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New Policies for the Dynamic Traveling Salesman Problem

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The *Dynamic Traveling Salesman Problem* is a variant of the classical *Traveling Salesman Problem* in which service requests arrive in a random fashion. In this paper we propose three new routing policies minimizing the expected total waiting time of all the demands in the system. In particular, we show that two policies are asymptotically optimal in heavy traffic while the third one has a constant factor guarantee of two. We also perform extensive simulations in order to compare the three policies under various arrival rates.

Keywords: Stochastic and Dynamic Vehicle Routing; Dynamic Fleet Management.

1 Introduction

The purpose of this paper is to describe and assess three new policies for the *Dynamic Traveling Salesman Problem* ($DTSP$). The $DTSP$ is a generalization of the classic *Traveling Salesman Problem* (TSP) in which customer requests arrive in an on-going fashion. Consequently, the salesman has to replan his route as soon as a new service request arrives. The objective is to minimize

the expected total waiting time, i.e. the sum of the expected waiting times of all the customer demands.

The DTSP falls under the broad category of real-time fleet management. A large part of the current literature is characterized by algorithms reacting to new requests only once they have occurred, while neglecting available stochastic information. Overviews of these problems can be found in Powell, Jaillet and Odoni [23], Psaraftis ([25], [26]), Gendreau and Potvin [12], and Ghiani et al [13]. Another line of research, which is particularly relevant to this paper, examines dispatching and routing policies whose performance can be determined analytically if specific assumptions are satisfied. See, e.g., Bertsimas and van Ryzin ([6], [7]), where demands are distributed in a bounded area in the plane and arrival times are modeled as a Poisson process. The authors identify optimal policies both in light and heavy traffic cases. Papastavrou [21] describes a routing policy that performs well both in light and heavy traffic, while Swihart and Papastavrou [28] examine a dynamic pickup and delivery extension. A related area of research, often referred to as Stochastic Vehicle Routing, examines problems in which demand becomes known at the beginning of each day. In this context the problem is to determine, on the basis of a probabilistic characterization of random data, a solution of least expected cost in which the order of customers is fixed regardless of the demand realization for a particular day (an a priori solution). Jaillet [18] introduced the Probablisitic Traveling Salesman Problem, Jaillet and Odoni [19] examined the capacitated case, while Bertsimas [5] introduced the multi-vehicle stochastic vehicle routing problem. A survey of the research in this area can be found in Powell, Jaillet and Odoni [23], Bertsimas and Simchi-Levi [8], and Gendreau, Laporte and Sguin [11]. Finally, we note the existence of a relatively recent line of research, known as anticipatory routing, in which general probability distributions are used in order to devise exact or heuristic policies. To our knowledge, four papers take this approach. Powell et al. [22] introduce a truckload dispatching problem, and Powell [24] provides formulations, solution methods, as well as numerical results. In these papers, future demand forecasts are used to determine which loads should be assigned to the vehicles in a truckload environment to account for forecasted capacity needs in the next period. In Thomas and White [29] a vehicle may serve several requests at a time and may wait for future demand both at a customer and non-customer locations. Not all requests have to be serviced and the objective function to be minimized is the expected value of a combination of travel costs, terminal costs, and revenue generated from a pickup. Bent and Van Hentenryck [4] consider a vehicle routing problem where customer locations and service times are random variables which are realized dynamically during plan execution. They develop a multiple scenario approach which continuously generates plans consistent with past decisions and anticipating future requests. Moreover, there

is another area of research related to online optimization and competitive analysis [2, 10, 14, 17, 20]. More specifically, some related problems have been solved by the so-called IGNORE-strategy [1, 15–17], which was proved to be 2.5-competitive for minimizing the makespan for the dial-a-ride problem in graphs [1] and to have performance bounds to minimizing the total waiting time in [15]. Moreover, it is proved in [16] that this strategy is asymptotically optimal for minimizing the total waiting time on trees in case of high load.
This paper exploits some of the results exposed in the second stream of research and extends them to define new routing policies for the *DTSP*. The sequel of the paper is organized as follows. In Section 2 we propose three new spatially unbiased policies and compare their performance in terms of lower bounds on the optimal solution value. Our computational experience is summarized in Section 4, followed by some concluding remarks in Section 5.

2 New Policies for the DTSP

We investigate the *Dynamic Traveling Salesman Problem* under the hypothesis that the requests occur in a plane. Moreover, we assume that service requests arrive according to a homogeneous Poisson process, and their locations are independently and uniformly distributed in a bounded region $\mathbf{A}$ having an area Σ. For the sake of simplicity the region is assumed to be a square whose median corresponds to a depot where the vehicle is based. Demands are served by a single vehicle that travels at a constant speed v. We also suppose that the vehicle operates in *heavy traffic* condition, i.e. the arrival rate $\lambda \to \infty$. We assume that there are no constraints on the vehicle's capacity and that the service time spent by the vehicle at each location is negligible. Our objective is to minimize the sum of the expected waiting times of all the demands in the system. Let $d_{i,i+1}$ be the time required by the vehicle to move from the i-th demand's location to the $(i+1)$-th demand's location. Moreover, we denote by D_i the service time of the i-th request. We also denote by t_i the arrival time of demand i, by $T_i = D_i - t_i$ the i-th demand's waiting time and with $T \equiv \lim_{i\to\infty} E[T_i]$ its steady state expected value.
In this section, we propose three spatially unbiased policies for the *DTSP*. A common feature of the first two policies is the partitioning of the service region $\mathbf{A}$ into a fixed number of subregions, whereas the third policy uses a simple but efficient insertion method. Moreover, in the light traffic condition, i.e. whenever no customer is waiting in the system, the idle vehicle is repositioned in the stochastic median of the service region and waits until a new demand arrives. More specifically, this last strategy is substantially the stochastic queue median policy described in [6] which was proved to be optimal in the light traffic condition.

2.1 *The Total Deferment Policy*

This policy represents an adaptation of the unbiased policy introduced in [6]. The service region $\mathbf{A}$ is partitioned into $k \in \mathbf{Z}_+$ subregions $\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_k$ of equal size, each having an area A such that:

$$Pr(\delta \in \mathbf{A}_i) = \frac{1}{k} \qquad (i = 1, .., k)\,,$$

where δ represents a generic demand to be serviced. The vehicle follows an a priori tour among zones arbitrarily designed. Within each subregion, the demands already known at the moment the vehicle enters in the subregion are serviced along an Hamiltonian path. As soon as the vehicle terminates the current path, it goes to the next subregion in the a priori tour. The process is then repeated until no demands are left. While the vehicle is moving along the current tour, new service requests may arrive. The locations of these requests may be in the current subregion (the subregion in which the vehicle is currently operating) or in another subregion. In the latter case these demands will be served as soon as the vehicle arrives into their subregion. In the former case the salesman has to decide whether to serve the new demands in the current tour or to postpone their service to the next tour. The *Total Deferment* policy (TD) implements the second strategy, i.e. it queues the new demands and services them in the next tour. Such a policy has the theoretical property of being asymptotically optimal as shown by the following theorem.

THEOREM 2.1 *For the DTSP with a spatial uniform demand distribution, the expected total system time T_{TD} under the TD policy is bounded by:*

$$\eta \frac{\beta^2}{2} \frac{\lambda \Sigma}{v^2} \;\leq\; T_{TD} \;\leq \eta \left(1 + \frac{1}{k}\right) \frac{\beta^2 \lambda \Sigma}{2v^2}\,,$$

where η is the expected total number of demands in the system waiting for service and β is the Euclidean TSP constant, which is estimated in [3] to be $\beta \approx 0.72$.

Proof The lower bound was derived in [6] and was proved for an arbitrary spatially unbiased policy. For the proof of the upper bound, let $b(t)$ be a random variable representing the number of demands for service left in subregion $\mathbf{A}_s$ $(1 \leq s \leq k)$ when the vehicle crosses over to $\mathbf{A}_{s+1}$. Denote by b the steady state expectation value of $b(t)$. Between two visits to $\mathbf{A}_{s+1}$ the vehicle has visited k subregions, therefore the average number of demands the vehicle finds in $\mathbf{A}_{s+1}$ is $\nu = kb$. In the same way there will be $(k-1)b$ demands in $\mathbf{A}_{s+2}$ and so on. As a result, the steady state average number of demands waiting

for service in the region **A** will be given by

$$\eta = b + 2b + 3b + \ldots + kb = \frac{k(k+1)b}{2}.$$

Therefore,

$$b = \frac{2\eta}{k(k+1)}$$

and the steady state average number of demands ν the vehicle finds in a subregion verifies:

$$\nu = kb = \frac{2\eta}{k+1}$$

Let h_1 and h_2 be two moments during which the vehicle performs two consecutive crossovers from $\mathbf{A}_s$ to $\mathbf{A}_{s+1}$. During the elapsed time $h_2 - h_1$, the vehicle should have performed k *TSP* paths (one for each subregion) and k travel distances lower than or equal to $\sqrt{5\Sigma/k}$ to go from a subregion to another. (Such a quantity becomes $((\sqrt{k}+1/\sqrt{k})\sqrt{5\Sigma}$ for odd values of k. In the sequel of this paper we consider, for simplicity, only even values of k.) Thus,

$$h_2 - h_1 \;\leq\; k\left(\frac{L_\nu}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right).$$

where L_ν is the length of an optimal TSP path through the ν points.
Let δ be a random generic demand in $\mathbf{A}_{s+1}$. The average waiting time for δ at time h_2 is $(h_2 - h_1)/2$, since δ is equally likely to arrive at any moment during $h_2 - h_1$. Moreover, it has to wait another period of time equal to $L_\nu/2v + \sqrt{5\Sigma/k}/2v$ in order to achieve the service of the ν demands in $\mathbf{A}_{s+1}$, since it is equally likely to be serviced at any order among the ν demands. Therefore, the steady state expected system time T_δ of demand δ is given by:

$$T_\delta \leq \frac{h_2 - h_1}{2} + \frac{1}{2}\left(\frac{L_\nu}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right) \;\leq\; \frac{h_2 - h_1}{2} + \frac{1}{2}\left(\frac{\tilde{L}_\nu}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right)$$

$$\leq\; \frac{k+1}{2}\left(\frac{\tilde{L}_\nu}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right) \;\approx\; \frac{k+1}{2}\left(\frac{\beta}{v}\sqrt{A\nu} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right)$$

$$= \frac{k+1}{2}\left(\frac{\beta}{v}\sqrt{\frac{\Sigma\,\nu}{k}} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right).$$

This expression has been derived by using a well known formula to estimate $\tilde{L}_\nu$, the length of the optimal *TSP* tour through the ν points and that verifies $L_\nu \leq \tilde{L}_\nu$. This formula is due to Beardwood *et al.* [3] who showed that, considering ν points independently and uniformly distributed in a bounded region $\mathbf{A}$ of area A, then:

$$\lim_{\nu\to\infty} \frac{\tilde{L}_\nu}{\sqrt{\nu}} = \beta\sqrt{A} \quad a.s.\,, \tag{1}$$

where $\beta \approx 0.7211$, and $'a.s.'$ means "almost surely" or "with probability one". Indeed, $\nu \to \infty$ as $\lambda \to \infty$ because:

$$\nu = \frac{2\,\eta}{k+1}.$$

In addition, by using Little's law for which $\eta = \lambda\, T_\delta$ and the lower bound defined in [6] for which

$$\frac{\beta^2}{2}\frac{\lambda\Sigma}{v^2} \leq T_\delta,$$

we have:

$$\nu = \frac{2\,\lambda\, T_\delta}{k+1} \geq \frac{\beta^2\lambda^2\Sigma}{(k+1)\,v^2}\,.$$

Thus, ν must be large as $\lambda \to \infty$ and consequently:

$$T_\delta \leq \frac{k+1}{2}\left\{\frac{\beta}{v}\sqrt{\frac{\Sigma}{k}\left(\frac{2\eta}{k+1}\right)} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right\},$$

or equivalently:

$$T_\delta \leq \frac{k+1}{2}\left\{\frac{\beta}{v\sqrt{k}}\sqrt{\frac{2\Sigma\,\lambda\,T_\delta}{(k+1)}} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right\}.$$

By solving for T_δ we get:

$$\sqrt{T_\delta} \leq \frac{\sqrt{2(k+1)\beta^2\lambda\Sigma} + \sqrt{2(k+1)\beta^2\lambda\Sigma + 8\sqrt{k}(k+1)v\sqrt{5\Sigma}}}{4v\sqrt{k}},$$

that is:

$$\sqrt{T_\delta} \leq \frac{\sqrt{2(k+1)\beta^2\lambda\Sigma + 8\sqrt{k}(k+1)v\sqrt{5\Sigma}}}{2v\sqrt{k}},$$

from which we get the result:

$$T_{TD} = \eta T_\delta \leq \eta\left(1 + \frac{1}{k}\right)\frac{\beta^2\lambda\Sigma}{2v^2}.$$

□

This result implies that, by choosing a large value of k, i.e. for $k \to \infty$, the Total Deferment policy results to be asymptotically optimal.

2.2 *The Partial Deferment Policy*

The *Partial Deferment* policy (*PD*) is a variant of the *Total Deferment* policy. The key difference between the two policies is the way in which the new demands are handled. When a new demand δ arrives in the same subregion in which the vehicle is currently operating, the new policy evaluates whether it is more convenient to serve it immediately or to allocate it to the subsequent tour. In case this request's service is delayed, the detour cost is due to the demand's expected waiting time before being served. In case of service during the current visit of the subregion, the detour cost includes the waiting time of the expected demands in the queue during the detour. Mathematically this can be expressed in terms of maximizing the insertion benefit defined as the difference between the profit and the cost produced by the detour. The profit is the saving that occurs from the service of the new demand δ immediately in the current path after, say, demand h. Let denote by n the random variable representing the number of demands to be served after δ in the tour in which it is inserted, then we have:

$$(L_{\eta+1} - d_{h,\delta})/v\,,$$

where η represents the steady state expectation of n and $L_{\eta+1}$ the length of the optimal *TSP* path through the η points plus δ. The cost of the detour is the

cost that incurs when the vehicle accepts to serve the new demand δ in the current tour causing a delay on the other demands in the queue. It is expressed as:

$$\frac{\eta}{v}\left(d_{h,\delta}+d_{\delta,h+1}-d_{h,h+1}\right).$$

The benefit is, thus, maximized as:

$$\max_{j\geq i}\left\{\left(\frac{L_{\eta+1}-d_{j,\delta}}{v}\right)-\frac{\eta}{v}\left(d_{j,\delta}+d_{\delta,j+1}-d_{j,j+1}\right)\right\},$$

which could be, without loss of generality since $L_{\eta+1}$ does not depend on the index j, written as:

$$\max_{j\geq i}\left\{\left(\frac{L_{\eta}-d_{j,\delta}}{v}\right)-\frac{\eta}{v},\left(d_{j,\delta}+d_{\delta,j+1}-d_{j,j+1}\right)\right\}$$

where i is the demand that the vehicle is at the point of serving along its current path when δ arrives, and j is a demand already scheduled in the current *TSP* path after i. In order to express such a benefit we use again formula (1) that can be written, for η points independently and uniformly distributed in a bounded region $\mathbf{A}$ of area Σ, as:

$$\lim_{\eta\to\infty}\frac{\tilde{L}_{\eta}}{\sqrt{\eta}}=\beta\sqrt{\Sigma}\quad a.s.. \tag{2}$$

Moreover because of the Little's law:

$$\eta=\lambda\tilde{L}_{\eta}/v\,. \tag{3}$$

and observing that if δ is served immediately in the current tour after, say, demand h, then:

$$L_{\eta}=\tilde{L}_{\eta}-d_{\delta,h+1}\,.$$

Therefore,without loss of generality, we can write

$$\max_{j\geq i}\left\{\left(\frac{L_{\eta}-d_{j,\delta}}{v}\right)-\frac{\eta}{v},\left(d_{j,\delta}+d_{\delta,j+1}-d_{j,j+1}\right)\right\}$$

or equivalently

$$\max_{j \geq i} \left\{ \left(\frac{\tilde{L}_\eta - d_{j,\delta}}{v} \right) - \frac{\eta}{v}, (d_{j,\delta} + d_{\delta,j+1} - d_{j,j+1}) \right\},$$

and, thus, the detour benefit becomes:

$$\max_{j \geq i} \left\{ (\lambda \beta^2 \Sigma / v^2 - \frac{d_{j,\delta}}{v}) - \lambda^2 \beta^2 \Sigma / v^2 \, (d_{j,\delta} + d_{\delta,j+1} - d_{j,j+1}) \right\}$$

for large values of η.
By using an estimation of the benefit we can define the best allocation of the new demand within the current subregion's *TSP* path. If the new demand results to be the last demand to be served in the current subregion *TSP* path, the policy will made a comparison between the resulting expected total system time to decide whether the demand is accepted in the current tour or referred to the next a priori designed tour.
Now we will discuss the optimality of such a policy. To this purpose we need to introduce first the following result.

LEMMA 2.2 *Let η be the total number of demands for service whose locations are independently and uniformly distributed in a square $\boldsymbol{A}$. Suppose that π ($\pi < \eta$) demands have been already served and denote by $n = (\eta - \pi)$ the remaining demands for service. The distribution among the n demands is not uniform anymore. Let $\tilde{\Lambda}_n$ be the length of the optimal TSP tour through these n points and $\tilde{L}_n$ be the length of the optimal TSP tour through the same points if they were uniformly distributed, then:*

$$\tilde{\Lambda}_n \leq \tilde{L}_n \, .$$

Proof Steele showed in [27] that if n points, having the same distribution $f(x)$, are independently but non-uniformly distributed in a square $\mathbf{A}$, then:

$$\lim_{n \to \infty} \frac{\tilde{\Lambda}_n}{\sqrt{n}} = \beta \int_{\mathbf{A}} \bar{f}(x)^{1/2} dx \quad a.s.$$

where $\bar{f}(x)$ is the density of the absolutely continuous part of the distribution $f(x)$. Moreover, by using Jensen's inequality [9] for concave functions, it follows

that:

$$\int_{\mathbf{A}} \bar{f}(x)^{1/2} dx \leq \left(\int_{\mathbf{A}} \bar{f}(x) dx \right)^{1/2} \leq \left(\int_{\mathbf{A}} f(x) dx \right)^{1/2}.$$

Therefore, by combining the formula (2) and Steele's property for $n \to \infty$, we obtain:

$$\tilde{\Lambda}_n \approx \beta\sqrt{n} \int_{\mathbf{A}} \bar{f}(x)^{1/2} dx \leq \beta\sqrt{n} \leq \beta\sqrt{\Sigma n} \approx \tilde{L}_n$$

□

Now the main result concerning the optimality of the *PD* policy can be derived.

THEOREM 2.3 *When applying the PD policy, an upper bound on the expected total system time T_{PD} is given by:*

$$T_{PD} \leq \eta \frac{(k+1)^2}{k^2} \frac{\beta^2 \lambda \Sigma}{2v^2}$$

where η is the expected total number of demands in the system waiting for service and β is the Euclidean TSP constant, which is estimated in [3] to be $\beta \approx 0.72$.

Proof Let $b(t)$ and $\mathbf{A}_s$ $(1 \leq s \leq k)$ defined as above. Let $b'(t)$ be a random variable representing the number of demands that the vehicle accepts during the current *TSP* path in $\mathbf{A}_s$. Denote by b and b' the steady state expectation values of $b(t)$ and $b'(t)$, respectively. Thus, the steady state average number of demands that are left for service in subregion $\mathbf{A}_s$ when the vehicle moves to $\mathbf{A}_{s+1}$ is $(b - b') \leq b$. Between two visits to $\mathbf{A}_{s+1}$ the vehicle has visited k subregions and consequently the average number of demands it finds in $\mathbf{A}_{s+1}$ is $\nu = (b - b') + (k-1)b$. Therefore, the steady state average number of demands to be served in the region $\mathbf{A}$ after a whole tour is:

$$\eta = (b - b') + (b - b') + b + \ldots + (b - b') + (k-1)b$$

$$= k(b - b') + \frac{k(k-1)b}{2}$$

and the steady state average number of demands ν the vehicle finds in a

subregion verifies:

$$\frac{\eta}{k} \leq \nu \leq \frac{2\eta}{k}.$$

Analogously to the proof of Theorem 1, and under the fact that the distribution is not uniform anymore since some of the demands have been already served, the difference $h_2 - h_1$ is bounded by:

$$h_2 - h_1 \leq k\left(\frac{\Lambda_{\nu+b'}}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right),$$

where $\Lambda_{\nu+b'} \leq \tilde{\Lambda}_{\nu+b'}$ is the length of a optimal TSP path through the $\nu + b'$ points.
Let δ be a random generic demand of $\mathbf{A}_{s+1}$ belonging to a set composed of the ν demands to be served in $\mathbf{A}_{s+1}$. As remarked in Theorem 1, the average waiting time for δ at time h_2 is $(h_2 - h_1)/2$. Moreover, it has to wait another period of time equal to $\Lambda_{\nu+b'}/2v + \sqrt{5\Sigma/k}/2v$ in order to finish the service of the $(\nu + b')$ demands in $\mathbf{A}_{s+1}$, since it is equally likely to be serviced at any order among the $(\nu + b')$ demands. Therefore, the steady state expected system time T_δ of demand δ is given by:

$$T_\delta = \frac{h_2 - h_1}{2} + \frac{1}{2}\left(\frac{\Lambda_{\nu+b'}}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right) \leq \frac{h_2 - h_1}{2} + \frac{1}{2}\left(\frac{\tilde{\Lambda}_{\nu+b'}}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right)$$

$$\leq \frac{k+1}{2}\left(\frac{\tilde{L}_{\nu+b'}}{v} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right) \approx \frac{k+1}{2}\left(\frac{\beta}{v}\sqrt{A(\nu+b')} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right)$$

$$= \frac{k+1}{2}\left(\frac{\beta}{v}\sqrt{\frac{\Sigma\,(\nu+b')}{k}} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}}\right).$$

This expression has been derived by using both formula (2) and Lemma (2.2) in order to estimate $\tilde{L}_{\nu+b'}$. Indeed, $(\nu + b') \to \infty$ as $\lambda \to \infty$ because:

$$\nu + b' \geq \frac{\eta}{k} + b'.$$

In addition, by noting that $\eta = \lambda\, T_\delta$ and by using Theorem (2.1) we have:

$$\nu + b' \;\geq\; \frac{\lambda\, T_\delta}{k} + b' \;\geq\; \frac{\beta^2 \lambda^2 \Sigma}{2\, k\, v^2} + b'\,.$$

Thus, $(\nu + b')$ is a large value as $\lambda \to \infty$ and consequently:

$$T_\delta \;\leq\; \frac{k+1}{2} \left\{ \frac{\beta}{v} \sqrt{\frac{\Sigma}{k}\left(\frac{2\eta}{k} + b'\right)} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}} \right\}\,,$$

or equivalently:

$$T_\delta \;\leq\; \frac{k+1}{2} \left\{ \frac{\beta}{vk} \sqrt{2\Sigma\, \lambda\, T_\delta} + \frac{1}{v}\sqrt{5\frac{\Sigma}{k}} + \frac{\beta}{v}\sqrt{\frac{\Sigma\, b'}{k}} \right\}.$$

By solving for T_δ we get:

$$\sqrt{T_\delta} \;\leq\; \frac{\sqrt{2(k+1)^2\beta^2\lambda\Sigma} + \sqrt{2(k+1)^2\beta^2\lambda\Sigma + 8k(k+1)v\sqrt{\Sigma\, k}(\sqrt{5} + \beta\sqrt{b'})}}{4kv},$$

that is:

$$\sqrt{T_\delta} \;\leq\; \frac{\sqrt{2(k+1)^2\beta^2\lambda\Sigma + 8k(k+1)v\sqrt{\Sigma\, k}(\sqrt{5} + \beta\sqrt{b'})}}{2kv},$$

and we get the result:

$$T_{PD} \;=\; \eta T_\delta \;\leq\; \eta \frac{(k+1)^2}{k^2} \frac{\beta^2 \lambda\, \Sigma}{2v^2}.$$

□

This result implies that the PD policy is asymptotically optimal. Indeed, by combining Theorem (2.1) and Theorem (2.3) we obtain:

$$\frac{T_{PD}}{T_{TD}} \;\leq\; \frac{\beta^2\lambda\eta\,(k+1)^2\Sigma + 4k(k+1)v\eta\sqrt{\Sigma\, k}(\sqrt{5} + \beta\sqrt{b'})}{\beta^2\lambda\eta\,\Sigma\, k^2}.$$

Therefore, as $\lambda \to \infty$ the report:

$$\frac{T_{PD}}{T_{TD}} \;\to\; 1 + \frac{1}{k^2} + \frac{2}{k}$$

indicating that even the *PD* policy is optimal for large values of k.

2.3 *The Insertion Policy*

Even though the previous two policies have nice theoretical properties, they may perform poorly under a finite arrival rate. This behaviour is because the rigid division of the service region into subregions may force the vehicle to postpone the service of a demand close to its position just because it belongs to another subregion. This happens, for example, when two demands are located close to the border of two adjacent subregions but can not be planned to be served consecutively within the a priori tour. In order to overcome this drawback the insertion policy (*Ins*) abandons the strategy of partitioning the service region **A** into subregions and makes use of a simple insertion technique. The idea behind this policy is to insert a new request into the current route by minimizing an insertion cost. In particular, a new request δ is inserted between two already scheduled requests in such a way the expected total system time:

$$\min_{j \geq i} \{E[T_1] + E[T_2] + \ldots + E[T_j] + E[T_\delta] + E[T_{j+1}] + \cdots\}$$

is minimized. The following property holds.

THEOREM 2.4 *Under heavy traffic, the insertion policy has a constant factor guarantee of two, i.e.:*

$$\frac{T_{Ins}}{T_{TD}} \leq 2\,, \qquad \lambda \to \infty$$

where T_{Ins} is the expected total system time associated to the insertion policy.

Proof Let δ be a new request. In addition, let η be the steady state average number of demands served during δ's waiting time. The distribution among the η demands is not uniform because some of the demands have already been served. Denote by x the steady state average number of demands in the path with an arrival time lower than t_δ and by y the steady state average number of demands with an arrival time greater than or equal to t_δ. Notice that by Little's law it follows that:

$$y \leq \frac{\lambda\, L_x}{v} \leq \frac{\lambda\, \tilde{L}_x}{v} \approx \frac{\lambda\, \beta\sqrt{\Sigma}\sqrt{x}}{v} \tag{4}$$

where we have used (2) to estimate $\tilde{L}_x$ because, as will be shown below, in heavy traffic x has to be large. Moreover, since δ is equally likely to arrive at

any moment during the service of the x demands, and is equally likely to be serviced at any order among the y demands then we can write:

$$\eta = \frac{x}{2} + \frac{y}{2} \leq \frac{x}{2} + \frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{2v}\,. \tag{5}$$

By Lemma (2.2) we obtain that T_δ, the steady state expected system time of δ, can be expressed as:

$$T_\delta = \Lambda_{(x+y)}/2v \leq \tilde{\Lambda}_{(x+y)}/2v \leq \tilde{L}_{(x+y)}/2v\,.$$

By using the fact that if x has to be large in heavy traffic then $(x+y)$ will be large too, we get:

$$T_\delta \leq \frac{\beta\sqrt{\Sigma}}{2v}\sqrt{x+y} \leq \frac{\beta\sqrt{\Sigma}}{2v}(\sqrt{x}+\sqrt{y})\,,$$

and by using the relation $\eta = \lambda\,T_\delta$ and equation (4) we obtain:

$$\eta \leq \frac{\lambda\,\beta\sqrt{\Sigma}}{2v}\sqrt{x} + \frac{\lambda\,\beta\sqrt{\Sigma}}{2v}\sqrt{\frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{v}}, \tag{6}$$

and, consequently, in heavy traffic we have:

$$x \leq \left\{\lambda\,\beta\sqrt{\Sigma}\sqrt{\frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{v}}\right\}/v\,. \tag{7}$$

Indeed, by (5) and by (6) it follows that:

$$\eta \leq min\left\{\frac{x}{2} + \frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{2v}, \frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{2v} + \frac{\lambda\,\beta\sqrt{\Sigma}}{2v}\sqrt{\frac{\lambda\,\beta\sqrt{\Sigma}\sqrt{x}}{v}}\right\}\,.$$

Now, suppose that (7) is true then we have:

$$1 \leq \frac{\lambda^3\beta^3\Sigma\sqrt{\Sigma}}{v^3}\,. \tag{8}$$

But as $\lambda \to \infty$ the quantity $(\lambda^3\beta^3\Sigma\sqrt{\Sigma}/v^3) \to \infty$, so (8) is true in heavy traffic, and is false in light traffic since it verifies $(\lambda^3\beta^3\Sigma\sqrt{\Sigma}/v^3) \to 0$ as $\lambda \to 0$.

Therefore by using (7) we obtain:

$$x \leq \lambda^2 \beta^2 \Sigma / v^2 \,. \tag{9}$$

Moreover, relations (5) and (9) ensure that:

$$T_\delta = \eta/\lambda \leq \beta^2 \lambda \, \Sigma / v^2, \tag{10}$$

and by Theorem (2.1) we have:

$$\frac{T_{Ins}}{T_{TD}} = \frac{\eta T_\delta}{T_{TD}} \leq 2 \qquad for \, \lambda \to \infty \,.$$

with η the expected total number of demands in the system. This prove has been based on the hypothesis that x is large. Now we shall proof that this assumption is true. By Theorem (2.1) we have:

$$\eta = \lambda \, T_\delta \geq \frac{\beta^2 \lambda^2 \Sigma}{2v^2} \,.$$

Thus, by using (5) we get:

$$\frac{x}{2} + \frac{\lambda \, \beta \sqrt{\Sigma}}{2v} \sqrt{x} \geq \frac{\beta^2 \lambda^2 \Sigma}{2v^2}$$

and by solving for x we obtain $x \geq \left\{ \left(1 - \sqrt{5}\right)^2 \lambda^2 \beta^2 \Sigma \right\} / 4$ which is a large quantity for $\lambda \to \infty$. □

3 Computational Results

We have compared the three policies under various (finite) arrival rates λ. The service region **A** is a 100 x 100 square (that has been divided into four equal subregions when the first two policies are used). We have carried out several sets of experiments by generating forty instances with a number of demands varying between 13 and 4500. The demands' arrival times and locations have been generated randomly according to an exponential and a uniform distributions, respectively. The speed v of the vehicle is assumed to be equal to 1. Parameters η and $\tilde{L}_\eta$ have been estimated by using expressions (2) and (3). The route within each subregion has been defined by using the *Nearest Neighbor Heuristic*. The algorithms have been implemented in C language and all the experiments have been performed on a Pentium IV computer with 504

Mb RAM. The results of our experiments are illustrated in Table 1. The first two columns report the values of arrival rate λ and the resulting number of demands for service. The third, sixth and tenth columns report the expected total system time obtained with the *TD*, the *PD* and the *Ins* policies, respectively. Moreover, the fifth and the eighth columns report the total number of tours, t, traversed by the vehicle in the first two policies, while the fourth, seventh and eleventh columns report the execution time *T(s)* of the three algorithms, respectively. Finally, the ninth and the twelfth columns report the relative improvement of policies *PD* and *Ins* with respect to policy *TD*, that we denote by I_1 and I_2, respectively.
The results reported in Table 1 show that policies *PD* and *Ins* outperform policy *TD* on most of the generated instances. Compared to *TD*, policy *PD* not only requires a lower number of a priori tours but also achieves an average improvement of 5%. Moreover, the results show clearly that policy *Ins* outperforms the other two policies. Indeed, the average improvement is 28% with respect to policy *TD* and 24% compared to policy *PD*. However, this improvement in the solution quality corresponds to a higher execution time especially for larger problems.

4 Conclusions

In this paper we have introduced and evaluated three new policies for the *DTSP*. The key difference between the *TD* and the *PD* policies is the way of inserting a new demand in the current subregion. On the other hand, the *Ins* policy makes use of a simple insertion technique. A computational study showed that the *Ins* policy achieves better results than the others two policies but requires a larger computing effort. Moreover, we have proved that the *TD* and *PD* policies are asymptotically optimal in heavy traffic, whereas the policy *Ins* has a constant factor guarantee of two. There are several issues that are still open for future research. Among them, we underline the *DTSP* under an heterogeneous arrival process.

Table 1. Experimental results

λ	*Demand*	*TD*	*T(s)*	*t*	*PD*	*T(s)*	*t*	I_1	*Ins*	*T(s)*	I_2
0.125	13	3551	0	2	2167	0	2	38	1948	1	45
0.125	14	2899	1	3	2382	0	3	17	1604	0	44
0.125	31	10666	0	2	8453	0	2	20	6468	1	39
0.125	27	7002	0	2	5991	1	2	14	4922	1	29
0.125	66	14640	1	4	7699	1	3	47	6671	2	54
0.125	60	19717	1	3	16689	1	3	15	15391	2	21
0.25	34	9245	0	2	8611	0	2	6	6864	0	25
0.25	28	7973	1	2	6610	0	2	17	5798	0	27
0.25	54	20474	0	2	16523	0	2	19	15178	1	25
0.25	60	24641	0	3	18495	0	2	24	16335	1	33
0.25	120	64268	0	3	51774	0	2	19	49516	6	22
0.25	112	55751	1	3	47606	0	2	14	39821	5	28
0.25	2514	4261804	13	9	3796442	48	7	10	3153404	1805	26
0.355	39	14956	0	2	11834	0	2	20	9243	1	38
0.355	46	17755	0	2	11457	0	2	35	11895	1	33
0.355	70	28453	0	2	27719	1	2	2	22978	2	19
0.355	80	34753	0	3	26670	1	2	23	25400	3	26
0.355	178	121928	1	3	103554	1	2	15	81333	12	33
0.355	172	117917	1	3	89897	1	2	23	87502	12	25
0.355	3418	7752157	17	8	7042203	46	6	9	5481657	4563	29
0.45	54	22179	0	2	17409	0	2	21	12977	2	41
0.45	105	48914	1	2	40716	0	2	16	41100	5	15
0.45	216	149115	1	3	133818	1	2	10	107601	16	27
0.45	211	152650	1	3	126415	2	3	17	108518	17	28
0.45	4500	12271333	22	7	11307125	81	5	7	8868317	5883	27
0.6	61	26057	1	2	23654	0	2	9	18480	2	29
0.6	142	92261	1	3	87369	1	2	5	70579	9	23
0.6	150	84771	1	3	79417	1	3	6	77078	9	9
0.6	265	199644	2	3	178264	2	2	10	152415	28	23
0.6	283	231862	1	2	207759	2	2	10	170558	31	26
0.8	112	66620	1	2	55231	1	2	17	47617	5	18
0.8	191	133952	1	3	115309	1	2	13	99007	14	26
0.8	413	320562	2	3	272694	2	2	14	210655	62	34
0.8	355	331389	2	3	312815	2	2	5	261303	49	21
0.8	4075	12208304	21	5	12117869	46	3	0.7	8845174	3906	27
0.99	247	190857	1	3	190995	2	2	0.07	151228	23	20
0.99	454	512846	3	3	431497	2	2	15	347267	21	32
0.99	486	539236	3	3	512307	3	2	4	400237	47	25
0.99	936	1435763	5	3	1323193	6	2	7	1096028	245	23
0.99	2117	4850611	6	3	4626169	15	2	4	3302173	1629	31
0.99	2657	6042570	13	4	6164919	15	3	1	4087637	1443	32

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