linear group. It is well known that in this case, $GL(F, A)$ can be identified with the group of all invertible $n \times n$ matrices with entries in F. The theory of finite dimensional linear groups is one of the most developed in group theory. It uses not only algebraic, but also topological, geometrical, combinatorial, and many other methods.

However, in the case when A has infinite dimension over F, the study of the subgroups of $GL(F, A)$ requires some additional restrictions. This case is more complicated and requires some additional restrictions allowing an effective employing of already developed techniques. The most natural restrictions here are the finiteness conditions. Finitary linear groups demonstrate the efficiency of such approach. We recall that a subgroup G of $GL(F, A)$ is called *finitary* if for each element $g \in G$ its centralizer $C_A(g)$ has finite codimension over F. The theory of finitary linear groups is now well-developed and many interesting results have been proved (see, for instance, the survey [1]). We begin with consideration on some generalizations of such groups.

1 On some generalizations of finitary linear groups

If G is a subgroup of $GL(F, A)$, we can consider the vector space A as a module over the group ring FG. We can obtain the following generalizations of finitary groups. Replacing the field F by the ring R, artinian and noetherian R-modules are natural generalizations of the concept of a finite dimensional vector space. Some related generalizations of finitary groups have been considered by B.A.F. Wehrfritz (see [2], [3], [4], [5]).

Let R be a ring, G a group and A an RG-module. Following B.A.F. Wehrfritz, a group G is called *artinian - finitary*, if for every element $g \in G$, the factor-module $A/C_A(g)$ is artinian as an R-module. In this case, we say that A is an *artinian - finitary* RG-module.

We observe that we can consider finitary linear groups as linear analogs of the FC-groups (we can define an FC-group G as a group such that $|G : C_G(x)|$ is finite for each element $g \in G$). Similarly, if $R = \mathbb{Z}$ and G is an artinian - finitary group, then the additive group of the factor-module $A/C_A(g)$ is Chernikov for every element $g \in G$. This shows that we can consider artinian - finitary groups as linear analogs of the groups with Chernikov conjugacy classes (shortly CC-groups).

One of the first important result of theory of FC-groups was a theorem due to B. H. Neumann that described the structure of FC- groups with bounded conjugacy classes. Following B. H. Neumann, a group G is called a BFC-group if there exists a positive integer b such that $|g^G| \leq b$ for each element $g \in G$. B. H. Neumann proved that a group G is a BFC-group if and only if the derived subgroup $[G, G]$ is finite ([6], Theorem 3.1).

A group $G \leq GL(F, A)$ is said to be a *bounded finitary linear group*, if there is a positive integer b such that $dim_F A/C_A(g) \leq b$ for each element $g \in G$. These groups are some linear analogs of BFC-groups. Let ωRG be the *augmentation ideal* of the group ring RG, i.e. the two-sided ideal of RG generated by the all elements $g - 1,\ g \in G$. The submodule $A(\omega FG)$ is called the derived submodule. We can consider the derived submodule as a linear analog of the derived subgroup. Note that in the general case we cannot obtain an analog of Neumann's theorem. It is not hard to construct an F_pG-module A over an infinite elementary abelian group G such that G is bounded finitary linear group but $A(\omega F_pG)$ has infinite dimension over F_p (see [7]). However, under some natural restrictions on the p-sections of a bounded finitary linear group, the finiteness of $dim_F(A(\omega FG))$ can be proved. Thus some linear analog of B. H. Neumann's theorem can be established. We considered a more general situation.

Let A be an artinian $\mathbb{Z}$-module. Then a set $\Pi(A)$ is finite. If D is a divisible part of A, then $D = K_1 \oplus \ldots \oplus K_d$ where K_j is a Prüfer subgroup, $1 \leq J \leq d$. The number d is an invariant of A. Another important invariant here is the order of A/D.

If D is a Dedekind domain, the structure of the artinian D-module A is very similar to that described above. Let D be a Dedekind domain. Put

$$Spec(D) = \{P\,|P \text{ is a maximal ideal of } D\}$$

Let P be a maximal ideal of D. Denote by A_P the set of all elements a such that $Ann_D(a) = P^n$ for some positive integer n. If A is a D-periodic module, then define

$$Ass_D(A) = \{P \in Spec(D)|A_P \neq < 0 >\}.$$

In this case, $A = \oplus_{P\in\pi} A_P$ where $\pi = Ass_D(A)$ (see, for instance, [8], Corollary 6.25). If A is an artinian D-module, then A is D-periodic and the set $Ass_D(A)$ is finite. Furthermore, $A = K_1 \oplus \cdots \oplus K_d \oplus B$ where K_j is a Prufer submodule, $1 \leq j \leq d$, B is a finitely generated submodule (see, for instance, [9], Theorem 5.7). Here the Prufer submodule is a D-injective evelope of a simple submodule. Observe that this decomposition is unique up to isomorphism. It follows that the number d is an invariant of the module A. Put $d = I_D(A)$. The submodule B has a finite series of submodules with D - simple factors. The Jordan-Holder Theorem implies that the length of this composition series is also an invariant of B, and hence of A. Denote this number by $I_F(A)$.

Let D be a Dedekind domain and G a group. The DG-module A is said to be a *bounded artinian finitary* if A is artinian finitary and there are positive integers b and d and a finite subset $\tau \subseteq Spec(D)$ such that $I_F(A/C_A(g)) \leq b$, $I_D(A/C_A(g)) \leq d$ and $Ass_D(A/C_A(g)) \subseteq b_\sigma(A)$. We will use the following notation:

$$\pi(A) = \{p|p = charD/P \quad \text{for all} \quad P \in b_\sigma(A)\}.$$

The group G is said to be *generalized radical* if G has an ascending series whose factors are either locally nilpotent or locally finite. Let p be a prime. We say that a group G has *finite section p-rank* $r_p(G) = r$ if every elementary abelian p-section U/V of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

In the paper [10], the following analog of Neumann's theorem has been obtained.

Theorem 1. (L.A.Kurdachenko, I.Ya.Subbotin, V.A.Chepurdya [10]) *Let D be a Dedekin domain, G a locally generalized radical group, and A a DG-module. Suppose that A is a bounded artinian finitary module. Assume also that there exists a positive integer r such that the section p-rank of G is at most r for all $p \in \pi(A)$. Then* *a) the submodule $A(\omega FG)$ is artinian as a D-module,*

b) the factor-group $G/C_G(A)$ has finite special rank.

Corollary 1. *Let F be a field, A a vector space over F, G a locally generalized radical subgroup of $GL(F, A)$. Suppose that there exists a positive integer r such that the section p-rank of G is at most r were $p = charF$. Then*

a) the submodule $A(\omega FG)$ is finite dimensional,

b) the factor-group $G/C_G(A)$ has finite special rank.

As we noted above the restriction on the section p-rank is essential.

2 Linear groups that are dual to finitary

Consider another analog of FC-groups which is dual in some sense to finitary linear groups. We introduce this concept not only for linear groups, but in a more general situation.

Let R be a ring, G a group and A an RG-module. If a is an element of A, then the set

$$aG = \{ag | g \in G\}$$

is called *the G-orbit of a*.

We say that G *has finite orbits on* A if the orbit aG is finite for all $a \in A$.

By the orbit stabilizer theorem, it is clear that in this situation, $|aG| = |G : C_G(a)|$ is finite, so we can think of aG as the analog of a conjugacy class.

Let F be a field and let G be a subgroup of $GL(F, A)$. Suppose that $dim_F(A)$ is finite and choose a basis $a_1, ..., a_n$ for the vector space A. Suppose that G has finite orbits on A. Then every element of $C_G(a_1) \cap ... \cap C_G(a_n)$ acts trivially on A, and hence $C_G(a_1) \cap ... \cap C_G(a_n) =< 1 >$. However, this intersection has finite index in G and hence G is finite. Thus, we can think of linear groups with finite orbits as generalizations of finite groups.

We say that G has *boundedly finite orbits on* A if there is a positive integer b such that $|aG| \leq b$ for each element $a \in A$. The smallest such b will be denoted by $lo_A(G)$.

Since $|aG| = |G : C_G(a)|$ for all $a \in A$, it is not hard to see that any group G in which $G/C_G(A)$ is finite has boundedly finite orbits on A. However, as the following example shows, the converse statement is far from being true.

Let A be a vector space over the field F admitting the basis $\{a_n | n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ the mapping $g_n : A \longrightarrow A$, given by

$$a_m g_n = \begin{cases} a_1 + a_m & \text{if } m = n+1 \\ a_m & \text{if } m \neq n+1 \end{cases}$$

is an F-automorphism of A. Then $G = \langle g_n | n \in \mathbb{N} \rangle$ is a subgroup of $GL(F, A)$. Clearly $[g_n, g_m] = 1$ whenever $n \neq m$, so that G is abelian. Moreover, if $char F = p > 0$, then G is an elementary abelian p-group. It follows in this case that $ag = a + ta_1$ for every $a \in A$, where $0 \leq t < p$. Consequently,

$$aG = \{a, a + a_1, a + 2a_1, ..., a + (p-1)a_1\}.$$

Therefore, $|aG| \leq p$ for each element $a \in A$, and G has boundedly finite orbits on A. However, it is clear that $C_G(A) =< 1 >$, so that $G/C_G(A)$ is infinite.

Let B be a vector space over a field F of characteristic $p > 0$ admitting the basis $\{b_n | n \in \mathbb{N}\}$. We define the mapping $x : B \longrightarrow B$ by the rule

$$b_m x = \begin{cases} b_m & \text{if } m \text{ is even} \\ b_{2n} + b_{2n+1} & \text{if } m = 2n+1. \end{cases}$$

Clearly, x is an F-automorphism of B and $B(\omega F < x >) = \oplus_{n \in \mathbb{N}} b_{2n} F$. In particular, the dimension of $B(\omega F < x >)$ is infinite. Since $|x| = p$, $|b < x >| \leq p$ for each element $b \in B$. Now let A and G be the vector space and the linear group from the first example, respectively. Then $L = G \times < x >$ acts on the vector space $C = A \oplus B$ in the natural way. Clearly, $|cL| \leq p^2$ for every element $c \in C$. However, the factor-group $L/C_L(C)$ is infinite and the dimension of $C(\omega FL)$ is infinite. In other words, we cannot have an analog of Neumann's theorem.

Next result describes linear groups acting with boundedly finite orbits.

Theorem 2. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, R a ring and A an RG-module. Suppose that G acts on A with boundedly finite G-orbits, and let $b = lo_A(G)$. Then*

i) $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite exponent such that G/L is finite.

ii) A contains an RG-submodule C such that C is finitely generated as an R-module and L acts trivially on C and A/C.

iii) There is a positive integer m such that m is a divisor of $b!$ and $mA(\omega RG) =< 0 >$.

Note that in the above statement the submodules of C need not be finitely generated. Therefore, we cannot deduce in this theorem that $A(\omega RG)$ is finitely generated as an R-module. However, if R is noetherian, then every finitely generated R-submodule is also noetherian. So in this case, every submodule of C is finitely generated. Even when R is a noetherian ring so that $A(\omega RG)$ is finitely generated, in general it appears that nothing can be deduced concerning its number of generators. We can now establish our next main theorem.

Theorem 3. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, R a noetherian ring and A an RG-module.*

i) Suppose that G acts on A with boundedly finite G-orbits, and let $b = lo_A(G)$. Then $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finitely generated.

ii) If a factor-group $G/C_G(A)$ has a normal subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finite, then G has boundedly finite orbits on A.

iii) If there is an integer b such that $R/b!\, R$ is finite and $b = lo_A(G)$, then $G/C_G(A)$ contains a normal abelian subgroup $L/C_G(A)$ of finite index and finite exponent such that $A(\omega RG)$ is finite.

Next we give some specific examples of rings satisfiyng the conditions of Theorem 3. Of course, one particular interesting example is the ring of integer.

Corollary 2. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group acting on the $\mathbb{Z}G$-module A. Then G has boundedly finite orbits on A if and only if G contains a normal subgroup L such that G/L and $A(\omega \mathbb{Z}L)$ are finite.*

Next result is a generalization of Corollary 2. An infinite Dedekin domain D is said to be a *Dedekind $\mathbb{Z}_0$-domain* if for every maximal ideal P of D, the quotient ring D/P is finite (see for instance [9], Chapter 6). If F is a finite field extension of $\mathbb{Q}$ and R is a finitely generated subring of F, then R is an example of a Dedekind $\mathbb{Z}_0$ domain.

Corollary 3. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, D a Dedekind $\mathbb{Z}_0$-domain and A a DG-module. Then G has boundedly finite orbits on A if and only if there exists a normal abelian subgroup $L/C_G(A)$ of $G/C_G(A)$ of finite index and finite exponent such that $A(\omega DG)$ is finite.*

For the case when the ring of scalars is a field, we obtain

Theorem 4. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, F a field of characteristic $p > 0$ and A an FG-module. Suppose that G acts on A with boundedly finite G-orbits. Then*

i) $G/C_G(A)$ contains a normal abelian p-subgroup $L/C_G(A)$ of finite exponent such that G/L is finite.

ii) A contains an FG-submodule C such that $dim_F(C)$ is finite and L acts trivially on C and A/C.

Next result deals with the situation when $G/C_G(A)$ is finite.

Theorem 5. (M.R.Dixon, L.A.Kurdachenko, J.Otal [11]) *Let G be a group, F a field and A an FG-module. Suppose that G acts on A with boundedly finite G-orbits. Assume that if $charF = p > 0$, then $G/C_G(A)$ is a p'-group. Then $G/C_G(A)$ is finite.*

In particular, if F is a field of characteristic 0, then G acts on the FG-module A with boundedly finite G-orbits if and only if $G/C_G(A)$ is finite.

We consider now the following generalization. If a group G acts on A with finite G-orbits, then an FG-submodule aFG has finite dimension over F.

Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. We say that G is a *linear group with finite dimensional G-orbits* (or that *A has finite dimensional G-orbits*) if the G-orbit aG generates a finite dimensional subspace for each element $a \in A$.

As we have seen above, if a group G has finite G-orbits then G has finite dimensional G-orbits, but the converse is false. Every ordinary finite dimensional linear group G is a group with finite dimensional G-orbits. But we have seen above that if a finite dimensional linear group G has finite G-orbits, then G is finite.

We say that a linear group G *has boundedly finite dimensional orbits on A* if there is a positive integer b such that $dim_F(aFG) \leq b$ for each element $a \in A$. Put

$$md(G) = max\{dim_F(aFG) \quad | \quad a \in A\}.$$

Every linear group G defined over a finite dimensional vector space A is a group with boundedly finite dimensional orbits.

In view of Neumann's result, a natural question arises: when is $dim_F(A(\omega FG))$ finite? An easy computation shows that $aFG \leq A(\omega FG) + aF$ for each $a \in A$, and hence if $dim_F(A(\omega FG)) \leq d$ then aFG is of F-dimension at most $d+1$. Thus, if $A(\omega FG)$ is finite dimensional, then G has boundedly finite dimensional orbits. However, as we showed above, even for linear groups having boundedly finite orbits on A, the converse is false. It would be interesting to know which conditions imposed on a group G implies that $A(\omega FG)$ is finite dimensional.

Let B be a subspace of A, then the *norm of B* in G is the subgroup

$$Norm_G(B) = \bigcap_{b \in B} N_G(bF).$$

Observe that $Norm_G(B)$ is the intersection of the normalizers of all F-subspaces of B, and that $G = Norm_G(A)$ if and only if every subspace of A is G-invariant.

The following theorem provides us with a description of linear groups having boundedly finite dimensional orbits on A.

Theorem 6. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Suppose that G has boundedly finite dimensional orbits on A and let $b = md(G)$. Then*

i) A has an FG-submodule D such that $dim_F(D)$ is finite and if $K = C_G(D)$, then $K \leq Norm_G(A/D)$. Moreover there exists a function f such that $dim_F(D) \leq f(b)$.

ii) K is a normal subgroup of G and has a G-invariant abelian subgroup T such that $A(\omega FT) \leq D$ and K/T is isomorphic to a subgroup of the multiplicative group of a field F.

iii) T is an elementary abelian p-subgroup if $char F = p > 0$ and is a torsion-free abelian group otherwise.

In particular, G is an extension of a metabelian group by a finite dimensional linear group.

We use Theorem 6 to establish several properties of groups with boundedly finite dimensional orbits that are analogs to corresponding results for finite dimensional linear groups. There are many applications of Theorem 6. Here we just select some of them. It is a well-known theorem of Schur that periodiic finite dimensional linear groups are locally finite.

Corollary 4. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Suppose that G has boundedly finite dimensional orbits on A.*

i) If G is periodic then G is locally finite.

ii) If G is locally generalized radical then G is locally (finite and soluble).

iii) If G is a periodic p'-group, where $p = char F$, then the centre of G includes a locally cyclic subgroup K such that G/K is soluble-by-finite.

Now we consider another topic: the reduction to the groups with finite dimensional orbits.

Let again G be a subgroup of $GL(F, A)$. We say that G is a *linear group with finite G-orbits of subspaces* if the set $cl_G(B) = \{Bg \quad | \quad g \in G\}$ is finite for each F-subspace B of A. Groups with this property are natural analogs of groups with finite G-orbits of elements. Since it is clear that $|cl_G(B)| = |G : N_G(B)|$, it follows that G has finite G-orbits of subspaces if and only if the indexes $|G : N_G(B)|$ are finite for all F-subspaces B of A. It is not hard to prove that if G has finite G-orbits of subspaces then $dim_F(aFG)$ is finite, for each element $a \in A$.

Observe that if every F-subspace B is G-invariant, then G is abelian. Linear groups with finite G-orbits of subspaces can be considered as natural generalizations of abelian linear groups.

For these groups we obtain the following result.

Theorem 7. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Suppose that G is a linear group with finite G-orbits of subspaces. Then the factor group $G/Norm_G(A)$ is finite and G is central-by-finite.*

We say that a group has *boundedly finite G-orbits of subspaces* if there is a positive integer b such that $|cl_G(B)| \leq b$ for all subspaces B of A.

Corollary 5. (M.R.Dixon, L.A.Kurdachenko, J.Otal [12]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$. Then G has finite G-orbits of subspaces if and only if G has boundedly finite G-orbits of subspaces.*

3 Linear groups with restriction on subgroups of infinite central dimension

If H is a subgroup of $GL(F, A)$, then H really acts on the factor-space $A/C_A(H)$. Following [13] we say that H has *finite central dimension*, if $dim_F(A/C_A(H))$ is finite. In this case $dim_F(A/C_A(H)) = centdim_F(H)$ will be called the *central dimension of the subgroup H*.

If H has finite central dimension, then $A/C_A(H)$ is finite dimensional. Put $C = C_G(A/C_A(H))$. Then, clearly, C is a normal subgroup of H and H/C is isomorphic to some subgroup of $GL_n(F)$ where $n = dim_F(A/C_A(H))$. Each element of C acts trivially on every factor of the series $< 0 > \leq C_A(H) \leq A$, so that C is an abelian subgroup. Moreover, if $charF = 0$, then C is torsion-free; if $charF = p > 0$, then C is an elementary abelian p-subgroup. Hence, the structure of H in general is defined by the structure of G/C, which is an ordinary finite dimensional linear group.

Let $G \leq GL(F, A)$ and let $L_{icd}(G)$ be the set of all proper subgroups of G having infinite central dimension. In the paper [13], it has been proved that if every proper subgroup of G has finite central dimension, then either G has finite central dimension or G is a Prufer p-group for some prime p (under some natural restrictions on G). This shows that it is natural to consider those linear groups G, in which the family $L_{icd}(G)$ is "very small" in some particular sense. But what means "very small" for infinite groups? One of the natural approaches possible here is to consider finiteness conditions. More precisely, it is natural to consider the groups in which the family $L_{icd}(G)$ satisfies a suitable strong finiteness condition. In the paper [14] we considered some of such situations. In particular, linear groups in which the family $L_{icd}(G)$ satisfies either the minimal or the maximal condition and some rank restriction were considered. The weak minimal and weak maximal conditions are natural group-theoretical generalizations of the ordinary minimal and maximal conditions. These conditions have been introduced by R.Baer [15] and D.I.Zaitsev [16]. The definition of the weak minimal condition in the most general form is the following.

Let G be a group and $\mathcal{M}$ a family of subgroups of G. We say that $\mathcal{M}$ satisfies the *weak maximal* (respectively *minimal*) condition (or that G satisfies the *weak maximal* (respectively *minimal*) condition for $\mathcal{M}$-subgroups), if for every ascending (respectively descending) chain $\{H_n \mid n \in \mathbb{N}\}$ of subgroups in the family M there exists a number $m \in \mathbb{N}$ such that the indexes $|H_{n+1} : H_n|$ (respectively $|H_n : H_{n+1}|$) are finite for all $n \geq m$.

Groups with the weak minimal or maximal conditions for some important families of subgroups have been studied by many authors (see, for instance, the book [17],5.1, and the survey [18]).

We say that a *group $G \leq GL(F, A)$ satisfies the weak maximal (respectively minimal) condition for subgroups of infinite central dimension, or shortly $Wmax - icd$ (respectively $Wmin - icd$)*, if the family $L_{icd}(G)$ satisfies the weak maximal (respectively minimal) condition.

The first results about linear groups satisfying the conditions $Wmin - icd$ and $Wmax - icd$ have been obtained in [19]. More precisely, this paper was devoted to the study of periodic groups with such properties. The main results are the following

Theorem 8. (J.M. Munoz-Escolano, J. Otal, N.N. Semko [19]) *Let F be a field, A a vector space over F and G a locally soluble periodic subgroup of $GL(F, A)$. Suppose that G has infinite central dimension and satisfies $Wmin - icd$ or $Wmax - icd$. The following assertions hold*

1) *If $char F = 0$, then G is a Chernikov group.*

2) *If $char F = p > 0$, then either G is a Chernikov group or G has a series of normal subgroups $H \leq D \leq G$ satisfying the following conditions:*

   2a) *H is a nilpotent bounded p-subgroup.*

   2b) *$D = H\lambda Q$ for some non-identity divisible Chernikov subgroup Q such that $p \notin \Pi(Q)$.*

   2c) *H has finite central dimension, Q has infinite central dimension.*

   2d) *If K is a Prufer q-subgroup of Q and K has infinite central dimension, then H has a finite K-composition series.*

   2e) *G/D is finite.*

Corollary 6. (J.M. Munoz-Escolano, J. Otal, N.N. Semko [19]) *Let F be a field, A a vector space over F and G a locally soluble periodic subgroup of $GL(F, A)$. Then the following conditions are equivalent:*

i) G satisfies the weak minimal condition on subgroups of infinite central dimension;

ii) G satisfies the weak maximal condition on subgroups of infinite central dimension;

iii) G satisfies the minimal condition on subgroups of infinite central dimension.

Corollary 7. (J.M.Munoz-Escolano, J.Otal, N.N.Semko [19]) *Let F be a field, A a vector space over F and G a locally nilpotent subgroup of $GL(F, A)$. Suppose that G has infinite central dimension. Then the following conditions are equivalent:*

i) G satisfies the weak minimal condition on subgroups of infinite central dimension;

ii) G satisfies the weak maximal condition on subgroups of infinite central dimension;

iii) G satisfies the minimal condition on subgroups of infinite central dimension;

iv) G is Chernikov; and

v) G satisfies the minimal condition on all subgroups.

For non-periodic groups, the situation is more complicated. The study of locally nilpotent linear groups satisfying $Wmin - icd$ and $Wmax - icd$ has been initiated in the papers [20], [21]. The first result shows that nilpotent groups with these conditions are minimax.

Theorem 9. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let F be a field, A a vector space over F and G a subgroup of $GL(F, A)$ having infinite central dimension. Suppose that H is a normal subgroup of G such that G/H is nilpotent. If G satisfies either $Wmin - icd$ or $Wmax - icd$, then G/H is minimax. In particular, if G is nilpotent, then G is minimax.*

Further results deal with to the case of prime characteristic.

Theorem 10. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let F be a field of prime characteristic, A a vector space over F and G a locally* *nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. if* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/Tor(G)$ *is minimax. In particular, if* $Tor(G)$ *has infinite central dimension, then* G *is minimax.*

Here $Tor(G)$ is the maximal normal periodic subgroup of G. If G is locally nilpotent group, then $Tor(G)$ consists of all elements of finite order, so that $G/Tor(G)$ is torsion-free.

Let $\mathcal{F}$ be the class of finite groups. If G is a group, then the intersection $G_{\mathcal{F}}$ of all subgroups of G, having finite index, is called the *finite reidual* of G.

Theorem 11. (L.A.Kurdachenko, J.M.Munoz-Escolano and J.Otal [20]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/G_{\mathcal{F}}$ *is minimax and nilpotent.*

Let $\mathcal{N}$ be the class of nilpotent groups. The intersection $G_{\mathcal{N}}$ of all normal subgroups H such that G/H is nilpotent, is called the *nilpotent residual* of G.

Theorem 12. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [20]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *satisfies either* $Wmin - icd$ *or* $Wmax - icd$*, then* $G/G_{\mathcal{N}}$ *is minimax.*

For the case of non-finitary linear groups, the following results were obtained.

Theorem 13. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal, N.N. Semko [21]) *Let* F *be a field,* A *a vector space over* F *and* G *a locally nilpotent subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *is not finitary and satisfies* $Wmin - icd$*, then* G *is minimax.*

For the case of hypercentral groups and prime characteristic the study was completed. In fact, the following holds

Theorem 14. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal, N.N. Semko [21]) *Let* F *be a field of prime characteristic,* A *a vector space over* F *and* G *a hypercentral subgroup of* $GL(F, A)$ *having infinite central dimension. If* G *nsatisfies* $Wmin - icd$*, then* G *is minimax.*

We observe that for the condition $Wmax - icd$ a similar result is not true. In the paper [21], a hypercentral linear group over the field of prime characteristic sarisfying $Wmax - icd$ which is not minimax was constructed.

The paper [22] began the study of soluble linear groups satisfying $Wmin - icd$. The following main result of this paper shows that their structure is rather similar to the structure of finite dimenional soluble groups.

Let $G \leq GL(F, A)$. We recall that an element $x \in G$ is called *unipotent* if there is a positive integer n such that $A(x - 1)^n = 0$. A subgroup H of G is called *unipotent* if every element of H is unipotent. A subgroup H of G is called *boundedly unipotent* if there is a positive integer n such that $A(x-1)^n = 0$ for each element $x \in H$.

Theorem 15. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [22]) *Let F be a field, A a vector space over F and G a soluble subgroup of $GL(F, A)$. Suppose that G has infinite central dimension and satisfies $Wmin - icd$. If G is not minimax, then G satisfies the following conditions:*

i) G has a normal boundedly unipotent subgroup L such that G/L is minimax;

ii) L has finite central dimension;

iii) if $charF = 0$, then L is nilpotent torsion-free subgroup;

iv) if $charF = p$ for some prime p, then L is a nilpotent bounded p-subgroup;

v) G is a finitary linear group.

If G is a subgroup of $GL(F, A)$, then G acts trivially on the factor-space $A/A(\omega FG)$. Hence G properly acts on the subspace $A(\omega FG)$. As in paper [23], we define the *augumentation dimension* of G to be the F-dimension of $A(\omega FG)$ and denote it by $augdim_F(G)$. This concept is opposite in some sense to the concept of central dimension. As for groups having finite central dimension, a group G of finite augmentation dimension contains a normal abelian subgroup C such that G/C is an ordinary finite dimensional group. Moreover, if $charF = 0$, then C is torsion-free, if $charF = p > 0$, then C is an elementary abelian p-subgroup. In the paper [23] linear groups in which the set of all subgroups having infinite augmentation dimension satisfies the minimal condition have been considered. In the paper [24] linear groups in which the set of all subgroups having infinite augmentation dimension satisfies some rank restrictions have been considered.

We can define finitary linear groups as the groups whose cyclic (and therefore finitely generated) subgroups have finite augmentation dimension. Therefore the following groups are the antipodes to finitary linear groups.

We say that a group $G \leq GL(F, A)$ is called *antifinitary linear group* if each proper infinitely generated subgroup of G has finite augmentation dimension (a subgroup H of an arbitrary group G is called *infinitely generated* if H has no a finite set of generators). These groups have been studied in the paper [25]. This study splits into two cases depending on whether or not the group is finitely generated.

Let $G \leq GL(F, A)$. Then the set

$$FD(G) = \{x \in G \quad | \quad <x> \quad \text{has finite augmentation dimension}\}$$

is a normal subgroup of G.

Let D be a divisible abelian group and G a subgroup of $Aut(D)$. Then D is said to be G-*divisibility irreducible* if D has no proper divisible G-invariant subgroups.

Theorem 16. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [25]) *Let F be a field, A a vector space over F and G a infinitely generated locally generalized radical subgroup of $GL(F, A)$. Suppose that G is not finitary and has infinite augmentation dimension. If G is not minimax, then G satisfies the following conditions:*

1) *If the factor-group $G/FD(G)$ is infinitely generated, then G is a Prüfer p-group for some prime p.*

2) *If $G/FD(G)$ is finitely generated, then G satisfies the following conditions:*

   2a) *$G = K < g >$ where K is a divisible abelian Chernikov subgroup and g is a p-element, where p is a prime such that $p = |G/FD(G)|$;*

   2b) *K is a normal subgroup of G;*

   2c) *K is G-divisibly irreducible;*

   2d) *K is a q-subgroup for some prime q;*

   2f) *if $q = p$, then K has finite special rank equal to $p^{m-1}(p-1)$ where $p^m = |< g > /C < g > (K)|$;*

   2g) *if $q \neq p$, then K has finite special rank $o(q, p^m)$ where as above $p^m = |< g > /C < g > (K)|$ and $o(q, p^m)$ is the order of q modulo p^m.*

Theorem 17. (L.A. Kurdachenko, J.M. Munoz-Escolano, J. Otal [25]) *Let F be a field, A a vector space over F and G a finitely generated radical subgroup of $GL(F, A)$. Suppose that G is not finitary and has infinite augmentation dimension. Then the following conditions holds:*

1) *$augdim_F FD(G)$ is finite;*

2) *G has a normal subgroup U such that G/U is polycyclic;*

3) *there is a positive integer m such that $A(x-1)m =< 0 >$ for each $x \in U$; in particular, U is nilpotent;*

4) *U is torsion-free if $charF = 0$ and is a bounded p-subgroup if $charF = p > 0$;*

5) *if*
$$< 0 >= Z_0 \leq Z_1 \leq \cdots \leq Z_m = U \tag{1}$$

is the upper central series of U, then $Z_1/Z_0, \ldots, Z_m/Z_{m-1}$ are finitely generated $\mathbb{Z} < g >$-modules for each element $g \in G\backslash FD(G)$. In particular, U satisfies the maximal condition on $< g >$-invariant subgroups for each element $g \in G\backslash FD(G)$.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 37–53. doi:10.1285/i15900932v30sn1p37

Recent Results on Generalized Baumslag-Solitar Groups

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Abstract. Generalized Baumslag-Solitar groups are the fundamental groups of finite graphs of groups with infinite cyclic vertex and edge groups. These groups have interesting group theoretic and algorithmic properties and they also have close connections with algebraic topology. Here we present an introduction to the theory with an account of recent results.

Keywords: Graph of groups, generalized Baumslag-Solitar group.

MSC 2000 classification: 20E06

1 Graphs of Groups

Let Γ be a connected graph, with loops and multiple edges allowed, and write

$$V(\Gamma) \quad \text{and} \quad E(\Gamma)$$

for the respective sets of vertices and edges of Γ. If $e \in E(\Gamma)$, we assign endpoints e^+, e^- and hence a direction to e,

$$\bullet_{e^-} \longrightarrow \bullet_{e^+}$$

To each $e \in E(\Gamma)$ and $x \in V(\Gamma)$ we assign groups H_e and G_x and we assume there are injective homomorphisms

$$\phi_{e^-} : H_e \to G_{e^-} \quad \text{and} \quad \phi_{e^+} : H_e \to G_{e^+}.$$

Then the system

$$\mathcal{G} = (\Gamma, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma),\ x \in V(\Gamma)),$$

ia called a *graph of groups.*

Next choose a maximal subtree T in Γ. Then the *fundamental group* of the graph of groups $\mathcal{G}$ is the group

$$G = \pi_1(\mathcal{G})$$

which is generated by the groups and elements

$$G_x \text{ and } t_e, \quad (x \in V(\Gamma),\ e \in E(\Gamma \backslash T)),$$

subject to the defining relations

$$h^{\phi_{e^-}} = h^{\phi_{e^+}}, \quad (e \in E(T)), \quad (h^{\phi_{e^+}})^{t_e} = h^{\phi_{e^-}}, \quad (e \in E(\Gamma \backslash T)),$$

for all $h \in H_e$. In the case where Γ is a tree, G is called a *tree product.*

The following result is fundamental – see [3], [5], [15].

(1.1). *Up to isomorphism the group $G = \pi_1(\mathcal{G})$ is independent of the choice of maximal subtree.*

Special cases of interest

(i) Let Γ have two vertices and a single edge e. Then G is the generalized free product

$$G = G_{e^-} *_H\ G_{e^+}$$

where the subgroup $H = H_e$ is amalgamated by means of the injective homomorphisms ϕ_{e^-} and ϕ_{e^+}.

(ii) Let Γ have one vertex x and one edge e, i.e., it is a loop. Then G is the HNN-extension

$$G =< t_e, G_x \mid (h^{\phi_{e^+}})^{t_e} = h^{\phi_{e^-}},\ h \in H_e > .$$

Here G_x is the base group, $H^{\phi_{e^-}}$ and $H^{\phi_{e^-}}$ are the associated subgroups, and t_e is the stable element.

We note an important property of graphs of groups.

(1.2). *Let $\mathcal{G} = (\Gamma, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma),\ x \in V(\Gamma))$ be a graph of groups and let Γ_0 be a connected subgraph of Γ. Define $G_0 = \pi_1(\mathcal{G}_0)$ where*

$$\mathcal{G}_0 = (\Gamma_0, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma_0), x \in V(\Gamma_0)).$$

Then the natural homomorphism from G_0 to G is injective. In particular each G_x is isomorphic with a subgroup of G.

For a detailed account of the theory of graphs of groups the reader may consult [3], [5], [15].

2 Generalized Baumslag-Solitar groups

A *Baumslag-Solitar group* is a 1-relator group with a presentation of the form

$$BS(m,n) =< t, x \mid (x^m)^t = x^n >,$$

where $m, n \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$: these groups seem to have first appeared in the literature in [1], but they may be of greater antiquity.

A similar type of 1-relator group is

$$K(m,n) =< x, y \mid x^m = y^n >,$$

where $m, n \in \mathbb{Z}^*$. When m and n are relatively prime, this is a *torus knot group*.

The groups $BS(m,n)$ and $K(m,n)$ are the fundamental groups of graphs of infinite cyclic groups where the graph is a 1-loop or a 1-edge respectively. There is a natural way to generalize these groups.

Let Γ be a finite connected graph. Associate infinite cyclic groups $< g_x >$ and $< u_e >$ to each vertex x and edge e and define injective homomorphisms

$$< u_e > \to < g_{e^-} > \text{ and } < u_e > \to < g_{e^+} >$$

by the assignments

$$u_e \ \mapsto \ (g_{e^-})^{\omega^-(e)} \text{ and } \ u_e \ \mapsto \ (g_{e^+})^{\omega^+(e)}$$

where $\omega^-(e), \omega^+(e) \in \mathbb{Z}^*$. So the edge e is assigned a *weight* $(\omega^-(e), \omega^+(e))$ and the graph of groups is determined by a *weight function*

$$\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*,$$

with values

$$\omega(e) = (\omega^-(e),\ \omega^+(e)).$$

We will write the weighted graph of infinite cyclic groups in the form

$$(\Gamma, \omega)$$

and refer to it as a *generalized Baumslag-Solitar graph* or *GBS-graph*.

Definition 1. A *generalized Baumslag-Solitar group*, or *GBS-group*, is the fundamental group of a GBS-graph (Γ, ω), in symbols

$$G = \pi_1(\Gamma, \omega).$$

To obtain a presentation of G choose a maximal subtree T in Γ; then G has generators

$$t_e,\ g_x,\quad e \in E(\Gamma\backslash T),\quad x \in V(\Gamma),$$

and defining relations

$$(g_{e^-})^{\omega^-(e)} = (g_{e^+})^{\omega^+(e)},\quad e \in E(T),$$

$$(g_{e^-})^{\omega^-(e)} = ((g_{e^+})^{\omega^+(e)})^{t_e},\quad e \in E(\Gamma\backslash T).$$

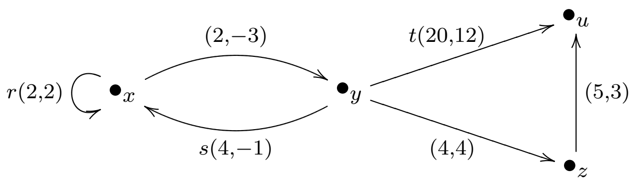
Note that up to isomorphism G does not depend on the choice of the maximal subtree.

Examples

1. If Γ is a 1-loop with weight (m, n), then $G = BS(n, m)$.

2. If Γ is a 1-edge with weight (m, n), then $G = K(m, n)$.

3. As a more complex example, consider the GBS-graph shown below.



Choose as the maximal subtree T the path $xyzu$ and let the stable letters be r, s, t as indicated. Then the corresponding GBS-group G has a presentation with generators

$$r, s, t, g_x, g_y, g_z, g_u$$

and relations

$$(g_x^2)^r = g_x^2,\ g_x^2 = g_y^{-3},\ g_y^4 = g_z^4,\ g_z^5 = g_u^3,\ (g_u^{12})^t = g_y^{20},\ (g_x^4)^s = g_y^{-1}.$$

3 Some Properties of GBS-groups

We list some known properties of GBS-groups. Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group.

(3.1). *The group G is finitely presented and torsion-free.*

For if F is a finite subgroup of G, it intersects each conjugate of a vertex group trivially, which implies that it is free and therefore trivial ([5], p.212).

(3.2). *If* Γ *is a tree, so that* G *is a GBS-tree product, then* G *is locally extended residually finite. Hence* G *is hopfian.*

Recall here that a group is *locally extended residually finite* (or LERF) if every finitely generated subgroup is closed in the profinite topology.

Proof of (3.2). Since Γ is a tree and $< g_x > \cap < g_y > \neq 1$ for all $x, y \in V(\Gamma)$, each vertex generator has a positive power lying in the centre. Hence $Z(G) =< z > \neq 1$ and $G/ < z >$ is the fundamental group of a graph of finite cyclic groups. It follows that $G/ < z >$ is virtually free (see Karrass, Pietrowski and Solitar [9]). If $n > 0$, then $G/ < z^n >$ is also virtually free. Since finitely generated free groups are LERF, (M. Hall [8]), G is LERF. □

Corollary 1. *The generalized word problem soluble in any GBS-tree product.*

GBS-tree products have another strong residual property.

(3.3). *A GBS-tree product* G *is conjugacy separable, i.e., if two elements are conjugate in every finite quotient of* G*, then they are conjugate in* G*.*

This follows from a theorem of Kim and Tang [10]: *if* G *is a (finite) tree product of groups each of which is finitely generated torsion-free nilpotent and if the amalgamations are cyclic, then* G *is conjugacy separable.*

Corollary 2. *The conjugacy problem is soluble in any GBS-tree product.*

Remark. In general $BS(m, n)$ is not hopfian, and hence is not even residually finite. For example, let $G =< t, g | \ (g^m)^t = g^n >$ where $\gcd(m, n) = 1$. Define an endomorphism θ of G by

$$t^\theta = t, \quad g^\theta = g^n.$$

Then θ is a surjective since $\mathrm{Im}(\theta)$ contains g^n and also $(g^n)^{t^{-1}} = g^m$, so $g \in \mathrm{Im}(\theta)$. But θ is not an automorphism of G if $m, n \neq \pm 1$, since $[g, g^{t^{-1}}]^\theta = 1$ and $[g, g^{t^{-1}}] \neq 1$.

The next result is an important characterization of GBS-groups due to Kropholler [11].

(3.4). *The non-cyclic GBS-groups are exactly the finitely generated groups of cohomological dimension 2 that have an infinite cyclic subgroup which is commensurable with its conjugates, i.e., intersecting each conjugate non-trivially.*

Kropholler also showed that there is a type of Tits alternative for GBS-groups, (Kropholler [11]).

(3.5). *The second derived subgroup of a GBS-group is free.*

Since free groups are residually soluble, we deduce from the last result:

Corollary 3. *Every GBS-group is residually soluble.*

The subgroups of a GBS-group are of very restricted type, as the next result shows.

(3.6). *Let H be a finitely generated subgroup of a GBS-group G. Then H is either free or a GBS-group.*

Proof. Assume that H is not free, so G is certainly non-cyclic. Now $\mathrm{cd}(H) \leq \mathrm{cd}(G) = 2$. If $\mathrm{cd}(H) = 1$, then by a result of Stallings and Swan the group H is free, since it is torsion-free: (for these results see [2], Chapter II). By this contradiction $\mathrm{cd}(H) = 2$. Now H must contain a commensurable element since otherwise it is free. Therefore by (3.4) H is a GBS-group. □

Corollary 4. *A GBS-group is coherent, i.e., all its finitely generated subgroups are finitely presented.*

Since GBS-groups have cohomological dimension 2 in general, it is natural to enquire about their (co)homology in dimensions 1 and 2. We begin with homology. Recall that

$$H_1(G) \simeq G_{ab} = G/G' \text{ and } H_2(G) \simeq M(G),$$

the Schur multiplier. We will investigate these groups in the next two sections.

4 The Abelianization of a GBS-group

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group defined with respect to a maximal subtree T of Γ. Then $G_{ab} = G/G'$ is the finitely generated abelian group with generators

$$t_e, \quad g_x \text{ where } e \in E(\Gamma \backslash T), \ x \in V(\Gamma),$$

and (abelian) defining relations

$$(g_{e^-})^{\omega^-(e)} = (g_{e^+})^{\omega^+(e)}, \ e \in E(\Gamma).$$

To find the complete structure of G_{ab} the *weight matrix* W must be transformed into Smith normal form. This matrix has rows indexed by edges and columns indexed by vertices: row e has entries

$$0, \ldots, 0, \ \omega^-(e) - \omega^+(e), \ 0 \ldots, 0$$

if e is a loop, and

$$0, \dots, 0,\ \omega^-(e), 0 \dots, 0, -\omega^+(e), 0, \dots 0$$

if e is not a loop.

Let

$$r_0(G)$$

denote the *torsion-free rank* of G_{ab}, i.e., the rank of G_{ab} modulo its torsion-subgroup. A formula for $r_0(G)$ can be found without resorting to the lengthy process of determining the Smith normal form of the matrix W. Since the t_e, $(e \in E(\Gamma\backslash T))$, are linearly independent, linear algebra shows that

$$r_0(G) = |E(\Gamma)| - |E(T)| + |V(\Gamma)| - \mathrm{rank}(W) = |E(\Gamma)| + 1 - \mathrm{rank}(W).$$

Let W_0 be the submatrix of W consisting of the rows which correspond to edges of the maximal subtree T. Then W_0 gives the structure of $(G_0)_{ab}$ where $G_0 = \pi_1(T, \omega)$. Since each pair of generators of G_0 is linearly dependent, we have $r_0(G_0) = 1$ and $\mathrm{rank}(W_0) = |V(\Gamma)| - 1$. Now $\mathrm{rank}(W) = \mathrm{rank}(W_0)$ or $\mathrm{rank}(W_0) + 1$, depending on whether each non-tree row of W is linearly dependent on the rows of W_0 or not. Therefore $r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon$ where $\epsilon = 1$ if $\mathrm{rank}(W) = \mathrm{rank}(W_0)$ and otherwise $\epsilon = 0$.

Tree dependence

Let $e \in E(\Gamma\backslash T)$ and put $e^- = x$ and $e^+ = y$; then there is a unique path from x to y in T. The defining relations associated with this path lead to a relation $x^h = y^k$, $(h, k \in \mathbb{Z}^*)$. (If $x = y$, then $h = k$). Let $\omega(e) = (m, n)$, so that $x^m \equiv y^n$ mod G'. We will say that e is *T-dependent* if (m, n) is a rational multiple of (h, k), (which means that $m = n$ if $e^- = e^+$). Otherwise e is *T-independent.* If every non-tree edge of Γ is T-dependent, then (Γ, ω) is said to be *tree dependent.* By (4.1) below this property does not dependent on the tree T. If (Γ, ω) is tree dependent, then $\mathrm{rank}(W) = \mathrm{rank}(W_0)$, and otherwise $\mathrm{rank}(W) = \mathrm{rank}(W_0) + 1$. Thus we obtain:

(4.1). *Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group defined relative to a maximal subtree T. Then*

$$r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon,$$

where $\epsilon = 0$ or 1, the rule being that $\epsilon = 1$ if and only if (Γ, ω) is tree dependent.

For example, consider Example 3 above. Here the maximal subtree is the path $xyzu$. All the non-tree edges with the exception of $\langle y, x\rangle$ are T-dependent, so (Γ, ω) is not tree dependent. Therefore $\epsilon = 0$ and $r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 = 3$ by (4.1).

5 The Schur multiplier of a GBS-Group

Next we consider how to compute the Schur multiplier of a GBS-group. First recall an inequality which is valid for any finitely presented group.

(5.1). *Let G be a finitely presented group with n generators and r relations. Then*

$$n - r \ \leq \ r_0(G) - d(M(G)),$$

where $d(H)$ denotes the minimum number of generators of H.

Proof. Let $1 \to R \to F \to G \to 1$ be a presentation of G where F is free of rank n and R is the normal closure of an r-element subset of F. Then $M(G)$ is given by Hopf's formula

$$M(G) \ \simeq \ (F' \cap R)/[F, R].$$

Now $r \geq d(R/[F, R])$, and since F/F' is free abelian, we have

$$d(R/[F, R]) = d((F' \cap R)/[F, R]) + d(F'R/F')$$

$$= d(M(G)) + n - r_0(F/F'R).$$

This shows that $r \geq \ d(M(G)) + n - r_0(G)$, from which the result follows. □

Observe the consequence that there is a least upper bound for the integer $n - r$ over all finite presentations of G: this is the *deficiency* of G,

$$\mathrm{def}(G).$$

Now apply (5.1) to a GBS-group $G = \pi_1(\Gamma, \omega)$, using the standard presentation with respect to a maximal subtree T. Here $n = |V(\Gamma)| + |E(\Gamma \backslash T)|$ and $r = |E(\Gamma)|$, so that

$$n - r = |V(\Gamma)| - |E(T)| = 1$$

and we have $\mathrm{def}(G) \geq 1$. Then $d(M(G)) \leq r_0(G) - (n - r) = r_0(G) - 1$ by (5.1). Therefore we have:

(5.2). *If G is a GBS-group, then $d(M(G)) \leq r_0(G) - 1$. Thus $M(G) = 0$ if $r_0(G) = 1$.*

Corollary 5. *If G is a GBS-tree product, then $M(G) = 0$.*

On the other hand, a Baumslag-Solitar group can have non-zero Schur multiplier.

(5.3). *Let $G = BS(m, n)$. Then $M(G) = 0$ if $m \neq n$ and $M(G) \simeq \mathbb{Z}$ if $m = n$.*

Proof. Suppose that $m \neq n$. Then $G_{ab} \simeq \mathbb{Z} \oplus \mathbb{Z}_{|m-n|}$, so that $r_0(G) = 1$ and $M(G) = 0$ by (5.2). Now assume that $m = n$. Note that $r_0(G) = 2$ in this case and hence $d(M(G)) \leq 2 - 1 = 1$; thus it is enough to show that $r_0(M(G)) = 1$. From the exact sequence $1 \to G' \to G \to G_{ab} \to 1$ we obtain the 5-term exact homology sequence

$$M(G) \to M(G_{ab}) \to G'/[G', G] \to G_{ab} \to G_{ab} \to 1.$$

Now $G'/[G', G]$ is finite since

$$[x, t]^m \equiv [x^m, t] \equiv x^{-m}(x^m)^t \equiv 1 \bmod [G', G].$$

Also $r_0(M(G_{ab})) = 1$, because $r_0(G) = 2$. Hence $\mathrm{Im}(M(G) \to M(G_{ab}))$ is infinite. Thus we have $1 \geq d(M(G)) \geq r_0(M(G)) \geq 1$, so that $r_0(M(G)) = 1$ and $M(G) \simeq \mathbb{Z}$. □

In fact there is a remarkably simple formula for the Schur multiplier of an arbitrary GBS-group ([14]).

(5.4). *If G is an arbitrary GBS-group, then $M(G)$ is free abelian of rank $r_0(G) - 1$.*

The proof of this result uses the 5-term homology sequence and the Mayer-Vietoris sequence for the homology of a generalized free product: for details see [14].

Corollary 6. *If G is any GBS-group, then* $\mathrm{def}(G) = 1$.

For by (5.1) and (5.2) we have

$$1 \leq \mathrm{def}(G) \leq r_0(G) - d(M(G)) = r_0(G) - r_0(G) + 1 = 1.$$

Corollary 7. *Let Γ be a bouquet of k loops. Then $M(G) \simeq \mathbb{Z}^\ell$ where $\ell = k + 1$ if each loop has equal weight values and otherwise $\ell = k$.*

The underlying reason here is that a bouquet of loops is tree dependent if and only if each loop has equal weight values.

For example, consider the GBS-group G in Example 3. Here $r_0(G) = 3$ and thus $M(G) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Central extensions

Knowledge of the Schur multiplier of a GBS-group G allows one to study central extensions of an arbitrary abelian group C by G. By the Universal Coefficients Theorem we have

$$H^2(G, C) \simeq \mathrm{Ext}(G_{ab}, C) \oplus \mathrm{Hom}(M(G), C).$$

Now(5.4) shows that $\mathrm{Hom}(M(G), C) \simeq \bigoplus C^{r_0(G)-1}$, while $G_{ab} \simeq \mathbb{Z}^{r_0(G)} \oplus F$ with F finite. Hence $\mathrm{Ext}(G_{ab}, C) \simeq \mathrm{Ext}(F, C)$, which can be computed if the structure of F is known. On the basis of these remarks we can characterize those GBS-groups G for which every central extension by G splits.

(5.5). *Every central extension by a generalized Baumslag-Solitar group G splits, i.e., is a direct product, if and only if G_{ab} is infinite cyclic.*

Proof. Let C be a trivial G-module and denote the periodic subgroup of G_{ab} by F; thus $G_{ab} \simeq \mathbb{Z}^{r_0(G)} \oplus F$ where F is finite. Since $H^2(G, C) \simeq \mathrm{Ext}(F, C) \oplus C^{r_0(G)-1}$, we have $H^2(G, C) = 0$ for every C if and only if $r_0(G) = 1$ and $\mathrm{Ext}(F, C) = 0$ for all C. By taking C to be $\mathbb{Z}$, we see that this happens precisely when $r_0(G) = 1$ and $F = 1$, i.e., $G_{ab} \simeq \mathbb{Z}$. □

6 Nilpotent quotients of GBS-Tree Products

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group where Γ is a tree and let $\bar{G}$ be a nilpotent quotient of G. Then $\bar{G}$ has a central cyclic subgroup $\bar{Z}$ which contains a positive power of every generator. Thus $\bar{G}/\bar{Z}$ is a finitely generated periodic nilpotent group, so it is finite. Clearly $r_0(G) = 1$, which implies that all lower central factors of $\bar{G}$ after the first are finite (by the usual tensor product argument for lower central factors). Hence $r_0(\bar{Z}) = 1$, which shows that $\bar{G}$ is central cyclic-by-finite, and hence finite-by-cyclic. Thus we have:

(6.1). *A nilpotent quotient of a GBS-tree product is finite-by-cyclic.*

Information about the second derived quotient group is also available.

(6.2). *If G is a GBS-tree product, then G/G'' is virtually abelian.*

Proof. Write $\bar{G} = G/G''$ and note that there exists an element $u \in G$ such that $G/< u > G'$ is finite. Next let x, y, z be generators of G; since G is a tree product, $< x > \cap < y > \cap < z > \neq 1$. Hence $([x, y]^{<z>})G''/G''$ is finitely generated and it follows that $[x, y]^G G''/G''$ is finitely generated, as is G'/G'' since G/G'' satisfies max-n. Also z^m centralizes G'/G'' for some $m > 0$, from which it follows that $< z^m > G'/G''$ is abelian and clearly it has finite index in G. □

Note that G'' is a free group by (3.5), so further derived factors may be complex. Furthermore the next result shows that one cannot expect to be able to say anything about finite factors of a GBS-group.

(6.3). *Every finite group is a quotient of a GBS-tree product.*

Proof. Let $F = \{f_1, \ldots, f_n\}$ be an arbitrary finite group with $m_i = |f_i|$. Let T be the line graph with edges $< f_1, f_2 >, < f_2, f_3 >, \ldots, < f_{n-1}, f_n >$, the weight of

edge $< f_i, f_{i+1} >$ being (m_i, m_{i+1}). By Von Dyck's theorem there is a surjective homomorphism from $\pi_1(T, \omega)$ to F such that $g_{f_i} \mapsto f_i$ since $f_i^{m_i} = 1 = f_{i+1}^{m_{i+1}}$.
□

7 Geometric quotients of GBS-groups

We now restrict attention to quotients of a GBS-group which arise in a natural way from the underlying GBS-graph. Let (Γ, ω) and $(\bar{\Gamma}, \bar{\omega})$ be GBS-graphs and let $G, \bar{G}$ be the corresponding GBS-groups defined with respect to the maximal subtrees $T, \bar{T}$. A pair of functions (γ, δ),

$$\gamma : V(\Gamma) \to V(\bar{\Gamma}), \quad \delta : E(\Gamma \backslash T) \to E(\bar{\Gamma} \backslash \bar{T})$$

is called a *vertex-edge pair* for $(\Gamma, \omega, T), (\bar{\Gamma}, \bar{\omega}, \bar{T})$ if
(i) $(\delta(e))^- = \gamma(e^-)$ and $(\delta(e))^+ = \gamma(e^+)$, $e \in E(\Gamma \backslash T)$;
(ii) if $< x, y > \in E(T)$ and $\gamma(x) \neq \gamma(y)$, then $< \gamma(x), \gamma(y) > \in E(\bar{T})$.
Thus non-tree edges of Γ are mapped to non-tree edges of $\bar{\Gamma}$ and an edge in T is mapped to an edge in $\bar{T}$ provided that γ has distinct values at the endpoints.

Definition 2. A homomorphism between the GBS-groups above

$$\theta : \pi_1(\Gamma, \omega) \to \pi_1(\bar{\Gamma}, \bar{\omega})$$

is called *geometric* if there is a vertex-edge pair (γ, δ) such that

$$g_x^\theta = g_{\gamma(x)}^{r(x)}, \ x \in V(\Gamma)$$
$$t_e^\theta = t_{\delta(e)}^{s(e)}, \ e \in E(\Gamma \backslash T),$$

where $r(x), s(e) \in \mathbb{Z}$. Thus θ is determined by the parameters

$$(\gamma, \delta, r(x), s(e) \mid x \in V(\Gamma), e \in E(\Gamma \backslash T)),$$

which are of course subject to certain restrictions.

A quotient group G/K of a GBS-group G is called a *geometric quotient* if $K = \mathrm{Ker}(\theta)$ where θ is a *surjective* geometric homomorphism from G to some GBS-group. (Note that in general the image of a geometric homomorphism need not be a GBS-group).

Some natural examples of geometric homomorphisms

1. *Loop deletion*

Suppose that the graph Γ has two loops e, e' through the same vertex and that e has weight (1,1). Then deleting e and mapping the associated generator t_e to 1 gives rise to a geometric homomorphism $\theta : \pi_1(\Gamma, \omega) \to \pi_1(\bar{\Gamma}, \bar{\omega})$ where $\bar{\Gamma}$ is Γ with e removed and $\bar{\omega}$ is the restriction of ω. Here the vertex pair fixes vertices and maps e and e' to e'.

2. *Loop identification*

Suppose that the graph Γ has two loops e, e' through a vertex and that they have the same weight. Identify the two loops to form a new graph $\bar{\Gamma}$, which is Γ with the loop e' removed. Map t_e and $t_{e'}$ to t_e: here the vertex pair fixes vertices and maps e and e' to e, with other edges fixed.

3. *Pinch maps*

Let $G = \pi_1(\Gamma, \omega)$ and let T be a maximal subtree in Γ. Choose any $e \in E(\Gamma)$ and write $m = \omega^-(e),\ n = \omega^+(e)$. Let d be a common divisor of m and n. Define a new weight function $\bar{\omega}$ on Γ by replacing the weight (m, n) by $(m/d, n/d)$, with all other weights unchanged. Write $\bar{G} = \pi_1(\Gamma, \bar{\omega})$. Then there is a surjective homomorphism

$$\theta : G \to \bar{G}$$

in which

$$x \mapsto \bar{x}, \quad y \mapsto \bar{y}.$$

Indeed $\bar{x}^{m/d} = \bar{y}^{n/d}$ implies that $\bar{x}^m = \bar{y}^n$, while $(\bar{x}^{m/d})^t = \bar{y}^{n/d}$ implies that $(\bar{x}^m)^t = \bar{y}^n$. Note that θ is a geometric homomorphism induced by the vertex-edge pair of identity functions. Also, if $e \in E(T)$, then

$$[x^{m/d}, y^{n/d}]^\theta = 1$$

and $[x^{m/d}, y^{n/d}] \neq 1$ if $d \neq \pm 1$. There is a similar discussion if $e \notin E(T)$. Hence θ is not an isomorphism if $d \neq \pm 1$. Call θ a *pinch map on e*.

4. *Edge contractions*

Let $G = \pi_1(\Gamma, \omega)$ and let T be a maximal subtree of Γ. Suppose that $e = < y, z > \in E(T)$ has relatively prime weights $m = \omega^-(e),\ n = \omega^+(e)$. We aim to define a *contraction along the edge* $e =< y, z >$. The diagram which follows exhibits a part of the graph Γ.

$$\bullet_x \xrightarrow{(p,q)} \bullet_y \xrightarrow{(m,n)} \bullet_z \xrightarrow{(r,s)} \bullet_u$$

Form a new graph $\bar{\Gamma}$ by deleting the edge e and adjusting the weights of adjacent edges appropriately: the relevant segment of $\bar{\Gamma}$ is

$$\bullet_a \xrightarrow{(p,qn)} \bullet_b \xrightarrow{(rm,s)} \bullet_c$$

Now define a vertex pair (γ, δ) by

$$\gamma(x) = a,\ \gamma(y) = b,\ \gamma(z) = b,\ \gamma(u) = c,$$

with other vertices fixed and δ preserving non-tree edges. A homomorphism

$$\theta : G \to \bar{G}$$

is defined by the rules

$${g_x}^\theta = g_a,\ {g_y}^\theta = {g_b}^n,\ {g_z}^\theta = {g_b}^m,\ {g_u}^\theta = g_c,\ \dots .$$

Then θ is a geometric homomorphism induced by (γ, δ): for example, the relation ${g_z}^r = {g_u}^s$ in G becomes ${g_b}^{rm} = {g_c}^s$ in $\bar{G}$. Since $\gcd(m, n) = 1$, we have $g_b \in \mathrm{Im}(\theta)$, so θ is surjective. Finally, note that

$$[g_y, g_z]^\theta = [{g_b}^n, {g_b}^m] = 1,$$

while if $|m| \neq 1$ and $|n| \neq 1$, then $[g_y, g_z] \neq 1$ and θ is not an isomorphism. Notice that edge contraction does not decrease weights in absolute value. (In a similar way it is possible to define a contraction along a loop.)

It is an important property of geometric homomorphisms that their composites are also geometric.

(7.1) *Let $G_i = \pi_1(\Gamma_i, \omega_i)$, $i = 1, 2, 3$, be GBS-groups with associated maximal subtrees T_i, and let $\phi_i : G_i \to G_{i+1}$, $i = 1, 2$, be geometric homomorphisms with parameters $(\gamma_i, \delta_i, r_i(x), s_i(e))$ relative to the T_i. Then the composite $\phi_1\phi_2$ is a geometric homomorphism from G_1 to G_3 with parameters*

$$(\gamma_2\gamma_1,\ \delta_2\delta_1,\ r_1(x)r_2(\gamma_1(x)),\ s_1(e)s_2(\delta_1(e)).$$

Proof. First observe that $(\gamma_2\gamma_1, \delta_2\delta_1)$ is a vertex-edge pair. For, if $e \in E(\Gamma_1 \backslash T_1)$, then $(\delta_2\delta_1(e))^\pm = \gamma_2(\delta_1(e)^\pm) = \gamma_2\gamma_1(e^\pm)$. Also, if $\langle x, y\rangle \in E(T_1)$ and $\gamma_2\gamma_1(x) \neq \gamma_2\gamma_1(y)$, then $\gamma_1(x) \neq \gamma_1(y)$, so $\langle \gamma_1(x), \gamma_1(y)\rangle \in E(T_2)$. Thus we have $\langle \gamma_2\gamma_1(x), \gamma_2\gamma_1(y)\rangle \in E(T_3)$. Next, if $x \in V(\Gamma_1)$, then

$$(g_x)^{\phi_1\phi_2} = (g_{\gamma_1(x)}^{r_1(x)})^{\phi_2} = (g_{\gamma_2\gamma_1(x)})^{r_1(x)r_2(\gamma_1(x))},$$

and there is a similar calculation for $(t_e)^{\phi_1\phi_2}$. $\square$

8 GBS-Simple Groups and GBS-Free groups

Every GBS-group has $\mathbb{Z}$ as a quotient, although not necessarily as a geometric quotient. We will say that a GBS-group is *GBS-simple* relative to a maximal subtree T if there are no surjective, geometric homomorphisms relative to T, with non-trivial kernel, from G to any non-cyclic GBS-group. Equivalently G has no proper, non-cyclic, geometric GBS-quotients. (Here a quotient is called proper if the associated normal subgroup is non-trivial). If a GBS-group has no proper, non-cyclic GBS-quotients at all, whether geometric or not, it is called *GBS-free*.

The following result, which is proved in [6], provides a complete classification of the GBS-groups which are GBS-simple: it also shows that the properties "GBS-free" and "GBS-simple" are identical.

(8.1). *Let* (Γ, ω) *be a GBS-graph and let* $G = \pi_1(\Gamma, \omega)$ *be the GBS-group defined with respect to a maximal subtree* T*. Then the following statements are equivalent:*

(a) *G is GBS-free;*
(b) *G is GBS-simple;*
(c) *there is a geometric isomorphism from G to one of the groups $BS(1,n), K(1,1), K(p,q), K(p,p^d)$, where $n \in \mathbb{Z}^*$, p, q are distinct primes and $d > 0$.*

Thus, for example, $K(2,4)$, $K(2,3)$, $BS(1,3)$ are GBS-free, but $K(4,9)$ and $BS(2,3)$ are not GBS-free. Notice that the theorem also shows that the property GBS-simple is independent of the maximal subtree T.

Sketch of proof of (8.1).
Assume G is GBS-simple, but not cyclic. The idea of the proof is to show there is a surjective, geometric homomorphism from G to a non-cyclic GBS-group whose underlying graph is either a 1-edge or a 1-loop. This will show that there is no loss in assuming the original graph to have one of these forms. Then these special cases can be dealt with. The geometric homomorphisms used will be composites of the special types (1)–(4) listed above: thus (7.1) is relevant.

Suppose first that Γ is a tree with more than one edge. Contract all edges with a weight vaue ± 1, which does not change G up to isomorphism. Thus we can assume that there are no such edges. There must be some edges left, otherwise the graph consists of a single vertex and G is infinite cyclic. If two or more edges are left, pinch and contract all edges but one, noting that after a pinch-contraction there are still no ± 1 labels. The resulting graph has a single edge and the group is non-cyclic, so we have reduced to the case of a 1-edge.

Now suppose Γ is not a tree and let T be a maximal subtree. Pinch and contract edges in T to a single vertex to get a bouquet of loops. Note that the group is non-cyclic.

From now on assume that Γ is a bouquet of $k \geq 2$ loops. Moreover, by pinching we can also assume that all the weights are relatively prime. The next step is to establish

(8.2). *If not all weights have absolute value 1, then G has a proper, non-cyclic geometric quotient and hence is not GBS-simple.*
Proof. We have

$$G =< t_1, \dots, t_k, x| \ (x^{m_i})^{t_i} = x^{n_i}, i = 1, \dots, k >$$

where $\gcd(m_i, n_i) = 1$. We can assume that $|m_i| \leq |n_i|$. Define

$$\ell = \ell\mathrm{cm}(n_1, \dots, n_k);$$

then the assignments

$$x \mapsto x^\ell, \ t_i \mapsto t_i$$

determine a geometric endomorphism θ of G, where the vertex pair consists of identity functions. We have to prove that θ is surjective. First G^θ contains t_i and $x^\ell = x^{(\ell/n_i)n_i}$, and hence $x^{(\ell/n_i)m_i}$. Since m_i, n_i are relatively prime, $x^{\ell/n_i} \in G^\theta$. Also the ℓ/n_i are relatively prime, so $x \in G^\theta$ and $\mathrm{Im}(\theta) = G$. Notice in addition that

$$[x, x^{t_i}]^\theta = [x^\ell, (x^\ell)^{t_i}] = [x^\ell, ((x^{n_i})^{t_i})^{\ell/n_i}] = [x^\ell, x^{m_i\ell/n_i}] = 1$$

and $[x, x^{t_i}] \neq 1$ if $|m_i| \neq 1$. On the other hand, if all the $|m_i| = 1$, then in a similar way $[x^{t_i^{-1}}, x^{t_i^{-1}t_j}] \in \mathrm{Ker}(\theta)$ and this is non-trivial if $j \neq i$. □

The discussion so far shows that we can assume that Γ is a bouquet of $k \geq 2$ loops where $|m_i| = 1 = |n_i|$ for all i. We can delete any loop with label (1,1). Then, if there are multiple loops with label $(1, -1)$, pass to a 1-loop quotient with $G = BS(1, -1)$ by identifying loops. The effect of the above analysis is to reduce to the case of a 1-loop. Thus it remains to deal with the cases of a 1-loop and a 1-edge. In these cases a complete description of all GBS-quotients is possible.

(8.3). *There is a surjective homomorphism from $G = K(m, n)$ to $\bar{G} = K(m', n')$, where $\bar{G}$ non-cyclic, if and only if there exist integers k, r, s such that either*

$$(i) \ m' = m/ks, \ n' = n/kr \ \textit{and} \ \gcd(r, m/k) = 1 = \gcd(s, n/k),$$

or

$$(ii) \ m' = n/kr, \ n' = m/ks \ \textit{and} \ \gcd(r, m/k) = 1 = \gcd(s, n/k).$$

The sufficiency of the conditions in the theorem is proved by observing that if m, n are relatively prime and p divides m, then there is a surjective geometric homomorphism

$$\theta : K(m,n) \longrightarrow K(\frac{m}{p}, n)$$

in which $x \mapsto \bar{x},\ y \mapsto \bar{y}^p$, where x, y and $\bar{x}, \bar{y}$ are the respective generators of the groups $G, \bar{G}$.

(8.4). *There is a surjective homomorphism θ from $BS(m,n)$ to $BS(\bar{m},\bar{n})$ if and only if $\bar{m} = m/q$ and $\bar{n} = n/q$ or $\bar{m} = n/q$ and $\bar{n} = m/q$ for some integer q dividing m and n.*

Sketch of proof
Let $G =< t, x >$ and $\bar{G} =< \bar{t}, \bar{x} >$ be the two groups and assume there is a surjective homomorphism from G to $\bar{G}$. To prove the result we will produce invariants of the groups. An obvious one is obtained from

$$G_{ab} \simeq \mathbb{Z} \times \mathbb{Z}_{|m-n|}.$$

Since θ maps G_{ab} onto $\bar{G}_{ab}$, we see that $\bar{m} - \bar{n}$ divides $m - n$. Assume that $m \neq n$: the case where $m = n$ requires a special argument.

Next we analyze the structure of G/T where $T/(x^G)'$ is the torsion-subgroup of $x^G/(x^G)'$. In fact

$$G/T \simeq < t > \ltimes A$$

where $A = \mathbb{Q}_\pi$ is the additive group of rational numbers with π-adic denominators, π being the set of primes involved in $\frac{n}{m}$ (after cancellation). Here t acts on A by multiplication by $\frac{n}{m}$, this being the additive version of the relation $(x^m)^t = x^n$.

Note that x^G is generated by all the elements commensurable with their conjugates, (i.e., elements g such that $< g > \cap < g >^h \neq 1$, for all $h \in G$). Therefore x^G is characteristic in G. Hence θ maps $< t > \ltimes A$ onto $< \bar{t} > \ltimes \bar{A}$, where $\bar{A} = \mathbb{Q}_{\bar{\pi}}$ is the additive group of rational numbers with $\bar{\pi}$-adic denominators, with $\bar{\pi}$ the set of primes involved in $\frac{\bar{n}}{\bar{m}}$. It follows that $\frac{n}{m} = \frac{\bar{n}}{\bar{m}}$ (or $\frac{\bar{m}}{\bar{n}}$, in which case a similar argument applies).

Let $d = \gcd(m,n)$ and write $m' = \frac{m}{d}$ and $n' = \frac{n}{d}$: similarly define $\bar{d}, \bar{m}', \bar{n}'$. Then $\frac{m'}{n'} = \frac{\bar{m}'}{\bar{n}'}$ and hence $m' = \bar{m}'$ and $n' = \bar{n}'$. Therefore $\bar{m} = \bar{d}m/d$ and $\bar{n} = \bar{d}n/d$, so that

$$\frac{m-n}{\bar{m}-\bar{n}} = \frac{d}{\bar{d}}$$

is an integer and $\bar{d}$ divides d. Writing $q = d/\bar{d}$, we have $\bar{m} = \frac{m}{q}$ and $\bar{n} = \frac{n}{q}$. Conversely, if $m, n, \bar{m}, \bar{n}$ satisfy the conditions, then by pinching we get a surjective homomorphism $G \to \overline{G}$. $\square$

The proof of (8.1) is now essentially complete: for fuller details see [6].

Remark. It follows from the discussions of (8.3) and (8.4) that (m, n) is an invariant of the groups $BS(m, n)$ and $K(m, n)$ up to multiplication by -1 (of either component in the second case) and interchange of components. It is more challenging to find invariants of arbitrary GBS-groups, although one example is the number of non-tree edges in the graph when the group is not $BS(1, -1)$.

We end with what is probably a hard question. *Is the isomorphism problem soluble for GBS-groups*, i.e., is there an algorithm which, when two GBS-graphs (Γ, ω) and $(\bar{\Gamma}, \bar{\omega})$ are given, decides if $\pi_1(\Gamma, \omega) \simeq \pi_1(\bar{\Gamma}, \bar{\omega})$? A positive answer is known in various special cases, particularly in the case of GBS-trees – for details see [4], [7], [12], [13].

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 55–72. doi:10.1285/i15900932v30sn1p55

L_{10}-free $\{p, q\}$-groups

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Abstract. If L is a lattice, a group is called L-free if its subgroup lattice has no sublattice isomorphic to L. It is easy to see that L_{10}, the subgroup lattice of the dihedral group of order 8, is the largest lattice L such that every finite L-free p-group is modular. In this paper we continue the study of L_{10}-free groups. We determine all finite L_{10}-free $\{p, q\}$-groups for primes p and q, except those of order $2^\alpha 3^\beta$ with normal Sylow 3-subgroup.

Keywords: subgroup lattice, sublattice, finite group, modular Sylow subgroup

MSC 2000 classification: 20D30

1 Introduction

This paper contains the results presented in the second part of our talk on "L_{10}-free groups" given at the conference "Advances in Group Theory and Applications 2009" in Porto Cesareo. The first part of the talk mainly contained results out of [6]. In that paper we introduced the class of L_{10}-free groups; here L_{10} is the subgroup lattice of the dihedral group D_8 of order 8 and for an arbitrary lattice L, a group G is called *L-free* if its subgroup lattice $L(G)$ has no sublattice isomorphic to L. It is easy to see that L_{10} is the unique largest lattice L such that every L-free p-group has modular subgroup lattice. So the finite L_{10}-free groups form an interesting, lattice defined class of groups lying between the modular groups and the finite groups with modular Sylow subgroups. Therefore in [6] we studied these groups and showed that every finite L_{10}-free group G is soluble and the factor group $G/F(G)$ of G over its Fitting subgroup is metacyclic or a direct product of a metacyclic $\{2, 3\}'$-group with the (non-metacyclic) group $Q_8 \times C_2$ of order 16. However, we were not able to determine the exact structure of these groups as had been done in the cases of L-free groups for certain sublattices L of L_{10} (and therefore subclasses of the class of L_{10}-free groups) in [2], [5] and [1].

In the present paper we want to determine the structure of L_{10}-free $\{p, q\}$-groups where p and q are different primes. As mentioned above, the Sylow subgroups of an L_{10}-free group have modular subgroup lattice. Hence a nilpotent group is L_{10}-free if and only if it is modular and the structure of these groups is well-known [4, Theorems 2.3.1 and 2.4.4]. So we only have to study non-nilpotent L_{10}-free $\{p,q\}$-groups G. The results of [6] show that one of the Sylow subgroups of G is normal – we shall choose our notation so that this is the Sylow p-subgroup P of G – and the other is cyclic or a quaternion group of order 8 or we are in the exceptional situation $p = 3$, $q = 2$. So there are only few cases to be considered (see Proposition 1 for details) and we handle all of them except the case $p = 3$, $q = 2$. Unfortunately, however, in the main case that $P = C_P(Q) \times [P,Q]$ where $[P,Q]$ is elementary abelian and Q is cyclic, the structure of G depends on the relation of q and $|Q/C_Q(P)|$ to $p-1$ (see Theorems 1–3). For example, if $q \nmid p-1$, then $C_P(Q)$ may be an arbitrary (modular) p-group, whereas $C_P(Q)$ usually has to be small if $q \mid p-1$. The reason for this and for similar structural peculiarities are the technical lemmas proved in §2, the most interesting being that a direct product of an elementary abelian group of order p^m and a nonabelian P-group of order $p^{n-1}q$ is L_{10}-free if and only if one of the ranks m or n is at most 2 (Lemma 3 and Theorem 2).

All groups considered are finite. Our notation is standard (see [3] or [4]) except that we write $H \cup K$ for the group generated by the subgroups H and K of the group G. Furthermore, p and q always are different primes, G is a finite $\{p,q\}$-group, $P \in \operatorname{Syl} p(G)$ and $Q \in \operatorname{Syl} q(G)$. For $n \in \mathbb{N}$,

C_n is the cyclic group of order n,
D_n is the dihedral group of order n (if n is even),
Q_8 is the quaternion group of order 8.

2 Preliminaries

By [6, Lemma 2.1 and Proposition 2.7], the Sylow subgroups of an L_{10}-free $\{p,q\}$-group are modular and one of them is normal. So we only have to consider groups satisfying the assumptions of the following proposition.

Proposition 1. *Let $G = PQ$ where P is a normal modular Sylow p-subgroup and Q is a modular Sylow q-subgroup of G operating nontrivially on P. If G is L_{10}-free, then one of the following holds.*

I. $P = C_P(Q) \times [P,Q]$ *where* $[P,Q]$ *is elementary abelian and* Q *is cyclic.*

II. $[P,Q]$ *is a hamiltonian* 2*-group and* Q *is cyclic.*

III. $p > 3$*,* $Q \simeq Q_8$ *and* $C_Q(P) = 1$*.*

IV. $p = 3$*,* $q = 2$ *and* Q *is not cyclic.*

Proof. Since Q is not normal in G, by [6, Proposition 2.6], Q is cyclic or $Q \simeq Q_8$ or $p = 3$, $q = 2$. By [6, Lemma 2.2], $[P, Q]$ is a hamiltonian 2-group or $P = C_P(Q) \times [P, Q]$ with $[P, Q]$ elementary abelian. In the first case, $q \neq 2$ and hence II. holds. In the other case, I. holds if Q is cyclic. And if $Q \simeq Q_8$, then clearly III. or IV. is satisfied or $C_Q(P) \neq 1$. In the latter case, $\phi(Q) \trianglelefteq G$ and $G/\phi(Q)$ is L_{10}-free with nonnormal Sylow 2-subgroup $Q/\phi(Q)$; again [6, Proposition 2.6] implies that $p = 3$ and hence IV. holds.

Definition 1. We shall say that an L_{10}-free $\{p, q\}$-group $G = PQ$ is of type I, II, III, or IV if it has the corresponding property of Proposition 1.

We want to determine the structure of L_{10}-free $\{p, q\}$-groups of types I–III. So we have to study the operation of Q on $[P, Q]$ and for this we need the following technical results. The first one is Lemma 2.8 in [6].

Lemma 1. *Suppose that $G = (N_1 \times N_2)Q$ with normal p-subgroups N_i and a cyclic q-group Q which operates irreducibly on N_i for $i = 1, 2$ and satisfies $C_Q(N_1) = C_Q(N_2)$. If G is L_{10}-free, then $|N_1| = p = |N_2|$ and Q induces a power automorphism in $N_1 \times N_2$.*

An immediate consequence is the following.

Lemma 2. *Suppose that $G = NQ$ with normal p-subgroup N and a cyclic q-group Q operating irreducibly on N. If G is L_{10}-free, then every subgroup of Q either operates irreducibly on N or induces a (possibly trivial) power automorphism in N; in particular, G is L_7-free.*

Proof. Suppose that $Q_1 \leq Q$ is not irreducible on N and let N_1 be a minimal normal subgroup of NQ_1 contained in N. Then $N = \langle N_1^x \mid x \in Q \rangle$ and so $N = N_1 \times \cdots \times N_r$ with $r > 1$ and $N_i = N_1^{x_i}$ for certain $x_i \in Q$. For $i > 1$, $C_{Q_1}(N_i) = C_{Q_1}(N_1)^{x_i} = C_{Q_1}(N_1)$ and hence Lemma 1 implies that a generator x of Q_1 induces a power automorphism in $N_1 \times N_i$. This power is the same for every i and thus x induces a power automorphism in N. This proves the first assertion of the lemma; that G then is L_7-free follows from [5, Lemma 3.1].

The following two lemmas yield further restrictions on the structure of L_{10}-free $\{p, q\}$-groups. In the proofs we have to construct sublattices isomorphic to L_{10} in certain subgroup lattices. For this and also when we assume, for a contradiction, that a given lattice contains such a sublattice, we use the standard notation displayed in Figure 1 and the following obvious fact.

Remark 1. Let L be a lattice.

(a) A 10-element subset $\{A, B, C, D, E, F, S, T, U, V\}$ of L is a sublattice isomorphic to L_{10} if the following conditions are satisfied :

(1.1) $D \cup S = D \cup T = S \cup T = A$ and $D \cap S = D \cap T = S \cap T = E$,

(1.2) $D \cup U = D \cup V = U \cup V = C$ and $D \cap U = D \cap V = U \cap V = E$, (1.3) $A \cup B = B \cup C = F$ and $A \cap B = A \cap C = B \cap C = D$,

(1.4) $S \cup U = S \cup V = T \cup U = T \cup V = F$.

(b) Conversely, every sublattice of L isomorphic to L_{10} contains 10 pairwise different elements $A, \ldots, V$ satisfying (1.1)–(1.4).

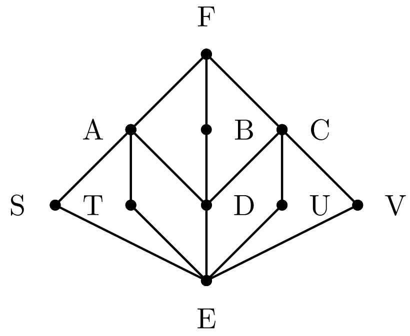

Figure 1

Lemma 3. *If $G = M \times H$ where M is a modular p-group with $|\Omega(M)| \geq p^3$ and H is a P-group of order $p^{n-1}q$ with $3 \leq n \in \mathbb{N}$, then G is not L_{10}-free.*

Proof. By [4, Lemma 2.3.5], $\Omega(M)$ is elementary abelian. So G contains a subgroup $F = F_1 \times F_2$ where $F_1 \leq M$ is elementary abelian of order p^3 and $F_2 \leq H$ is a P-group of order p^2q; let $F_1 = \langle a, b, c\rangle$ and $F_2 = \langle d, e\rangle\langle x\rangle$ where a, b, c, d, e all have order p, $o(x) = q$ and x induces a nontrivial power automorphism in $\langle d, e\rangle$. We let $E = 1$ and define every $X \in \{A, B, C, D, U, V\}$ as a direct product $X = X_1 \times X_2$ with $X_i \leq F_i$ in such a way that (1.2) and (1.3) hold for the X_i in F_i $(i = 1, 2)$ and then of course also for the direct products in F. For this we may take $A_1 = \langle a, b\rangle$, $B_1 = \langle a, bc\rangle$, $U_1 = \langle c\rangle$, $V_1 = \langle ac\rangle$, hence $D_1 = \langle a\rangle$ and $C_1 = \langle a, c\rangle$, and similarly $A_2 = \langle d, e\rangle$, $B_2 = \langle d, ex\rangle$, $U_2 = \langle x\rangle$, $V_2 = \langle dx\rangle$, and hence $D_2 = \langle d\rangle$ and $C_2 = \langle d, x\rangle$. Since $q \mid p-1$, we have $p > 2$ and so we finally may define $S = \langle ae, bd\rangle$ and $T = \langle ae^2, bd^2\rangle$.

Then $A = \langle a, b, d, e\rangle$ is elementary abelian of order p^4 and $D = \langle a, d\rangle$; therefore $D \cup S = D \cup T = S \cup T = A$. Since S, T, D all have order p^2, it follows that $D \cap S = D \cap T = S \cap T = 1$ and so also (1.1) holds. Now x and dx operate in the same way on A and do not normalize $\langle ae^i\rangle$ or $\langle bd^i\rangle$ (i=1,2); hence all the groups $S \cup U$, $S \cup V$, $T \cup U$, $T \cup V$ contain $A = S \cup S^x = T \cup T^x$. Since $A \cup U = A \cup V = F$, also (1.4) holds. Thus $\{A, \ldots, V\}$ is a sublattice of $L(G)$ isomorphic to L_{10}.

We remark that Theorem 2 will show that if $|\Omega(M)| \leq p^2$ or $n \leq 2$ in the group G of Lemma 3, then G is L_{10}-free.

Lemma 4. *Let $k, l, m \in \mathbb{N}$ such that $k \leq l < m$ and $q^m \mid p-1$. Suppose that $G = PQ$ where $P = M_1 \times M_2 \times M$ is an elementary abelian normal p-subgroup of G with $|M_i| \geq p$ for i=1,2 and $|M| \geq p^2$ and where Q is cyclic and induces power automorphisms of order q^k in M_1, q^l in M_2, and of order q^m in M. Then G is not L_{10}-free.*

Proof. We show that $G/C_Q(P)$ is not L_{10}-free and for this we may assume that $C_Q(P) = 1$, that is, $|Q| = q^m$. Then G contains a subgroup $F = AQ$ where $A = \langle a, b, c, d \rangle$ is elementary abelian of order p^4 with $a \in M_1$, $b \in M_2$ and $c, d \in M$. We let $E = 1$, $D = \langle a, c \rangle$, $S = \langle acd, bcd^{-1} \rangle$, $T = \langle acd^2, bc^{-1}d^{-1} \rangle$, $U = Q$, $V = Q^{ac}$, $C = DQ$, $B = DQ^{bd}$ and claim that these groups satisfy (1.1)–(1.4).

This is rather obvious for (1.1) since $|D| = |S| = |T| = p^2$ and, clearly, $D \cup S = D \cup T = S \cup T = A$. By [4, Lemma 4.1.1], $Q \cup Q^{ac} = [ac, Q]Q$ and $Q \cap Q^{ac} = C_Q(ac)$; since Q induces different powers in $\langle a \rangle$ and $\langle c \rangle$, we have $[ac, Q] = \langle a, c \rangle$ and $C_Q(ac) = C_Q(c) = 1$. It follows that (1.2) is satisfied. Since $G/D \simeq \langle b, d \rangle Q$ and $Q \cap Q^{bd} = C_Q(bd) = 1$, we have $B \cap C = D$ and so (1.3) holds. Finally, since a generator of Q (or of Q^{ac}) induces different powers in M_i and M, $S \cup U$ and $S \cup V$ contain $\langle a, cd, b, cd^{-1} \rangle = A$; similarly $T \cup U$ and $T \cup V$ both contain $\langle a, cd^2, b, c^{-1}d^{-1} \rangle = A$. Thus also (1.4) holds and $\{A, \ldots, V\}$ is a sublattice of $L(G)$ isomorphic to L_{10}.

To show that the groups in our characterizations indeed are L_{10}-free, we shall need the following simple properties of sublattices isomorphic to L_{10}.

Lemma 5. *Let M and N be lattices. If M and N are L_{10}-free, then so is $M \times N$.*

Proof. This follows from the fact that L_{10} is subdirectly irreducible; see [5, Lemma 2.2] the proof of which (for $k = 7$) can be copied literally.

Lemma 6. *Let G be a group and suppose that $A, \ldots, V \in L(G)$ satisfy (1.1)–(1.4). If $W \leq G$ such that $F \not\leq W$, then either $S \not\leq W$ and $T \not\leq W$ or $U \not\leq W$ and $V \not\leq W$.*

Proof. Otherwise there would exist $X \in \{S, T\}$ and $Y \in \{U, V\}$ such that $X \leq W$ and $Y \leq W$. But then $F = X \cup Y \leq W$, a contradiction.

Lemma 7. *Let $\overline{P} \trianglelefteq G$ such that $|G : \overline{P}|$ is a power of the prime q and suppose that Q_0 is the unique subgroup of order q in G. If $\overline{P}$ and G/Q_0 are L_{10}-free, then so is G.*

Proof. Suppose, for a contradiction, that $\{A, \ldots, V\}$ is a sublattice of $L(G)$ isomorphic to L_{10} and satisfying (1.1)–(1.4). Since $\overline{P}$ is L_{10}-free, $F \not\leq \overline{P}$. By Lemma 6, either S and T or U and V are not contained in $\overline{P}$ and therefore have order divisible by q. Hence either $Q_0 \leq S \cap T = E$ or $Q_0 \leq U \cap V = E$; in both cases, G/Q_0 is not L_{10}-free, a contradiction.

In the inductive proofs that the given $\{p,q\}$-group $G = PQ$ is L_{10}-free, the above lemma will imply that $C_Q(P) = 1$. And the final result of this section handles a situation that shows up in nearly all of these proofs.

Lemma 8. *Let $G = PQ$ where P is a normal Sylow p-subgroup of G and Q is a nontrivial cyclic q-group or $Q \simeq Q_8$; let $Q_0 = \Omega(Q)$ be the minimal subgroup of Q.*

Assume that every proper subgroup of G is L_{10}-free and that there exists a minimal normal subgroup N of G such that $P = N \times C_P(Q_0)$; in addition, if $Q \simeq Q_8$, suppose that every subgroup of order 4 of Q is irreducible on N.

Then G is L_{10}-free.

Proof. Suppose, for a contradiction, that G is not L_{10}-free and let $\{A, \dots, V\}$ be a sublattice of $L(G)$ isomorphic to L_{10}; so assume that (1.1)–(1.4) hold. Since every proper subgroup of G is L_{10}-free, $F = G$.

By assumption, $G = NC_G(Q_0)$; hence $Q_0^G \le NQ_0$ and $[P, Q_0] \le N$. Since $P = [P, Q_0]C_P(Q_0)$ (see [4, Lemma 4.1.3]), it follows that

$$[P, Q_0] = N \quad \text{and} \quad Q_0^G = NQ_0. \tag{1}$$

Suppose first that E is a p-group. By Lemma 6, we have $S, T \not\le P\phi(Q)$ or $U, V \not\le P\phi(Q)$; say $U, V \not\le P\phi(Q)$. Then U and V both contain Sylow q-subgroups of G, or subgroups of order 4 of G if $Q \simeq Q_8$. Since $U \cap V = E$ is a p-group, $C = U \cup V$ contains two different subgroups of order q and hence by (1), $C \cap N \ne 1$. Since U is irreducible on N, it follows that $N \le C$. Therefore $Q_0^G = NQ_0 \le C$ and so C contains every subgroup of order q of G. Since $S \cap C = T \cap C = E$ is a p-group, it follows that S and T are p-groups. Hence $A = S \cup T \le P$; but then also $B \cap C = D \le A$ is a p-group and therefore $B \le P$. So, finally, $G = A \cup B \le P$, a contradiction.

Thus E is not a p-group and therefore contains a subgroup of order q. If we conjugate our L_{10} suitably, we may assume that

$$Q_0 \le E. \tag{2}$$

Every subgroup X of G containing Q_0 is of the form $X = (X \cap P)Q_1$ where $Q_0 \le Q_1 \in \operatorname{Syl} q(X)$; since $X \cap P = [X \cap P, Q_0]C_{X \cap P}(Q_0)$ and $[X \cap P, Q_0] \le X \cap N$, it follows that

$$X \le C_G(Q_0) \ \text{ if } \ Q_0 \le X \ \text{ and } \ X \cap N = 1. \tag{3}$$

Since $G = A \cup B = A \cup C = B \cup C$, at least two of the three groups A, B, C are not contained in $P\phi(Q)$ and hence contain Sylow q-subgroups of G, or subgroups of order 4 of G if $Q \simeq Q_8$. Similarly, two of the groups A, B, C are not contained in $C_G(Q_0)$ and hence, by (2) and (3), have nontrivial intersection with N. So there exists $X \in \{A, B, C\}$ having both properties. Since the Sylow q-subgroups of X are irreducible on N, it follows that $N \leq X$. Let $Y, Z \in \{A, B, C\}$ with $Y \neq X \neq Z$ such that $Y \cap N \neq 1$ and Z contains a Sylow q-subgroup of G, or a subgroup of order 4 of G if $Q \simeq Q_8$. Then $1 < Y \cap N \leq Y \cap X = D$ and hence also $Z \cap N \neq 1$. Thus $N \leq Z$ and so

$$N \leq X \cap Z = D. \tag{4}$$

Therefore $S \cap N = S \cap D \cap N = E \cap N$ and $U \cap N = E \cap N$; so if $E \cap N = 1$, then (2) and (3) would imply that $G = S \cup U \leq C_G(Q_0)$, a contradiction. Thus $E \cap N \neq 1$. Again by Lemma 6, $U, V \nleq P\phi(Q)$, say. So $U \cap N \neq 1 \neq V \cap N$ and U and V are irreducible on N; it follows that $N \leq U \cap V = E$. But by assumption, $G = NC_G(Q_0)$ and $N \cap C_G(Q_0) = 1$ so that $G/N \simeq C_G(Q_0)$ is L_{10}-free, a final contradiction.

3 Groups of type I

Unfortunately, as already mentioned, this case splits into three rather different subcases according to the relation of q and $|Q/C_Q(P)|$ to $p-1$. We start with the easiest case that q does not divide $p-1$. In the whole section we shall assume the following.

Hypothesis I. Let $G = PQ$ where P is a normal p-subgroup of G with modular subgroup lattice, Q is a cyclic q-group and $P = C_P(Q) \times [P, Q]$ with $[P, Q]$ elementary abelian and $[P, Q] \neq 1$.

Theorem 1. *Suppose that G satisfies Hypothesis I and that $q \nmid p - 1$.*
Then G is L_{10}-free if and only if $P = C_P(Q) \times N_1 \times \cdots \times N_r$ $(r \geq 1)$ and for all $i, j \in \{1, \ldots, r\}$ the following holds.

(1) Every subgroup of Q operates trivially or irreducibly on N_i.

(2) $C_Q(N_i) \neq C_Q(N_j)$ for $i \neq j$.

Proof. Suppose first that G is L_{10}-free. By Maschke's theorem, Q is completely reducible on $[P, Q]$ and hence $[P, Q] = N_1 \times \cdots \times N_r$ with $r \geq 1$ and Q irreducible on N_i for all $i \in \{1, \ldots, r\}$. By Lemma 2, every subgroup of Q either is irreducible on N_i or induces a power automorphism in N_i. But since $q \nmid p - 1$, there is no power automorphism of order q of an elementary abelian p-group and hence all these induced power automorphisms have to be trivial. Thus (1) holds and (2) follows from Lemma 1.

To prove the converse, we consider a minimal counterexample G. Then G satisfies (1) and (2) but is not L_{10}-free. Every subgroup of G also satisfies (1) and (2) or is nilpotent with modular subgroup lattice; the minimality of G implies that every proper subgroup of G is L_{10}-free.

If $C_Q(P) \neq 1$, then $Q_0 := \Omega(Q)$ would be the unique subgroup of order q in G and again the minimality of G would imply that G/Q_0 would be L_{10}-free. Since also P is L_{10}-free, Lemma 7 would yield that G is L_{10}-free, a contradiction. Thus $C_Q(P) = 1$ and hence there is at least one of the N_i, say N_1, on which Q_0 acts nontrivially and hence irreducibly. By (2), Q_0 centralizes the other N_j so that $P = N_1 \times C_P(Q_0)$. By Lemma 8, G is L_{10}-free, a final contradiction.

We come to the case that G satisfies Hypothesis I and $q \mid p-1$. Then again by Maschke's theorem, $[P,Q] = N_1 \times \cdots \times N_r$ $(r \geq 1)$ with irreducible $GF(p)Q$-modules N_i; but this time some of the N_i might be of dimension 1. In fact, if the order of the operating group $Q/C_Q(P)$ divides $p-1$, then $|N_i| = p$ for all i (see [3, II, Satz 3.10]). Therefore a generator x of Q induces power automorphisms in all the N_i and $[P,Q]$ is the direct product of nontrivial eigenspaces of x. We get the following result in this case.

Theorem 2. *Suppose that G satisfies Hypothesis I and that $|Q/C_Q(P)|$ divides $p-1$; let $Q = \langle x \rangle$.*

Then G is L_{10}-free if and only if $P = C_P(Q) \times M_1 \times \cdots \times M_s$ $(s \geq 1)$ with eigenspaces M_i of x satisfying (1) and (2).

(1) $C_Q(M_s) < C_Q(M_{s-1}) < \cdots < C_Q(M_1) < Q$

(2) One of the following holds:

(2a) $|M_i| = p$ for all $i \in \{1, \ldots, s\}$,

(2b) $|M_1| \geq p^2$, $|M_i| = p$ for all $i \neq 1$ and $|\Omega(C_P(Q))| \leq p^2$,

(2c) $|M_2| \geq p^2$, $|M_i| = p$ for all $i \neq 2$ and $C_P(Q)$ is cyclic.

Proof. Suppose first that G is L_{10}-free. As mentioned above, since $|Q/C_Q(P)|$ divides $p-1$, $[P,Q]$ is a direct product of eigenspaces $M_1, \ldots, M_s$ of x. By Lemma 1, $C_Q(M_i) \neq C_Q(M_j)$ for $i \neq j$ and we can choose the numbering of the eigenspaces in such a way that (1) holds.

If $|M_i| = p$ for all i, then (2a) is satisfied. So suppose that $|M_k| \geq p^2$ for some $k \in \{1, \ldots, s\}$. Then by (1), $K := C_Q(M_k) < C_Q(M_i)$ for all $i < k$. Therefore if $k \geq 3$, then x would induce power automorphisms of pairwise different orders $|Q/C_Q(M_i)|$ in M_i for $i \in \{1, 2, k\}$, contradicting Lemma 4. So $k \leq 2$, that is, $|M_i| = p$ for all $i > 2$; and if $k = 2$, again Lemma 4 implies that also $|M_1| = p$.

Let $K < Q_1 \leq Q$ such that $|Q_1 : K| = q$. Then $K \leq Z(H)$ if we put $H = (C_P(Q) \times M_1 \times \cdots \times M_k)Q_1$ and M_kQ_1/K is a P-group of order $p^{n-1}q$

with $n \geq 3$. So if $k = 2$, then by (1), $Q_1 \leq C_Q(M_1)$ and hence $H/K = (C_P(Q) \times M_1)K/K \times M_2Q_1/K$; by Lemma 3, $|\Omega(C_P(Q) \times M_1)| \leq p^2$. Thus $C_P(Q)$ is cyclic and (2c) holds. Finally, if $|M_2| = p$, then $k = 1$ and Lemma 3 applied to H/K yields that $|\Omega(C_P(Q))| \leq p^2$. So (2b) is satisfied and G has the desired structure.

To prove the converse, we again consider a minimal counterexample G. Then G satisfies (1) and (2) and $L(G)$ contains 10 pairwise different elements $A, \ldots, V$ satisfying (1.1)–(1.4).

Every subgroup of G is conjugate to a group $H = (H \cap P)\langle y \rangle$ with $y \in Q$. By (1) there exists $k \in \{0, \ldots, s\}$ such that y has $M_{k+1}, \ldots, M_s$ as nontrivial eigenspaces; and (2) implies that if $|H \cap M_i| \geq p^2$ for some $i \in \{k+1, \ldots, s\}$, then either $k = 0$ or $k = 1$ and $i = 2$. In the first case, H trivially satisfies (1) and (2); in the other case, G satisfies (2c) and (2b) holds for H. The minimality of G implies :

$$\text{Every proper subgroup of } G \text{ is } L_{10}\text{-free and } F = G. \tag{3}$$

Again let $Q_0 := \Omega(Q)$. If $C_Q(P) \neq 1$, then G/Q_0 and, by Lemma 7, also G would be L_{10}-free, a contradiction. Thus

$$C_Q(P) = 1. \tag{4}$$

By (1), $C_Q(M_s) = C_Q(P) = 1$ and Q_0 centralizes $M_1, \ldots, M_{s-1}$; furthermore Q_0 induces a power automorphism of order q in M_s. Thus

$$P = M_s \times C_P(Q_0) \text{ and } Q_0^G = M_sQ_0 \text{ is a } P\text{-group}. \tag{5}$$

If $|M_s| = p$, then by Lemma 8, G would be L_{10}-free, a contradiction. Thus $|M_s| > p$ and hence $s \leq 2$, by (2); in fact, (2) implies that there are only two possibilities for the M_i.

Let $M_0 := C_P(Q)$. Then one of the following holds : (6)

(6a) $P = M_0 \times M_1$ where $|\Omega(M_0)| \leq p^2$ and $|M_1| \geq p^2$,

(6b) $P = M_0 \times M_1 \times M_2$ where M_0 is cyclic, $|M_1| = p$ and $|M_2| \geq p^2$.

By Lemma 6, either $S, T \not\leq P\phi(Q)$ or $U, V \not\leq P\phi(Q)$; say $U, V \not\leq P\phi(Q)$. Then

$$U \text{ and } V \text{ contain Sylow } q\text{-subgroups of } G. \tag{7}$$

We want to show next that $E = 1$. For this note that by (5), $G = M_sC_G(Q_0)$ and $M_s \cap C_G(Q_0) = 1$. Since every subgroup of M_s is normal in G, the map

$$\phi : L(M_s) \times [C_G(Q_0)/Q_0] \longrightarrow [G/Q_0]; (H, K) \longmapsto HK$$

is well-defined. Every $L \in [G/Q_0]$ is of the form $L = (L \cap P)Q_1$ where $Q_0 \leq Q_1 \in \mathrm{Syl}\, q(L)$; since $M_s = [P, Q_0]$, we have $L \cap P = (L \cap M_s)C_{L\cap P}(Q_0)$. Hence $L = (L \cap M_s)C_L(Q_0)$ and the map

$$\psi : [G/Q_0] \longrightarrow L(M_s) \times [C_G(Q_0)/Q_0]; L \longmapsto (L \cap M_s, C_L(Q_0))$$

is well-defined and inverse to ϕ. Thus $[G/Q_0] \simeq L(M_s) \times [C_G(Q_0)/Q_0]$. By (3), $C_G(Q_0)$ is L_{10}-free and then Lemma 5 implies that also $[G/Q_0]$ is L_{10}-free. So $[G/Q_0^g]$ is L_{10}-free for every $g \in G$ and this implies that E is a p-group.

Now suppose, for a contradiction, that $E \neq 1$. By (6), the M_i are eigenspaces (and centralizer) of every Sylow q-subgroup of G. Therefore by (7), $U \cap P$ and $V \cap P$ are direct products of their intersections with the M_i and hence this also holds for $(U \cap P) \cap (V \cap P) = E \cap P = E$. The minimality of G implies that $E_G = 1$. Hence $E \cap M_1 = E \cap M_2 = 1$ and so $E \leq M_0$ and $|\Omega(M_0)| = p^2$. If two of the groups S, T, U, V would contain $\Omega(M_0)$, then $\Omega(M_0) \leq E$, contradicting $E_G = 1$. Hence there are $X \in \{S, T\}$ and $Y \in \{U, V\}$ such that $X \cap M_0$ and $Y \cap M_0$ are cyclic. Since $E \leq M_0$, it follows that $E \trianglelefteq X \cup Y = G$, a contradiction. We have shown that

$$E = 1 \tag{8}$$

and come to the crucial property of G.

(9) Let $X, Y \leq G$ such that Y contains a Sylow q-subgroup of G; let $|X| = p^j q^k$ where $j, k \in \mathbb{N}_0$. Then $|X \cup Y| \leq p^{j+2}|Y|$.

Proof. Conjugating the given situation suitably, we may assume that $Q \leq Y$. Suppose first that X is a p-group and let $H = M_0$ and $K = M_1$ if (6a) holds, whereas $H = M_0 \times M_1$ and $K = M_2$ if (6b) holds. Then $X \leq P = H \times K$ where H is modular of rank at most 2 and K is elementary abelian. Let $X_1 = XK \cap H$, $X_2 = XH \cap K$ and $X_0 = (X \cap H) \times (X \cap K)$. Then by [4, 1.6.1 and 1.6.3], $X_1/X \cap H \simeq X_2/X \cap K$ and X/X_0 is a diagonal in the direct product $(X_1 \times X_2)/X_0 = X_1X_0/X_0 \times X_2X_0/X_0$. Since $X_2/X \cap K$ is elementary abelian and $X_1/X \cap H$ has rank at most 2, we have $|(X_1 \times X_2) : X| = |X_1/X \cap H| \leq p^2$.

Now $X \cup Y \leq (X_1 \times X_2) \cup Y$. Since $L(P)$ is modular, any two subgroups of P permute [4, Lemma 2.3.2]; furthermore, Q normalizes X_2. So if Q also normalizes X_1, then $X_1 \times X_2$ permutes with Y and $|X \cup Y| \leq |X_1 \times X_2| \cdot |Y| \leq |X| \cdot p^2 \cdot |Y|$, as desired. If Q does not normalize X_1, then (6b) holds and X_1 is cyclic since every subgroup of $H = M_0 \times M_1$ containing M_1 is normal in G. Then $X_1/X \cap H$ is cyclic and elementary abelian and hence $|(X_1 \times X_2) : X| = |X_1/X \cap H| \leq p$. It follows that $|X \cup Y| \leq |(X_1M_1 \times X_2)Y| \leq |X| \cdot p^2 \cdot |Y|$. Thus (9) holds if X is a p-group.

Now suppose that X is not a p-group; so $X = (X \cap P)Q_1^a$ where $1 \neq Q_1 \leq Q$ and $a \in [P, Q]$. If (6a) holds, then by (4), $M_0 = C_P(Q_1)$ and M_1 is a nontrivial eigenspace of Q_1; hence $X \cap P = (X \cap M_0) \times (X \cap M_1)$. Since every subgroup of M_0 is permutable and every subgroup of M_1 is normal in G, we have that $\langle a \rangle \trianglelefteq G$ and $X \cup Y = (X \cap P)(Y \cap P)(Q \cup Q_1^a) \leq (X \cap P)Y\langle a \rangle$; thus $|X \cup Y| \leq p^j \cdot |Y| \cdot p$. Finally, if (6b) holds, then $C_P(Q_1) = M_0$ or $C_P(Q_1) = M_0 \times M_1 = H$; in any case, $X \cap P = (X \cap H) \times (X \cap M_2)$. Since P is abelian, $(X \cap H)M_1$, $X \cap M_2$ and $Y \cap P$ are normal in G and $a = a_1 a_2$ with $a_i \in M_i$. Hence $X \cup Y \leq ((X \cap H)M_1 \times (X \cap M_2))(Y \cap P)Q\langle a_2 \rangle$ and so $|X \cup Y| \leq p^{j+1} \cdot |Y| \cdot p$, as claimed.

Since U and V contain Sylow q-subgroups of G, we may apply (9) with $X \in \{S, T\}$ and $Y \in \{U, V\}$. Then since $X \cap C = E = 1$, we obtain, if $|X| = p^j q^k$, that $p^j q^k |C| = |XC| \leq |G| = |X \cup Y| \leq p^{j+2}|Y|$ and hence

$$|C : Y| \leq \frac{p^2}{q^k} \quad \text{for} \quad Y \in \{U, V\}. \tag{10}$$

Similarly, $A \cap Y = 1$ and therefore $|A||Y| = |AY| \leq |G| = |X \cup Y| \leq p^{j+2}|Y|$; hence $|A| \leq p^{j+2}$, that is

$$|A : X| \leq \frac{p^2}{q^k} \quad \text{for} \;\; X \in \{S, T\}. \tag{11}$$

Since $S \cap T = 1 = D \cap T$, we have $|S|, |D| \leq |A : T|$ and $|T| \leq |A : S|$; similarly $|U| \leq |C : V|$ and $|V| \leq |C : U|$. Thus (10) and (11) yield that

$$S, T, D, U, V \quad \text{all have order at most } p^2. \tag{12}$$

In particular, $|S| \leq p^2$ and $|U| \leq pq^m$ where $q^m = |Q|$ and hence by (9), $|G| = |S \cup U| \leq p^5 q^m$. If $|P| = p^2$, then since $|M_s| \geq p^2$, we would have that $G = M_1 Q$; by [5, Lemma 3.1], G then even would be L_7-free, a contradiction. Thus

$$p^3 \leq |P| \leq p^5. \tag{13}$$

Now suppose, for a contradiction, that $A \not\leq P$. Since $A = S \cup T$, one of these subgroups, say S, has to contain a Sylow q-subgroup of A; so if we take $X = S$ above, then $k \geq 1$ in (10) and (11). By (10), $|C : V| < p^2$ and since $|C : V|$ is a power of p, it follows that $|C : V| = p$. Hence $|U| \leq p$ and since $q^m \mid |U|$, we have $|U| = q^m$. By (11), $|A : S| < p^2$ and since $|A : S|$ is a power of p, also $|A : S| = p$ and hence $|T| \leq p$. If T would be a q-group, then by (9), $|G| = |T \cup U| \leq p^2 q^m$, contradicting (13). Thus $|T| = p$ and $|G| = p^3 q^m$. But then $P = H \times M_s$ where $H \trianglelefteq G$ and $|H| = p$; it follows that $HT \trianglelefteq G$ and then $|G| = |HTU| \leq p^2 q^m$, again contradicting (13). Thus A is a p-group. Hence $L(A)$ is modular and so by (8), $|A| = |S||T| = |S||D| = |T||D|$. Therefore $|S| = |T| = |D|$ and by (13),

$$|A| = p^2 \quad \text{or} \quad |A| = p^4. \tag{14}$$

Suppose first that $|A| = p^2$. Then $|S| = |D| = p$ and by (12), $|U| \leq pq^m$. It follows from (9) that $|G| = |S \cup U| \leq p^4q^m$. So $|C_P(Q)| \leq p^2$ and hence P is abelian. Since $A \leq P$ and $G = A \cup B$, also B contains a Sylow q-subgroup of G; hence $B \cap P \trianglelefteq G$ and $C \cap P \trianglelefteq G$ and so $D = (B \cap P) \cap (C \cap P) \trianglelefteq G$. Therefore $C = DU$ and so $|C : U| = |D| = p$. It follows that $|V| = q^m$ and $|G| = |S \cup V| = p^3q^m$, by (9) and (13). Then again $P = H \times M_s$ with $H \trianglelefteq G$ and $|H| = p$ so that $|G| = |HSV| \leq p^2q^m$, a contradiction. Thus

$$|A| = p^4 \quad \text{and} \quad |S| = |T| = |D| = p^2. \tag{15}$$

Suppose first that $|U| = q^m$ or $|V| = q^m$, say $|U| = q^m$. Then by (9), $|G| = |S \cup U| \leq p^4q^m$ and since $|A| = p^4$, we have $A = P \trianglelefteq G$. Therefore $D = A \cap B \trianglelefteq B$ and $D \trianglelefteq C$ so that again $D \trianglelefteq G$. Furthermore $|V| = |G : A| = q^m$ and so $C = U \cup V \leq Q^G$. Since $|B : D| = |G : A| = q^m$, also $B \leq Q^G$; hence $G = B \cup C \leq Q^G$ so that $M_0 = 1$, by (6). By [5, Lemma 3.1], M_1Q is L_{10}-free; hence (6b) holds and $|M_2| = p^3$. It follows that Q induces a power automorphism either in D or in A/D; but in both groups $C = DU$ and $G/D = (A/D)(C/D)$ there exist two Sylow q-subgroups generating the whole group, a contradiction. So $|U| \neq q^m \neq |V|$ and by (12), $|U| = |V| = pq^m$. Since $A \cap U = E = 1$, it follows that $A < P$; so (13) and (15) yield that

$$|G| = p^5q^m \quad \text{and} \quad |U| = |V| = pq^m. \tag{16}$$

Since $L(P)$ is modular, $L(S) \simeq [A/D] \simeq L(T)$. So if S would be cyclic, then A would be of type (p^2, p^2) and hence by (6), $A \cap M_s = 1$ and $|P| \geq p^6$, a contradiction. Thus S and T are elementary abelian and so P is generated by elements of order p; by [4, Lemma 2.3.5], P is elementary abelian.

Now if (6a) holds, then $M_0S \trianglelefteq G$ and hence $G = M_0SU$. Since $|M_0| \leq p^2$, it follows from (16) that $|M_0| = p^2$ and $U \cap M_0 = 1$. Since $U \cap P \trianglelefteq G$, we have $U \cap P \leq M_1$ and so $U \leq Q^G = M_1Q$. Similarly, $V \leq Q^G$ and hence $C = U \cup V \leq Q^G$. Since $|C| \geq |D||U| = p^3q^m$ and $|M_1| = p^3$, it follows that $C = Q^G \trianglelefteq G$. But then $|B : D| = |G : C| = p^2$, so $|B| = p^4$ and $G = A \cup B \leq P$, a contradiction.

So, finally, (6b) holds and $P = M_0 \times M_1 \times M_2$ where $|M_0 \times M_1| \leq p^2$. This time $(M_0 \times M_1)S \trianglelefteq G$ and it follows from (16) that $|M_0 \times M_1| = p^2$ and $U \cap P \leq M_2$ and $V \cap P \leq M_2$. So $|M_2| = p^3$ and since $U \cap V = 1$, we have either $M_2 \leq C$ or $C \cap M_2 = (U \cap P) \times (V \cap P)$. In the first case, by (5), C

would contain every subgroup of order q of G; since $B \cap C = D$ is a p-group, it would follow that $B \leq P$ and hence $G = A \cup B \leq P$, a contradiction. So $|C \cap M_2| = p^2$ and if C_0, U_0, V_0 are the subgroups generated by the elements of order q of C, U, V, respectively, then by (5), C_0 is a P-group of order p^2q and U_0, V_0 are subgroups of order pq in C_0. So $U_0 \cap V_0 \neq 1$, but by (8), $U \cap V = 1$, the final contradiction.

We come to the third possibility for a group satisfying Hypothesis I.

Theorem 3. *Suppose that G satisfies Hypothesis I and that $q \mid p - 1$ but $|Q/C_Q(P)|$ does not divide $p - 1$; let $k \in \mathbb{N}$ such that q^k is the largest power of q dividing $p - 1$.*

Then G is L_{10}-free if and only if there exists a minimal normal subgroup N of order p^q of G such that one of the following holds.

(1) $P = C_P(Q) \times N$ where $|\Omega(C_P(Q))| \leq p^2$.

(2) $P = C_P(Q) \times N_1 \times N$ where $N_1 \trianglelefteq G$, $|N_1| = p$ and $C_P(Q)$ is cyclic.

(3) $q = 2$, $k = 1$ and $P = M \times N$ where $|M| = p^2$, Q is irreducible on M and $C_Q(N) < C_Q(M)$.

(4) $P = M \times N$ where M is elementary abelian of order p^2 and Q induces a power automorphism of order q in M.

(5) $P = N_1 \times N_2 \times N$ where $N_i \trianglelefteq G$, $|N_i| = p$ for $i = 1, 2$ and where $C_Q(N_1) < C_Q(N_2) = \phi(Q)$.

Proof. Suppose first that G is L_{10}-free. Again by Maschke's theorem, $[P, Q] = N_1 \times \cdots \times N_r$ $(r \geq 1)$ with Q irreducible on N_i and we may assume that $C_Q(N_r) \leq C_Q(N_i)$ for all i. Then $K := C_Q(P) = C_Q(N_r)$ and since $|Q/K|$ does not divide $p - 1$, we have that $|N_r| > p$. By Lemma 2 and [5, Lemma 3.1], $|N_r| = p^q$ and $|Q/K| = q^{k+1}$, or $|Q/K| \geq q^{k+1} = 4$ in case $q = 2$, $k = 1$. We let $N := N_r$ and have to show that G satisfies one of properties (1)–(5).

For this put $M := C_P(Q) \times N_1 \times \cdots \times N_{r-1}$, so that $P = M \times N$, and let $Q_1 \leq Q$ such that $K < Q_1$ and $|Q_1 : K| = q$. By Lemma 2, Q_1 induces a power automorphism of order q in N; by Lemma 1, $C_Q(N) < C_Q(N_i)$ for all $i \neq r$ and hence Q_1 centralizes M. So $PQ_1/K = MK/K \times NQ_1/K$ where NQ_1/K is a P-group of order $p^q q$. By Lemma 3, $|\Omega(M)| \leq p^2$; in particular, $r \leq 3$.

If $r = 1$, then $M = C_P(Q)$ and (1) holds. If $r = 2$, then either $|N_1| = p$ and $C_P(Q)$ is cyclic, that is (2) holds, or $|N_1| = p^2$ and $C_P(Q) = 1$. In this case, since Q is irreducible on N_1 and, by Lemma 1, induces automorphisms of different orders in N and N_1, again Lemma 2 and [5, Lemma 3.1] imply that $q = 2$ and $k = 1$; thus (3) holds.

Finally, suppose that $r = 3$. Since $|\Omega(M)| \leq p^2$, it follows that $M = N_1 \times N_2$, $|N_1| = |N_2| = p$ and $C_P(Q) = 1$. If $q = 2$ and $k = 1$, then $Q = \langle x \rangle$ induces automorphisms of order 2 in N_1 and N_2; thus $a^x = a^{-1}$ for all $a \in M$ and (4) holds. So suppose that $q > 2$ or $q = 2$ and $k > 1$. Then $|Q/K| = q^{k+1}$ as mentioned above and so $|\phi(Q) : K| = q^k$ divides $p - 1$. Thus $H := P\phi(Q)$ is one of the groups in Theorem 2 and by Lemma 2, $\phi(Q)$ induces a power automorphism of order q^k in N. Since $[P, \phi(Q)] \leq [P, Q] = N_1 \times N_2 \times N$ and $C_Q(N) < C_Q(N_i)$ for $i \in \{1, 2\}$, N is one of the eigenspaces of x^p in $[P, \phi(Q)]$. Hence H satisfies (2b) or (2c) of Theorem 2. In the first case, $N = M_1$ in the notation of that theorem and $N_1 \times N_2 \leq C_P(\phi(Q))$ since $C_{\phi(Q)}(M_1)$ is the largest centralizer of a nontrivial eigenspace of x^p. So $C_Q(N_1) = \phi(Q) = C_Q(N_2)$ and by Lemma 1, Q induces a power automorphism of order q in $N_1 \times N_2$; thus (4) holds. In the other case, $N = M_2$ and $|M_1| = p$, so that $M_1 = N_1$, say, and then $N_2 \leq C_P(\phi(Q))$. Thus (5) holds and G has the desired properties.

To prove the converse, we again consider a minimal counterexample G. Then G has a minimal normal subgroup N of order p^q and satisfies one of the properties (1)–(5) but is not L_{10}-free. As in the proof of Theorem 1, by Lemma 7, $C_Q(P) = 1$.

Let H be a proper subgroup of G. Then either H contains a Sylow q-subgroup of G or $H \leq P\phi(Q)$. In the first case, $N \leq H$ or $H \cap N = 1$. Hence H satisfies the assumptions of Theorem 3 or Theorem 2 or is nilpotent; the minimality of G implies that H is L_{10}-free. So suppose that $H = P\phi(Q)$. A simple computation shows (see [5, p. 523]) that if $q > 2$ or if $q = 2$ and $k > 1$, then q^{k+1} is the largest power of q dividing $p^q - 1$. Therefore in these cases, by [3, II, Satz 3.10], a generator x of Q operates on $N = (GF(p^q), +)$ as multiplication with an element of order q^{k+1} of the multiplicative group of $GF(p^q)$. The q-th power of this element lies in $GF(p)$ and therefore fixes every subgroup of N. Thus $\phi(Q)$ induces a power automorphism of order q^k in N. So if G satisfies (1) or (4), then H satisfies $s = 1$ and (2b) of Theorem 2; the same holds if G satisfies (2) and $\phi(Q)$ centralizes N_1. If G satisfies (2) and $[\phi(Q), N_1] \neq 1$ or G satisfies (5), then (2c) of Theorem 2 holds for H. Finally, if $q = 2$ and $k = 1$, then either $\phi(Q)$ is irreducible on N or $|Q| = 4$; hence H satisfies the assumptions of Theorem 3 or 2. In all cases, Theorem 2 and the minimality of G imply that H is L_{10}-free.

Finally, $Q_0 = \Omega(Q)$ induces a power automorphism of order q in N and centralizes the complements of N in P given in (1)–(5). So $P = N \times C_P(Q_0)$ and by Lemma 8, G is L_{10}-free, the desired contradiction.

Note that in Theorem 1 and in (2a) of Theorem 2, $C_P(Q)$ may be an arbitrary modular p-group since by Iwasawa's theorem [4, Theorem 2.3.1], a direct product of a modular p-group with an elementary abelian p-group has modular subgroup lattice. In all the other cases of Theorems 2 and 3, Lemma 3 implied that $|\Omega(C_P(Q))| \leq p^2$; in (2b) of Theorem 2 and (1) of Theorem 3, $C_P(Q)$ may be an arbitrary modular p-group with this property.

4 Groups of type II and III

We now determine the groups of type II. Theorem 4 shows that modulo centralizers the only such group is $SL(2,3) \simeq Q_8 \rtimes C_3$.

Theorem 4. *Let $G = PQ$ where P is a normal Sylow 2-subgroup of G, Q is a cyclic q-group, $2 < q \in \mathbb{P}$, and $[P,Q]$ is hamiltonian.*

Then G is L_{10}-free if and only if $G = M \times NQ$ where M is an elementary abelian 2-group, $N \simeq Q_8$ and Q induces an automorphism of order 3 in N.

Proof. Suppose first that G is L_{10}-free. Then $L(P)$ is modular and since $[P,Q]$ is hamiltonian, it follows from [4, Theorems 2.3.12 and 2.3.8] that $P = H \times K$ where H is elementary abelian and $K \simeq Q_8$. Hence $\phi(P) = \phi(K)$ and $\Omega(P) = H \times \phi(P)$. By Maschke's theorem there are Q-invariant complements M of $\phi(P)$ in $\Omega(P)$ and $N/\phi(P)$ of $\Omega(P)/\phi(P)$ in $P/\phi(P)$. Then $\Omega(N) = \Omega(P) \cap N = \phi(P)$ implies that $N \simeq Q_8$ and since $[P,Q] \not\leq \Omega(P)$, Q operates nontrivially on N. Therefore $q = 3$ and Q induces an automorphism of order 3 in N.

Since P is a 2-group, $G/\phi(P)$ is an L_{10}-free $\{p,q\}$-group of type I with $q \nmid p-1$. By Theorem 1, $P/\phi(P) = C_{P/\phi(P)}(Q) \times N_1 \times \cdots \times N_r$ with nontrivial $GF(2)Q$-modules N_i satisfying (1) and (2) of that theorem. By (1), the subgroup of order 3 of $Q/C_Q(N_i)$ is irreducible on N_i; therefore $|N_i| = 4$ and hence $C_Q(N_i) = \phi(Q)$ for all i. But then (2) implies that $r = 1$. It follows that $N_1 = N/\phi(P)$ and $[M,Q] \leq M \cap N = 1$; thus $G = M \times NQ$ as desired.

To prove the converse, we again consider a minimal counterexample G; let $\{A,\ldots,V\}$ be a sublattice of L(G) isomorphic to L_{10} and satisfying (1.1)–(1.4). The minimality of G implies that $F = G$ and, together with Lemma 7, that $C_Q(P) = 1$; hence $|Q| = 3$.

If A or C, say C, contains two subgroups of order 3, then $NQ \leq C$ and hence $C \trianglelefteq G$. Then $D = A \cap C = B \cap C \trianglelefteq A \cup B = G$ and $A/D \simeq G/C \simeq B/D$ are 2-groups; therefore G/D is a 2-group. Similarly, $E = S \cap D = U \cap D \trianglelefteq S \cup U = G$ and $S/E \simeq G/C$ and $U/E \simeq C/D$ are 2-groups. Thus G/E is a modular 2-group and hence L_{10}-free, a contradiction.

So A and C both contain at most one subgroup of order 3 and therefore are nilpotent. By Lemma 6, we have $U, V \not\leq P$, say; so U and V contain the subgroup Q_1 of order 3 of C and it follows that $Q_1 \leq U \cap V = E \leq A$. Hence $G = A \cup C \leq C_G(Q_1)$, a final contradiction.

We finally come to groups of type III; more generally, we determine all L_{10}-free $\{p,2\}$-groups in which Q_8 operates faithfully on P.

Theorem 5. *Let $G = PQ$ where P is a normal Sylow p-subgroup with modular subgroup lattice, $Q \simeq Q_8$ and $C_Q(P) = 1$.*

Then G is L_{10}-free if and only if $P = M \times N$ where $|N| = p^2$, Q operates irreducibly on N and one of the following holds :

(1) $p \equiv 3\ (mod\ 4)$, $M = C_P(Q)$ and $|\Omega(M)| \leq p^2$,

(2) $M = C_P(Q) \times M_1$ where $C_P(Q)$ is cyclic, $M_1 \trianglelefteq G$ and $|M_1| = 3$,

(3) $C_P(Q) = 1$ and $M = C_P(\Omega(Q))$ is elementary abelian of order 9.

Proof. Suppose first that G is L_{10}-free. By [6, Lemma 2.2], $P = C_P(Q) \times [P,Q]$ and $[P,Q]$ is elementary abelian; by Maschke's theorem, $[P,Q] = N_1 \times \cdots \times N_r$ with irreducible $GF(p)Q$-modules N_i. As $C_Q(P) = 1$, there exists $i \in \{1,\ldots,r\}$ such that $C_Q(N_i) = 1$; we choose the notation so that $i = r$ and let $N = N_r$, $M = C_P(Q) \times N_1 \times \cdots \times N_{r-1}$ and $Q_0 = \Omega(Q)$.

Clearly, $|N| \geq p^2$ and since $C_N(Q_0)$ is Q-invariant, $C_N(Q_0) = 1$; hence N is inverted by Q_0. It follows that if X is a maximal subgroup of Q, then $C_X(W) = 1$ for every minimal normal subgroup W of NX. By Lemma 1, either X is irreducible on N or it induces a power automorphism in N. Since Q is irreducible on N, at most one maximal subgroup of Q can induce power automorphisms in N and hence there are at least two maximal subgroups of Q which are irreducible on N. It follows that $|N| = p^2$ and $p \equiv 3\ (mod\ 4)$.

If there would exist $i \in \{1,\ldots,r-1\}$ such that $C_Q(N_i) = 1$, then there would exist a maximal subgroup X of Q which is irreducible on both N_i and N; but then $(N_i \times N)X$ would be L_{10}-free, contradicting Lemma 1. Thus $N = N_r$ is the unique N_i on which Q is faithful; it follows that $M = C_P(Q_0)$.

Since NQ_0 is a P-group of order $2p^2$, Lemma 3 yields that $|\Omega(M)| \leq p^2$. So if $r = 1$, then (1) holds; therefore assume that $r \geq 2$. Then $C_G(Q_0)/Q_0 = MQ/Q_0$ is L_{10}-free and has non-normal elementary abelian Sylow 2-subgroups of order 4. By [6, Proposition 2.6], $p = 3$. It follows that (2) holds if $r = 2$ and (3) holds if $r = 3$.

To show that, conversely, all the groups with the given properties are L_{10}-free, we consider a minimal counterexample G to this statement and want to apply Lemma 8.

Again since Q is irreducible on N and $|N| = p^2$, it follows that N is inverted by $Q_0 = \Omega(Q)$. By assumption, M is centralized by Q_0 and therefore we have that $P = N \times C_P(Q_0)$. Furthermore every subgroup of order 4 of Q is faithful on N and hence irreducible on N since $4 \nmid p-1$. So it remains to be shown that every proper subgroup H of G is L_{10}-free.

If $8 \nmid |H|$, then $H \leq PQ_1$ for some maximal subgroup Q_1 of Q. Since Q_1 is irreducible and faithful on N, the group PQ_1 is L_{10}-free by Theorem 3; thus also H is L_{10}-free. So suppose that H contains a Sylow 2-subgroup of G, say $Q \leq H$. Then either $N \leq H$ or $H \cap N = 1$ and then $H \leq MQ$. In the first case, the minimality of G implies that H is L_{10}-free. In the second case, we may assume that $H = MQ$. This group even is modular if (1) holds and by [6, Lemma 4.5], it is L_{10}-free if (2) is satisfied. So suppose that (3) holds. Then H/Q_0 is a group of order 36 so that it is an easy exercise to show that it is L_{10}-free (see also Remark 2); by Lemma 7, then also H is L_{10}-free. Thus every proper subgroup of G is L_{10}-free and Lemma 8 implies that G is L_{10}-free, the desired contradiction.

Remark 2. (a) Part (1) of Theorem 5 characterizes the L_{10}-free $\{p, q\}$-groups of type III and shows that also for $p = 3$ the corresponding groups are L_{10}-free.

(b) In addition, parts (2) and (3) of Theorem 5 show that for $p = 3$ there are exactly three further types of L_{10}-free $\{2, 3\}$-groups in which Q_8 operates faithfully. In these, $MQ/\Omega(Q)$ is isomorphic to

(i) $C_{3^n} \times D_6 \times C_2$ $(n \geq 0)$, or

(ii) $H \times C_2$ where H is a P-group of order 18, or

(iii) $D_6 \times D_6$.

(c) The groups in (ii) and (iii) both are subgroups of the group G in Example 4.7 of [6] and therefore are L_{10}-free.

Proof of (b). Clearly, the four group $Q/\Omega(Q)$ can only invert M_1 in (2) of Theorem 5; so we get the groups in (i). If (3) holds, then $M = M_1 \times M_2$ where $M_i \trianglelefteq MQ$ and $|M_i| = 3$. So if $C_Q(M_1) = C_Q(M_2)$, we obtain (ii) and if $C_Q(M_1) \neq C_Q(M_2)$, then $M_1C_Q(M_2)$ and $M_2C_Q(M_1)$ centralize each other modulo $\Omega(Q)$ and hence (iii) holds.

We finally mention that by Lemma 7, to characterize also the L_{10}-free $\{2, 3\}$-groups with Sylow 2-subgroup Q_8 operating non-faithfully on a 3-group P, it remains to determine the L_{10}-free $\{2, 3\}$-groups having a four group as Sylow 2-subgroup. This, however, is the crucial case in the study of L_{10}-free $\{2, 3\}$-groups since by [6, Lemma 2.9], in every such group PQ we have $|\Omega(Q/C_Q(P))| \leq 4$.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 73–91. doi:10.1285/i15900932v30sn1p73

A Conjecture of Brian Hartley and developments arising

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Abstract. Around 1980 Brian Hartley conjectured that if the unit group of a torsion group algebra FG satisfies a group identity, then FG satisfies a polynomial identity. In this short survey we shall review some results dealing with the solution of this conjecture and the extensive activity that ensued. Finally, we shall discuss special polynomial identities satisfied by FG (or by some of its subsets) and the corresponding group identities satisfied by its unit group (or by some of its subsets).

Keywords: Group Algebras, Polynomial Identities, Group Identities, Lie Structure, Involutions.

MSC 2000 classification: 16S34, 16R50, 16W10, 16U60.

1 A Conjecture of Brian Hartley

Let $\langle x_1, x_2, \ldots \rangle$ be the free group on a countable set of generators. If S is any subset of a group G, we say that S satisfies a group identity if there exists a non-trivial reduced word $w(x_1, \ldots, x_n) \in \langle x_1, x_2, \ldots \rangle$ such that $w(g_1, \ldots, g_n) = 1$ for all $g_i \in S$. For elements $y_1, \ldots, y_n$ of a group G, set $(y_1, y_2) = y_1^{-1} y_2^{-1} y_1 y_2$, the *group commutator* of y_1 and y_2, and inductively $(y_1, \ldots, y_n) = ((y_1, \ldots, y_{n-1}), y_n)$. Obviously, abelian groups and nilpotent groups are examples of groups satisfying a group identity ((x_1, x_2) and $(x_1, \ldots, x_c)$ for some c, respectively).

In an attempt to give a connection between the additive and the multiplicative structure of a group algebra FG of a group G over a field F, Brian Hartley made the following famous conjecture.

Conjecture 1. *Let G be a torsion group and F an infinite field. If the unit group $\mathcal{U}(FG)$ of FG satisfies a group identity, then FG satisfies a polynomial identity.*

We recall that a subset R of FG satisfies a polynomial identity (PI) if there exists a non-trivial element $f(x_1, \ldots, x_n)$ in the free algebra $F\{x_1, x_2, \ldots\}$ on non-commuting indeterminates $x_1, x_2, \ldots$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_i \in R$. The conditions under which FG satisfies a polynomial identity were determined in classical results due to Passman and Isaacs-Passman (see Corollaries 5.3.8 and 5.3.10 of [41]) summarized in the following

Theorem 1. *Let F be a field of characteristic $p \geq 0$ and G a group. Then FG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index.*

For the sake of completeness recall that, for any prime p, a group G is said to be p-abelian if its commutator subgroup G' is a finite p-group, and that 0-abelian means abelian.

The Hartley's Conjecture was first studied by Warhurst in his PhD thesis [48] where special words satisfied by $\mathcal{U}(FG)$ were investigated. Pere Menal [39] suggested a possible solution for some p-groups. When the field is infinite, Goncalves and Mandel [21] verified it in the special case that the group identity is actually a semigroup identity (that is, an identity of the form $x_{i_1}x_{i_2}\cdots x_{i_k} = x_{j_1}x_{j_2}\cdots x_{j_l}$). Giambruno, Jespers and Valenti [11] handled the characteristic 0 case as well as the characteristic $p > 0$ case when G has no elements of p-power order. In fact, under these assumptions FG is semiprime and the fact that $\mathcal{U}(FG)$ satisfies a group identity forces G to be abelian. By using the Menal's construction, Giambruno, Sehgal and Valenti [18] solved the conjecture, by proving the following

Theorem 2. *Let G be a torsion group and F an infinite field. If $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

A positive answer to Hartley's Conjecture having been established, it was natural to look for necessary and sufficient conditions for $\mathcal{U}(FG)$ to satisfy a group identity. Clearly, satisfying a polynomial identity cannot be sufficient. We see from Theorem 1 that if G is finite, then FG always satisfies a polynomial identity, but if $\operatorname{char} F = 0$, then $\mathcal{U}(FG)$ does not satisfy a group identity unless G is abelian. The question was solved by Passman [42], by using the results of [18], in the following

Theorem 3. *Let F be an infinite field of characteristic $p > 0$ and G a torsion group. Then the following are equivalent:*

(i) *$\mathcal{U}(FG)$ satisfies a group identity;*

(ii) *$\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;*

(iii) *G has a normal p-abelian subgroup of finite index and G' is a p-group of bounded exponent.*

The fact that F is assumed to be infinite allowed the authors to apply a *Vandermonde determinant argument* (see, for instance, Proposition 1 of [11] and the roles played by its implications in [18] and [42]). On the other hand, by Theorem 3, for any non-abelian finite group G, if $\mathcal{U}(FG)$ satisfies a group identity then G is p-abelian. This is obviously no longer true if F has finitely many elements: in this case, for any finite group G, $\mathcal{U}(FG)$ is finite, hence it satisfies a group identity. Subsequently a lot of work has been done to generalize the above results to

- arbitrary fields
- arbitrary groups
- special subsets of $\mathcal{U}(FG)$

1.1 Arbitrary Fields F

By modifying the original proof of [18], Liu [36] confirmed the Hartley's Conjecture for fields of all sizes. His arguments were decisive to generalize the results of [42] to group algebras over non-necessarily infinite fields. This was done by Liu and Passman in [37]. It turns out that the solution is different if G' is not a p-group. Their main results are the following.

Theorem 4. *Let F be a field of characteristic $p > 0$ and G a torsion group. If G' is a p-group, then the following are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;

(iii) G has a p-abelian subgroup of finite index and G' has bounded exponent.

Theorem 5. *Let F be a field of characteristic $p > 0$ and G a torsion group. If G' is not a p-group, then the following are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ has bounded exponent;

(iii) G has a p-abelian subgroup of finite index, G has bounded exponent and F is a finite field.

1.2 Non-torsion Groups

In general, the Hartley's Conjecture is not expected to hold for arbitrary groups. For instance, if G is a torsion-free nilpotent group, then the only units in FG are *trivial*, namely αg, with $0 \neq \alpha \in F$ and $g \in G$, and $\mathcal{U}(FG)$ is nilpotent. But FG need not satisfy a polynomial identity. The main obstruction in trying to characterize group algebras of non-torsion groups whose units satisfy a group identity is the difficulty in handling the torsion free part of the group. It is worth noting that for any such result, a restriction will be required for the sufficiency, pending a positive answer to the following very famous (and difficult) conjecture due to Kaplansky.

Conjecture 2. *If G is a torsion-free group and F a field, then the only units in FG are trivial.*

Anyway, for groups with elements of infinite order the question was studied by Giambruno, Sehgal and Valenti in [20]. They proved that, if $\mathcal{U}(FG)$ satisfies a group identity, then the torsion elements of G form a subgroup, T. For the converse, a suitable restriction upon G/T is required, namely that it is a u.p. (unique product) group, i.e., for every pair of non-empty finite subsets S_1 and S_2 of G/T, there exists an element $g \in G/T$ that can be uniquely written as $g = s_1 s_2$, with each $s_i \in S_i$. We have to separate two cases according as FG is semiprime (by virtue of Theorems 4.2.12 and 4.2.13 of [41] this means that either $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 0$ and G has no normal subgroups with order divisible by p) or not.

Theorem 6. *Let FG be semiprime and suppose that F is infinite or G has an element of infinite order. If $\mathcal{U}(FG)$ satisfies a group identity then*

(1) all the idempotents of FG are central;

(2) T is an abelian p'-subgroup of G.

Conversely, if G is a group satisfying (1) and (2) and G/T is nilpotent of class c, then $\mathcal{U}(FG)$ satisfies the group identity $((x_1, \ldots, x_c), (x_{c+1}, \ldots, x_{2c})) = 1$.

The characteristic zero case having been dealt with, in the next result assume that F is a field of characteristic $p \geq 2$.

Theorem 7. *Suppose that F is infinite or G has an element of infinite order. We have the following*

(1) If $\mathcal{U}(FG)$ satisfies a group identity then P, the set of the p-elements of G, is a subgroup.

(2) If P is of unbounded exponent and $\mathcal{U}(FG)$ satisfies a group identity then

(a) G contains a p-abelian subgroup of finite index;

(b) G' is of bounded p-power exponent.

Conversely, if P is a subgroup and G satisfies (a) and (b), then $\mathcal{U}(FG)$ satisfies a group identity.

(3) If P is of bounded exponent and $\mathcal{U}(FG)$ satisfies a group identity then

(a′) P is finite or G contains a p-abelian subgroup of finite index;

(b′) $T(G/P)$ is an abelian p'-subgroup and so T is a group;

(c′) every idempotent of $F(G/P)$ is central.

Conversely, if P is a subgroup, G satisfies (a′), (b′) and (c′) and G/T is a u.p. group, then $\mathcal{U}(FG)$ satisfies a group identity.

1.3 Special Subsets of $\mathcal{U}(FG)$

A natural question of interest is to ask if group identities satisfied by some special subset of the unit group of a group algebra FG can be lifted to $\mathcal{U}(FG)$ or force FG to satisfy a polynomial identity. A motivation for this study is the classical theorem of Amitsur regarding an identity on symmetric elements of a ring with involution forcing an identity of the whole ring. In this framework, the *symmetric units* have been the subject of a good deal of attention.

Let FG be the group ring of a group G over a field F of characteristic different from 2. If G is endowed with an involution $\star$, then it can extended F-linearly to an involution of FG, also denoted by $\star$. An element $\alpha \in FG$ is said to be *symmetric with respect to* $\star$ if $\alpha^\star = \alpha$. We write FG^+ for the set of symmetric elements, which are easily seen to be the linear combinations of the terms $g + g^\star$, for all $g \in G$. Let $\mathcal{U}^+(FG)$ denote the set of symmetric units. Prior to the last couple of years, attention had largely been devoted to the *classical involution* induced from the map $g \mapsto g^{-1}$ on G. Giambruno, Sehgal and Valenti [19] confirmed a stronger version of Hartley's Conjecture by proving

Theorem 8. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

Under the same restrictions as in the above theorem, they also obtained necessary and sufficient conditions for $\mathcal{U}^+(FG)$ to satisfy a group identity. They get different answers depending on whether G contains a copy of the quaternion group Q_8. More precisely, it is effected by the presence in G of a copy of a *Hamiltonian 2-group*. We recall that a non-abelian group G is a Hamiltonian group if every subgroup of G is normal. It is well-known that in this case $G = O \times E \times Q_8$, where O is an abelian group with every element of odd order and E is an elementary abelian 2-group. In fact, a crucial remark for the classification of torsion group algebras FG whose symmetric units satisfy a group identity is that, for any commutative ring R and Hamiltonian 2-group H, RH^+ is commutative. The main result of [19] is the following

Theorem 9. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from 2 endowed with the classical involution.*

(a) If $\operatorname{char} F = 0$, *$\mathcal{U}^+(FG)$ satisfies a group identity if and only if G is either abelian or a Hamiltonian 2-group.*

(b) If $\operatorname{char} F = p > 2$, *then $\mathcal{U}^+(FG)$ satisfies a group identity if and only if FG satisfies a polynomial identity and either $Q_8 \not\subseteq G$ and G' is of bounded exponent p^k for some $k \geq 0$ or $Q_8 \subseteq G$ and*

(1) the p-elements of G form a (normal) subgroup P of G and G/P is a Hamiltonian 2-group;

(2) G is of bounded exponent $4p^s$ for some $s \geq 0$.

Obviously, group identities on $\mathcal{U}^+(FG)$ do not force group identities on $\mathcal{U}(FG)$. To see this it is sufficient to observe that, for any infinite field F of characteristic $p > 2$, FQ_8^+ is commutative, hence $\mathcal{U}^+(FQ_8)$ satisfies a group identity but, according to Theorem 3, $\mathcal{U}(FG)$ does not satisfy a group identity.

The above results were extended to non-torsion groups in [44] under the usual restriction for the only if part related to Kaplansky's Conjecture. We do not review here the statements of that paper, but we confine ourselves to report the following result, which goes in the direction of the Hartley's Conjecture and Theorem 8.

Theorem 10. *Let FG be the group algebra of a group G with an element of infinite order over an infinite field F of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then the set P of p-elements of G forms a normal subgroup and, if P is infinite, then FG satisfies a polynomial identity.*

Recently, there has been a considerable amount of work on involutions of FG obtained as F-linear extension of arbitrary group involutions on G other than the classical one. In particular, Broche Cristo, Jespers, Polcino Milies and Ruiz Marin have studied the interesting question as to when FG^+ and $FG^- = \{\alpha \mid \alpha \in FG \quad \alpha^\star = -\alpha\}$ the Lie subalgebra of the *skew-symmetric* elements of FG are commutative ([25] and [5]). Goncalves and Passman [22] considered the existence of bicyclic units u in the integral group rings such that the group $\langle u, u^\star \rangle$ is free. Marciniak and Sehgal in [38] had proved that, with respect to the classical involution, $\langle u, u^\star \rangle$ is always free if $u \neq 1$.

In the classification results on group algebras whose symmetric units with respect to the classical involution satisfy a group identity in some sense the exceptional cases turned out to involve Hamiltonian 2-groups, because they are non-abelian groups such that the symmetric elements in the group rings commute. When one works with linear extensions of arbitrary involutions of the base group of the group algebra, one finds a larger class of groups such that the symmetric elements of the related group algebra have the same property. In order to state the next results, a definition is required. We recall that a group G is said to be an LC-group (that is, it has the "lack of commutativity" property) if it is not abelian, but if $g, h \in G$, and $gh = hg$, then at least one of g, h and gh must be central. These groups were introduced by Goodaire. By Proposition III.3.6 of [23], a group G is an LC-group with a unique non-identity commutator (which must, obviously, have order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. Here, $\zeta(G)$ denotes the centre of G.

Definition 1. A group G endowed with an involution $*$ is said to be a special LC-group, or SLC-group, if it is an LC-group, it has a unique non-identity commutator z, and for all $g \in G$, we have $g^* = g$ if $g \in \zeta(G)$, and otherwise, $g^* = zg$.

The SLC-groups arise naturally in the following result proved by Jespers and Ruiz Marin [25] for an arbitrary involution on G.

Theorem 11. *Let R be a commutative ring of characteristic different from 2, G a non-abelian group with an involution $*$ which is extended linearly to RG. Then the following are equivalent:*

(i) RG^+ is commutative;

(ii) RG^+ is the centre of RG;

(iii) G is an SLC-group.

This is crucial for the classification of torsion group algebras endowed with an involution induced from an arbitrary involution on G with symmetric units satisfying a group identity. The question was originally studied by Dooms and Ruiz [8] and completely solved by Giambruno, Polcino Milies and Sehgal [14].

Theorem 12. *Let F be an infinite field of characteristic $p \neq 2$, G a torsion group with an involution $*$ which is extended linearly to FG. Then the symmetric units of FG satisfy a group identity if and only if one of the following holds:*

(a) FG is semiprime and G is abelian or an SLC-group;

(b) FG is not semiprime, the p-elements of G form a (normal) subgroup P, G has a p-abelian normal subgroup of finite index, and either

(1) G' is a p-group of bounded exponent, or

(2) *G/P is an SLC-group and G contains a normal $*$-invariant p-subgroup B of bounded exponent, such that P/B is central in G/B and the induced involution acts as the identity on P/B.*

2 Lie Properties in FG

Any associative algebra A over a field F may be regarded as a Lie algebra by defining the Lie multiplication

$$[a,b] = ab - ba \qquad \forall a,b \in A.$$

For any two subspaces S and T of A, we define $[S,T]$ to be the additive subgroup of A generated by all the Lie products $[s,t]$ with $s \in S$ and $t \in T$. Obviously $[S,T]$ is a F-subspace of A. We can define inductively the *Lie central series* and the *Lie derived series* of A by

$$A^{[1]} = A, \qquad A^{[n+1]} = [A^{[n]}, A]$$

and

$$\delta^{[0]}(A) = A, \qquad \delta^{[n+1]}(A) = [\delta^{[n]}(A), \delta^{[n]}(A)],$$

respectively. One may also enlarge the terms of this series by making them associative at every stage. More precisely, we define by induction the series

$$A^{(1)} = A, \qquad A^{(n+1)} = \langle [A^{(n)}, A] \rangle$$

and

$$\delta^{(0)}(A) = A, \qquad \delta^{(n+1)}(A) = \langle [\delta^{(n)}(A), \delta^{(n)}(A)] \rangle,$$

where, for any two associative ideals S, T of A, $\langle [S,T] \rangle$ denotes the associative ideal of A generated by $[S,T]$.

We say that A is *Lie nilpotent* if $A^{[n]} = 0$ for some integer n and, similarly, A is *Lie solvable* if $\delta^{[m]}(A) = 0$ for some integer m. In a similar fashion, A is said to be *strongly Lie nilpotent* (*strongly Lie solvable*, respectively) if $A^{(n)} = 0$ (if $\delta^{(n)}(A) = 0$, respectively) for some integer n. If A is strongly Lie nilpotent, the smallest integer m such that $A^{[m+1]} = 0$ ($A^{(m+1)} = 0$, respectively) is called the *Lie nilpotency class* (the *strong Lie nilpotency class*, respectively) of A and is denoted by $cl_L(A)$ ($cl^L(A)$, respectively). We make at once the following simple observations: an algebra A which is strongly Lie nilpotent (solvable, respectively) is Lie nilpotent (solvable, respectively) and the (strong) Lie nilpotency property implies the (strong) Lie solvable property. It is apparent that algebras which are Lie solvable satisfy a certain multilinear polynomial identity.

At the beginning of 70s, thanks to the classification by Passman and Isaacs of PI group algebras, Passi, Passman and Sehgal [40] solved the question of when a group algebra FG of a group G over a field F is Lie solvable and Lie nilpotent by proving the following

Theorem 13. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is Lie nilpotent if and only if G is nilpotent and p-abelian.*

Theorem 14. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is Lie solvable if and only if either G is p-abelian or $p = 2$ and G contains a 2-abelian subgroup of index 2.*

For the sake of completness we recall that the original results of [40] were established for arbitrary group rings over commutative rings with identity. For an overview we refer to Chapter V of [43], where the conditions so that a group algebra satisfies the strong Lie identities were also stated, namely

Theorem 15. *Let FG be the group algebra of a group G over a field F. Then FG is strongly Lie nilpotent if and only if FG is Lie nilpotent.*

Theorem 16. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is strongly Lie solvable if and only if G is p-abelian.*

Another question of interest was to find necessary and sufficient conditions so that a group algebra FG is *bounded Lie Engel*. We recall that, for a positive integer n, a ring R (or a subset of it) is said to be Lie n-Engel if

$$[a, \underbrace{b, \dots, b}_{n \text{ times}}] = 0$$

for all $a, b \in R$. A ring R is *bounded Lie Engel* if it is Lie n-Engel for some positive integer n. This was done by Sehgal (Theorem V.6.1 of [43]).

Theorem 17. *Let FG be the group algebra of a group G over a field F. If* $\operatorname{char} F = 0$, *then FG is bounded Lie Engel if and only if G is abelian. If* $\operatorname{char} F = p > 0$, *then FG is bounded Lie Engel if and only if G is nilpotent and G has a p-abelian normal subgroup of finite p-power index.*

We have already seen in Section 1 the connection between group identities on units and polynomial identities on the group algebra. Furthermore it is possible frequently to reduce problems concerning specific group identities to problems concerning specific Lie identities. This is evident in particular when the group algebra is Lie nilpotent. To this purpose, let us consider the unit group $\mathcal{U}(FG)$ of a group algebra FG and let $u, v \in \mathcal{U}(FG)$. Then

$$(u, v) - 1 = u^{-1}v^{-1}[u, v].$$

A consequence of this fact is that, for any positive integer n,

$$\gamma_n(\mathcal{U}(FG)) - 1 \subseteq FG^{(n)}, \tag{1}$$

where $\gamma_n(\mathcal{U}(FG))$ denotes the n-th term of the lower central series of the group $\mathcal{U}(FG)$. It immediately follows that if FG is strongly Lie nilpotent then $\mathcal{U}(FG)$ is nilpotent and, if $cl(\mathcal{U}(FG))$ denotes the nilpotency class of $\mathcal{U}(FG)$, $cl(\mathcal{U}(FG)) \leq cl^L(FG)$. Gupta and Levin [24] improved the result of (1) by proving that

$$\gamma_n(\mathcal{U}(FG)) - 1 \subseteq \langle FG^{[n]} \rangle$$

and, consequently, if FG is Lie nilpotent then $cl(\mathcal{U}(FG)) \leq cl_L(FG)$. The implication between the Lie nilpotency property of FG and the nilpotency of $\mathcal{U}(FG)$ is true also in the other direction, at least in the modular case (if char $F = p > 0$, a group algebra FG is said to be *modular* if G contains at least one element of order p) as established by Khripta [26].

Theorem 18. *Let FG be the modular group algebra of a group G over a field F. Then $\mathcal{U}(FG)$ is nilpotent if and only if FG is Lie nilpotent.*

The semiprime case was settled by Fisher, Parmenter and Sehgal [10] and involves more conditions.

Theorem 19. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Suppose that G has no elements of order p (if $p > 0$). Then $\mathcal{U}(FG)$ is nilpotent if and only if G is nilpotent and one of the following holds:*

(a) T, the set of the elements of finite order of G, is a central subgroup of G;

(b) $|F| = 2^\beta - 1$ is a Mersenne prime, T is an abelian subgroup of G of exponent $p^2 - 1$ and, for all $x \in G$ and $t \in T$, $x^{-1}tx = t$ or t^p.

At the end of 1980s, Shalev (see [45] for a general discussion) proposed a systematic study of the nilpotency class of the unit group of a group algebra of a finite p-group G over the field with p elements $\mathbb{F}_p$. Even in the case in which the group G is rather simple, $\mathcal{U}(\mathbb{F}_pG)$ is a finite p-group whose structure is rather complicated and its nilpotency class in some way measures its complexity. For a long time, a line of research has been that of considering the existence of a given groups L *involved* in $\mathcal{U}(\mathbb{F}_pG)$. Using this approach, Coleman and Passman [7] proved that, if G is non-abelian, then the wreath product $C_p \wr C_p$, where C_p is the group of order p, is involved in $\mathcal{U}(\mathbb{F}_pG)$, from which it follows that $cl(\mathcal{U}(\mathbb{F}_pG)) \geq p$. Subsequently Baginski [2] has proved the equality in the case in which the commutator subgroup of G has order p. Based on the original idea of Coleman and Passman, Shalev conjectured that $\mathcal{U}(\mathbb{F}_pG)$ always possesses a section isomorphic to the wreath product $C_p \wr G'$ and proved the result in [46] when G' is cyclic and p is odd.

A fundamental contribution in this framework was given by the solution of Jennings' Conjecture on radical rings by Du [9], which allowed to conclude that, for any field F of characteristic p and finite p-group G,

$$cl_L(FG) = cl(\mathcal{U}(FG)). \tag{2}$$

In this way, group commutator computations were replaced by ones involving Lie commutators, which are considerably easier. But this is not the only advantage. Indeed, in [3] Bahandari and Passi proved for an arbitrary Lie nilpotent group algebra FG that

$$cl_L(FG) = cl^L(FG)$$

provided $\operatorname{char} F = p > 3$ (it is an *open question* to decide if the equality holds for arbitrary p). Thus, according to (2), when $\operatorname{char} F = p > 3$ and G is a finite p-group the computation of the nilpotency class of $\mathcal{U}(FG)$ is reduced to that of $cl^L(FG)$. For this an extension of Jennings's theory provides a rather satisfactory formula based on the size of the Lie dimension subgroups of the underlying group G. In confirmation of all this, the most prominent results in this direction, presented in [47], were just deduced on the basis of the breakthrough of Du and Bhandari and Passi.

The equality (2) is easily seen to be satisfied when G is a (not necessarily finite) p-group. Recently, Catino, Siciliano and Spinelli [6] settled the case in which G is an arbitrary torsion group by proving the following

Theorem 20. *Let F be a field of positive characteristic p and G a torsion group containing an element of order p such that $\mathcal{U}(FG)$ is nilpotent. Then $cl_L(FG) = cl(\mathcal{U}(FG))$.*

One cannot expect that Theorem 20 is valid for arbitrary modular group algebras. In fact, Theorems 4.3, 4.4 and 5.2 of [4] provide examples in which the equality does not hold.

2.1 Amitsur Theorem and Lie identities for FG^+ and FG^-

Let A be an F-algebra with involution $*$. A question of general interest is which properties of A^+ or A^- can be lifted to A. One of the most celebrated results in this direction is the following theorem due to Amitsur [1] dealing with algebras satisfying a $*$-polynomial identity. We recall that an F-algebra A with involution $*$ satisfies a $*$-polynomial identity if there exists a non-zero polynomial $f(x_1, x_1^*, \ldots, x_t, x_t^*)$ in $F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$, the *free associative algebra with involution* on the countable set of variables $\{x_1, x_2, \ldots\}$, such that $f(r_1, r_1^*, \ldots, r_t, r_t^*) = 0$ for all $r_i \in A$.

Theorem 21. *Let F be a field and A an F-algebra with involution (with or without an identity). If A satisfies a $*$-polynomial identity, then A satisfies a polynomial identity.*

Obviously if A^+ or A^- satisfy a polynomial identity, then A satisfies a $*$-polynomial identity and, by the above theorem, it is PI.

Since the second half of the 90s there have a been a number of papers devoted to investigate the extent to which the Lie identities satisfied by the symmetric and the skew-symmetric elements of a group algebra FG with respect to the classical involution determine the Lie identities satisfied by the whole group ring. Work on Lie nilpotence was begun by Giambruno and Sehgal in [16] with the following

Theorem 22. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ or FG^- are Lie nilpotent if and only if FG is Lie nilpotent.*

It is easy enough to see that the above result does not hold if G has 2-elements. Indeed, as observed in Section 1.3, if G is a Hamiltonian 2-group, then the symmetric elements of FG commute. But Theorem 13 tells us that FG is not Lie nilpotent. Moreover, if D_8 denotes the dihedral group of order 8, for any field F of odd characteristic FD_8^- is commutative, but again FD_8 is not Lie nilpotent. In [27] Lee showed that Theorem 22 can be extended to groups not containing the quaternions, and then classified the groups G containing Q_8 such that FG^+ is Lie nilpotent.

Theorem 23. *Let FG be the group algebra of a group G not containing Q_8 over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ is Lie nilpotent if and only if FG is Lie nilpotent.*

Theorem 24. *Let FG be the group algebra of a group G containing Q_8 over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^+ is Lie nilpotent if and only*

(a) $p = 0$ and $G \cong Q_8 \times E$, where E is an elementary abelian 2-group, or

(b) $p > 2$ and $G \cong Q_8 \times E \times P$, where E is an elementary abelian 2-group and P is a finite p-group.

Work on group algebras of groups containing 2-elements whose Lie subalgebra of skew-symmetric elements is nilpotent is much more complicated and took a rather long time. It was begun by Giambruno and Polcino Milies in [12] and recently completed by Giambruno and Sehgal [17] with the proof of the following

Theorem 25. *Let FG be the group algebra of a group G over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^- is Lie nilpotent if and only*

(a) G has a nilpotent p-abelian subgroup H with $(G \setminus H)^2 = 1$, or

(b) G has an elementary abelian 2-subgroup of index 2, or

(c) the p-elements of G form a finite normal subgroup P and G/P is an elementary abelian 2-group.

The same questions concerning the bounded Lie Engel property were investigated a bit later by Lee [28]. Also in this case for the symmetric elements the answer depends on the fact that G contains Q_8 or not.

Theorem 26. *Let FG be the group algebra of a group G not containing Q_8 over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ is bounded Lie Engel if and only if FG is bounded Lie Engel.*

Theorem 27. *Let FG be the group algebra of a group G containing Q_8 over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^+ is bounded Lie Engel if and only*

(a) $p = 0$ and $G \cong Q_8 \times E$, where E is an elementary abelian 2-group, or

(b) $p > 2$ and $G \cong Q_8 \times E \times P$, where E is an elementary abelian 2-group and P is a p-group of bounded exponent having a p-abelian subgroup of finite index.

Up to now the best known result as when the skew-symmetric elements of a group algebra are Lie n-Engel is again in the same paper by Lee [28]. It deals with groups without elements of even order and is in the same direction as Theorem 22.

Theorem 28. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic different from 2 endowed with the classical involution. Then FG^- is bounded Lie Engel if and only if FG is bounded Lie Engel.*

For any F-algebra with involution A it is easy to see that $[A^+, A^+] \subseteq A^-$. Thus, as A^- is a Lie subalgebra of A, if it is Lie solvable then so is A^+. This simple observation is very useful for the classification of group algebras whose skew and symmetric elements are Lie solvable. The question has been recently investigated by Lee, Sehgal and Spinelli in [31]. It was solved under a restriction upon the orders of the group elements. Their first theorem deals with the characteristic zero case and two different prime characteristic cases.

Theorem 29. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic $p \neq 2$ endowed with the classical involution. Suppose either that $p = 0$ or else $p > 2$ and either*

(a) G has only finitely many p-elements, or

(b) G contains an element of infinite order.

Then the following are equivalent:

(i) FG^+ is Lie solvable;

(ii) FG^- is Lie solvable;

(iii) FG is Lie solvable.

We can assume now that the group G is torsion. No result is known that completely covers the remaining case, but the following theorem, also from [31], gives a partial answer.

Theorem 30. *Let F be a field of characteristic $p > 2$. Let G be a torsion group containing an infinite p-subgroup of bounded exponent, but no non-trivial elements of order dividing $p^2 - 1$. Let FG have the classical involution. Then the following are equivalent:*

(i) FG^+ is Lie solvable;

(ii) FG^- is Lie solvable;

(iii) FG is Lie solvable.

No result is currently known for groups with 2-elements except for what concerns the skew-symmetric elements. In fact, if $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 2$ and G has only finitely many p-elements, Lee, Sehgal and Spinelli (Theorem 1.2 of [31]) classified the groups G containing 2-elements such that FG^- is Lie solvable. They also observed that, in order to remove the condition that G contains an infinite p-subgroup of bounded exponent in Theorem 30, it is sufficient to consider the case in which G has a normal subgroup A which is a direct product of finitely many copies of the quasicyclic p-group, C_{p^∞}, and $G/A = \langle Ag \rangle$, where the order of g is a prime power. This case, however, remains open. Indeed, the restriction can be dropped whenever G does not have C_{p^∞} as a subhomomorphic image.

Of course, for any field F of characteristic different from 2, FQ_8^+ is Lie solvable (being commutative) but, according to Theorem 14, FQ_8 is not. Unfortunately, the usual criterion that G does not contain Q_8 will not be sufficient.

Indeed, as observed after Theorem 22, if G is the dihedral group of order 8, then FG^- is commutative; hence, FG^+ is Lie solvable. However, it seems reasonable to conjecture that if G has no 2-elements, then the conclusions of Theorem 29 hold without any other restriction.

Work on Lie identities for symmetric elements is very useful also in discussing the corresponding group identities for the symmetric units. We do not review the details of this in the present survey, but we confine ourselves to report the principal results showing how, in some sense, polynomial identities satisfied by FG^+ reflect group identities satisfied by $\mathcal{U}^+(FG)$ and the latter ones can be lifted to the whole unit group of FG. For the following result see [34].

Theorem 31. *Let F be an infinite field of characteristic $p > 2$. Let G be a group containing an infinite p-subgroup of bounded exponent, but no non-trivial elements of order dividing $p^2 - 1$. Let FG have the classical involution. Then the following are equivalent:*

(i) $\mathcal{U}^+(FG)$ *is solvable;*

(ii) $\mathcal{U}(FG)$ *is solvable;*

(iii) FG^+ *is Lie solvable;*

(iv) FG *is Lie solvable.*

If one replaces the hypothesis that G contains an infinite p-subgroup of bounded exponent with the weaker assumption that G contains infinitely many p-elements, Lee and Spinelli (Theorem 4 of [34]) proved that (i), (ii) and (iv) are equivalent. The case in which G contains finitely many p-elements was completely solved again in [34], but for the details we refer the reader to the original paper.

Along this line, Lee and Spinelli [35] determined the conditions under which the subgroup generated by $\mathcal{U}^+(FG)$, $\langle\mathcal{U}^+(FG)\rangle$, is bounded Engel, when G is torsion and F infinite.

Theorem 32. *Let FG be the group algebra of a torsion group G not containing Q_8 over an infinite field F of characteristic different from 2 endowed with the classical involution. Then the following are equivalent:*

(i) $\langle\mathcal{U}^+(FG)\rangle$ *is bounded Engel;*

(ii) $\mathcal{U}(FG)$ *is bounded Engel;*

(iii) FG^+ *is bounded Lie Engel;*

(iv) FG *is bounded Lie Engel.*

Theorem 33. *Let FG be the group algebra of a torsion group G containing Q_8 over an infinite field F of characteristic different from 2 endowed with the classical involution. Then $\langle \mathcal{U}^+(FG)\rangle$ is bounded Engel if and only if FG^+ is bounded Lie Engel.*

The result by Jespers and Ruiz Marin (Theorem 11) on group algebras FG endowed with F-linear extensions of arbitrary group involutions whose symmetric elements commute is fundamental for the investigation of more general properties of FG^+. The first results of this type were obtained by Giambruno, Polcino Milies and Sehgal [13] for groups without 2-elements. They confirm that Theorem 22 and Theorem 26 can be extended to this general setting.

Theorem 34. *Let F be a field of characteristic different from 2, G a group without 2-elements with an involution $*$ and let FG have the induced involution. Then FG^+ is Lie nilpotent (bounded Lie Engel, respectively) if and only if FG is Lie nilpotent (bounded Lie Engel, respectively).*

Obviously we cannot expect that the result is true for an arbitrary group G. According to the discussion after Theorem 10, the answer will depend on the presence of SLC-groups in G. A complete answer has been given by Lee, Sehgal and Spinelli [32] with the proof of the following

Theorem 35. *Let F be a field of characteristic $p > 2$, G a group with an involution $*$ and let FG have the induced involution. Suppose that FG is not Lie nilpotent. Then FG^+ is Lie nilpotent if and only if G is nilpotent, and G has a finite normal $*$-invariant p-subgroup N such that G/N is an SLC-group.*

Theorem 36. *Let F be a field of characteristic $p > 2$, G a group with an involution $*$ and let FG have the induced involution. Suppose that FG is not bounded Lie Engel. Then FG^+ is bounded Lie Engel if and only if G is nilpotent, G has a p-abelian $*$-invariant normal subgroup A of finite index, and G has a normal $*$-invariant p-subgroup N of bounded exponent, such that G/N is an SLC-group.*

As when FG is endowed with the classical involution, the link between Lie identities satisfied by FG^+ and group identities satisfied by $\mathcal{U}^+(FG)$ appears strong. In confirmation of this, by using Theorem 12, Lee, Sehgal and Spinelli [33] found necessary and sufficient conditions so that $\mathcal{U}^+(FG)$ is nilpotent by proving the following

Theorem 37. *Let F be an infinite field of characteristic different from 2, G a torsion group with an involution $*$ and let FG have the induced involution. Then $\mathcal{U}^+(FG)$ is nilpotent if and only if FG^+ is Lie nilpotent.*

We recall that Theorem 37 was originally proved for group algebras over arbitrary fields (non-necessarily infinite) endowed with the classical involution by Lee [29]. The same question was investigated for non-necessarily torsion groups by Lee, Polcino Milies and Sehgal in [30].

Finally, work on group algebras whose skew-symmetric elements satisfy a certain Lie identity is rather complicated. Recently Giambruno, Polcino Milies and Sehgal [15] have classified the torsion groups G with no 2-elements for which FG^- is Lie nilpotent. It turns out that the conclusion is much more involved than for the classical involution (Theorem 22). Their main result is the following.

Theorem 38. *Let F be a field of characteristic $p \neq 2$ and G a torsion group with no elements of order 2. Let $*$ be an involution on FG induced by an involution of G. Then the Lie algebra FG^- is nilpotent if and only if FG is Lie nilpotent or $p > 2$ and the following conditions hold:*

(1) the set P of p-elements in G is a subgroup;

(2) $*$ is trivial on G/P;

(3) there exist normal $*$-invariant subgroups A and B, $1 \leq B \leq A$ such that B is a finite central p-subgroup of G, A/B is central in G/B with both G/A and $\{a \mid a \in A \quad aa^* \in B\}$ finite.

Acknowledgements. The author wants to thank the organizers of the conference *Advances in Group Theory 2009* for the invitation and E. Spinelli for his help with the preparation of these notes. The research was supported by NSERC Canada.

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Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) suppl. n. 1, 93–99. doi:10.1285/i15900932v30sn1p93

Quasinormal subgroups of finite p-groups

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Abstract. The distribution of quasinormal subgroups within a group is not particularly well understood. Maximal ones are clearly normal, but little is known about minimal ones or about maximal chains. The study of these subgroups in finite groups quickly reduces to p-groups. Also within an abelian quasinormal subgroup, others (quasinormal in the whole group) abound. But in non-abelian quasinormal subgroups, the existence of others can be dramatically rare.

Keywords: Finite p-groups, quasinormal subgroups.

MSC 2000 classification: 20E07, 20E15

1 Introduction and development of the theory

A subgroup Q of a group G is said to be *quasinormal* in G if $QH = HQ$ (the subgroup generated by Q and H) for every subgroup H of G. In this situation we write Q *qn* G. (The term *permutable* has also been used on occasions; but then the natural implication, when referring to two permutable subgroups, is that they simply permute with each other under multiplication.) Clearly normal subgroups are quasinormal, but not conversely in general. However, Ore [11] proved that *quasinormal subgroups of finite groups are subnormal*; and in separate unpublished work, Napolitani and Stonehewer showed that *quasinormal subgroups of infinite groups are ascendant in at most $\omega + 1$ steps.* For most of what follows, however, we shall restrict ourselves to finite groups.

The simplest examples of non-normal quasinormal subgroups are to be found in the non-abelian groups of order p^3 and exponent p^2 (for an odd prime p); and there they are of course abelian of order p. Indeed Itô and Szép [7] proved that *a quasinormal subgroup of a finite group is always nilpotent modulo its normal core.* In 1967 and 1968, respectively, Thompson [16] and Nakamura [9] gave core-free examples of nilpotency class 2. Then in 1971 and 1973, Bradway, Gross and Scott [2] and Stonehewer [13] showed that *any nilpotency class is possible.* Following this (in [14]), examples of core-free quasinormal subgroups were constructed *with derived length d, for any positive integer d.* But by this time a significant improvement on the Itô - Szép result had been established by Maier and Schmid [8], viz.

if Q is a quasinormal subgroup of a finite group G, with normal core Q_G, then Q/Q_G lies in the hypercentre of G/Q_G.
It follows easily from this result that the problems associated with quasinormal subgroups of finite groups reduce quickly to p-groups. For, if Q is quasinormal in G and $Q_G = 1$, then it is easy to show that each Sylow p-subgroup P of Q is also quasinormal in G. (See [12], Lemma 6.2.16.) Also, by [8], each p'-element of G commutes with P elementwise. Therefore if S is a Sylow p-subgroup of G, then

P is quasinormal in S

and the complexities of Q's embedding in G reduce to those of P's embedding in S.

It is worth pointing out that there is a very good reason for studying quasinormal subgroups in finite p-groups, apart from their curiosity value. This relates to *modular* subgroups. Recall that a subgroup M of a group G is *modular* if, for each subgroup U of G, the map $H \mapsto U \cap H$ is a lattice isomorphism from $[\langle U, M\rangle/M]$ to $[U/(U \cap M)]$. Since modular subgroups are invariant under lattice isomorphisms, and since the quasinormal and modular subgroups of finite p-groups coincide ([12], Theorem 5.1.1), it follows that in finite p-groups

quasinormal subgroups are invariant under lattice isomorphisms.

Of course normal subgroups do *not* satisfy this property.

2 Abelian quasinormal subgroups

In trying to understand more about quasinormal subgroups, it is surely natural to begin with the abelian ones, even the cyclic ones. Indeed if Q is a cyclic quasinormal subgroup of a group G, then

every subgroup of Q is also quasinormal in G.

(See [12], Lemma 5.2.11.) This result is true for all groups G, not just the finite ones. Following a conjecture by Busetto and Napolitani, much more was discovered about the cyclic case by Cossey and Stonehewer in [3]:-

If Q is a cyclic quasinormal subgroup of odd order in a finite group G, then $[Q, G]$ is abelian and Q acts by conjugation on $[Q, G]$ as power automorphisms. Thus the normal closure Q^G is abelian-by-cyclic.
A key situation in establishing this result showed that each element of $[Q, G]$ has the form $[q, g]$, with $q \in Q$ and $g \in G$. Note that Q does not have to be core-free here. The case when Q has even order is considerably more complicated and is

dealt with in [4] and [5]. There are examples where $[Q, G]$ is *not* abelian, but it is always nilpotent of class at most 2.

In moving from cyclic to abelian quasinormal subgroups Q, it is clear that not all subgroups of Q will be quasinormal. But there are surprisingly many of them:-

Let Q be an abelian quasinormal subgroup of G (finite or infinite). Then Q^n is quasinormal in G, provided n is odd or divisible by 4.

(See [15].) Again it is not necessary for Q to be core-free here. But there are examples with G finite where Q^2 is not quasinormal. However, now that chains of quasinormal subgroups are beginning to appear, it is natural to ask if, given Q *qn* G, a finite p-group, there are maximal chains of quasinormal subgroups of G, passing through Q, that are composition series of G. We may even ask if *all* maximal chains of quasinormal subgroups in a finite p-group have to be composition series. We shall see below that the answers here are "no". But for abelian Q, a lot can be said in a positive direction.

Recall that Q *qn* G if and only if $QX = XQ$ for all cyclic subgroups X of G. Thus a significant partial stage, on the way to understanding more about quasinormal subgroups, is to be able to make statements about quasinormal subgroups Q of groups G of the form

$$G = QX, \tag{1}$$

where X is cyclic. Nakamura showed in [10] that in this situation when G is a finite p-group, Q always contains a quasinormal subgroup of G of order p. For the moment, we shall assume that (1) holds. Moreover we shall assume that

$$G \text{ is a finite } p\text{-group and } Q_G = 1.$$

Then clearly X contains a non-trivial normal subgroup of G and so

$$\Omega_1(X) \leqslant Z(G).$$

Here $Z(G)$ is the centre of G and $\Omega_1(G)$ denotes the subgroup generated by the elements of order p. More generally $\Omega_i(G)$ will denote the subgroup generated by the elements of order at most p^i. Then the following result, due to Cossey and Stonehewer, will appear in the Journal of Algebra in the volume dedicated to the memory of Karl Gruenberg:-

Theorem 1. *Let Q qn $G = QX$, with G a finite p-group, $Q_G = 1$, Q abelian and X cyclic. Then*

(a) *$W_i = \Omega_i(Q)$ qn G, for all $i \geqslant 1$;*

(b) *there exists a composition series of G, passing through the W_i's, in which every subgroup is quasinormal in G; and*

(c) *if p is odd, then there is a composition series of G, passing through the p^i-th powers of Q, in which every subgroup is quasinormal in G.*

Removing the hypothesis (1) is not easy. All that we can say to date is the following (see [6]):-

Theorem 2. *If Q is a quasinormal subgroup of order p^2 in a finite p-group G (with p an odd prime), then there is a quasinormal subgroup of G of order p lying in Q.*

Unfortunately there is nothing canonical about this subgroup of order p, and its existence was established only by an exhaustive survey of all possibilities. Thus for our final section we shall revert to the hypothesis (1).

3 Non-abelian quasinormal subgroups

There is a universal embedding theorem for the situation (1), due to Berger and Gross [1]:-

Given a prime p and an integer $n \geqslant 1$, there exists a finite p-group

$$G = QX$$

such that

(i) *Q qn G, $Q_G = 1$ (so $Q \cap X = 1$) and $X = \langle x \rangle$ is cyclic of order p^n;*

(ii) *if Q^* qn $G^* = Q^*X^*$, a finite p-group, with $Q^*_{G^*} = 1$ and $X^* = \langle x^* \rangle$ is cyclic of order p^n, then G^* embeds in G uniquely such that Q^* embeds in Q and x^* maps to x.*

The group G has exponent p^n and Q has exponent p^{n-1}. Moreover

$$\Omega_1(G) = \Omega_1(Q) \times \Omega_1(X)$$

is elementary abelian and an indecomposable X-module. Let $G_n = G$. Then $G_n/\Omega_1(G_n) \cong G_{n-1}$. Also $\Omega_i(G_n)$ has exponent p^i. Berger and Gross define G_n as a permutation group on the integers modulo p^n and Q is the stabiliser of $\{0\}$.

Assume (for simplicity) that p is odd. In Canberra in 2007, Cossey, Stonehewer and Zacher began to study these groups G_n for small values of n. In Warwick in 2009, Cossey and Stonehewer have continued this work for arbitrary n and a succession of results has been obtained, giving a fairly complete picture, with a somewhat surprising conclusion.

The first non-trivial case is when $n = 2$. Here Q is elementary abelian of rank $p - 2$. Also $\Omega_1(G) = Q \times \Omega_1(X)$ is a uniserial X-module. Thus there is a unique chief series of G between $\Omega_1(X)$ and $\Omega_1(G)$ and *the intersections of its terms with Q are precisely the quasinormal subgroups of G lying in Q.* So they form part of a composition series of G passing through Q. This is a special case of Theorem 1 above.

Now suppose that $n = 3$. Here $\Omega_1(G) = \Omega_1(Q) \times \Omega_1(X)$ is an indecomposable X-module of rank $p(p-1) = r+1$, say. Again *the quasinormal subgroups of* G*, of exponent* p *and lying in* Q*, are precisely the non-trivial intersections with* Q *of the terms of the unique chief series of* G *between* $\Omega_1(G)$ *and* $\Omega_1(X)$. Denote these intersections by

$$\Omega_1(Q) = W_r > W_{r-1} > \cdot \cdot \cdot > W_1 \ (> W_0 = 1).$$

Modulo $\Omega_1(G)$, $\Omega_2(G)$ is an indecomposable X-module of rank $p-1$. Let $X_2 = \Omega_2(X)$ and let Q_1 be a quasinormal subgroup of G, of exponent p^2, lying in Q. Then

(i) $\Omega_1(Q_1) = W_i$*, some* $i \geqslant r - p$*; and*

(ii) Q_1X_2 *modulo* $\Omega_1(G)$ *is an* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$.

Conversely, for any i, j *with* $r \geqslant i \geqslant r-p$ *and* $p-1 \geqslant j \geqslant 2$*, there is a quasinormal subgroup* Q_1 *of* G*, of exponent* p^2 *lying in* Q*, with* $\Omega_1(Q_1) = W_i$ *and* Q_1X_2 *modulo* $\Omega_1(G)$ *the* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$ *of rank* j.

It follows that there are maximal chains of quasinormal subgroups of G, passing through Q, which are composition series of G.

Next we consider $G = G_n$ *for* $n = 4$. Here $\Omega_1(G)$ has rank $p^2(p-1) = s+1$, say. Again *the quasinormal subgroups of* G*, of exponent* p *lying in* Q*, are the non-trivial intersections with* Q *of the terms of the unique chief series of* G *between* $\Omega_1(G)$ *and* $\Omega_1(X)$*, but only those of rank* $\leqslant p^2-1$. If Q_1 is a quasinormal subgroup of G, of exponent p^2 lying in Q, then $\Omega_1(Q_1)X_1$ *is an* X*-submodule of* $\Omega_1(G)$ *of rank* $i+1$*, with* $i \geqslant s - p(p-1)$*; and* Q_1X_2 *modulo* $\Omega_1(G)$ *is an* X*-submodule of* $\Omega_2(G)/\Omega_1(G)$ *of rank* $j+1$*, with* $j \geqslant p-2$*. Moreover there are quasinormal subgroups* Q_1 *of this form for each of the above values of* i *and* j. But now we see that we have a ‘gap’ in a maximal chain of quasinormal subgroups passing through Q. Indeed the largest quasinormal subgroup of exponent p has rank p^2-1, and the smallest of exponent p^2 is elementary abelian of rank $p(p-1)^2-1$ extended by an elementary abelian group of rank $p-2$. The former is contained in the latter and has index

$$p^{p^2(p-3)+2(p-1)}.$$

So *there is no composition series of* G (=G_4) *passing through* Q *and consisting of quasinormal subgroups of* G.

How big can this gap be in general? If H is a quasinormal subgroup of G (a finite p-group) and K is a quasinormal subgroup of G maximal subject to being properly contained in H, then clearly $K \lhd H$. But is H/K restricted in some way? The answer is 'no'! Indeed G_5 has no quasinormal subgroups of exponent p^2 lying in Q. In fact G_n, for $n \geqslant 5$, has no quasinormal subgroups of exponent p^2 lying in Q. Since $G_5 \cong G_6/\Omega_1(G_6)$, G_6 has no quasinormal subgroups of exponent p^3 lying in Q; and so on. The situation is as follows:-

Theorem 3. *The only non-trivial quasinormal subgroups of G_n $(n \geqslant 2)$, lying in Q, have exponent*

$$p,\ p^{n-2} \text{ and } p^{n-1}.$$

Thus there is a 'black hole' between exponent p and exponent p^{n-2}. To sum up, let Q_1 be a quasinormal subgroup of G_n lying in Q and let $X_i = \Omega_i(X)$, all i. Then for each i with $p^i \leqslant$ exponent of Q_1,

$\Omega_i(Q_1)X_i$ modulo $\Omega_{i-1}(G)$ is a submodule of the indecomposable X-module $\Omega_i(G)/\Omega_{i-1}(G)$.

If the rank of this submodule is r_i, then r_i is restricted to a known range of values. Moreover, there is a quasinormal subgroup Q_1 for each choice of values of r_i within each range. The somewhat lengthy proofs will appear elsewhere.

Acknowledgements. The author is grateful to the Australian National University for financial support during the initial stages of the work leading to the new results described here.

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ISSN: 1123-2536 (print version), **e-ISSN** 1590-0932 (electronic version).

Note di Matematica

Volume 30, suppl. number 1, 2010