Limited Recourse in Two-Stage Stochastic Linear Programs

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Abstract

In several real-world applications, modelled by two-stage stochastic problems, first and second-stage decisions (or some of their components) represent identical variables of the problem that is modelled. In these cases an appropriate solution of the problem might require that the second-stage decisions do not differ substantially from the corresponding first-stage ones. In this paper we propose a parametric approach to control the variability of the first and second-stage decisions and present a suitable solution framework. The advantage of the new approach is illustrated by considering two specific applications in electric power management and financial planning.

1 Introduction

Two-stage stochastic linear programs provide a suitable framework for modelling decision problems under uncertainty arising in several real-world applications. The flexibility of these models is related to their "dynamic" nature: besides the first-stage variables, representing decisions made in face of uncertainty, the model includes second-stage decisions, i.e. recourse actions, which may be taken once a specific realization of the random quantities is observed.

The model can be formulated as follows:

$$(\mathsf{SLP\_2S}) \qquad \min \quad c_0^T x + \sum_{l=1}^{N} \pi_l c_l^T y_l \tag{1}$$

$$\text{s.t.} \quad A_0 x = b_0, \tag{2}$$

$$T_l x + W_l y_l = h_l, \qquad l = 1, \ldots, N \tag{3}$$

$$x \geq 0, \quad y_l \geq 0. \qquad l = 1, \ldots, N \tag{4}$$

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Uncertainty in the model's parameters is represented by a set of scenarios $\Omega = \{1,\ldots,N\}$, each occurring with a given probability $\pi_l \geq 0$ such that $\sum_{l=1}^{N} \pi_l = 1$. According to the robust optimization terminology [13] equations (2) represent the *structural* constraints whose coefficients $A_0 \in R^{m_1 \times n_1}$, $b_0 \in R^{m_1}$ are unaffected by uncertainty, whereas equations (3) depict the *control* constraints with scenario dependent coefficients ($T_l \in R^{m2 \times n_1}, W_l \in R^{m2 \times n_2}, h_l \in R^{m_2}$, for $l \in \Omega$). For an introduction to two-stage stochastic programming models and overview of the main solution methods, refer to [4, 8, 16].

An important issue in the stochastic programming framework is the stability of the time-staged solutions. The need to have recomendations which do not vary substantially over different instances of the input data have been recognized in several domains of applications (see [13, 15] for a detailed survey). Different approaches for addressing the issue have been proposed in the literature. Besides reactive approaches such as postoptimality analysis and contamination technique [6] we cite two recently proposed proactive frameworks: *robust optimization* [13] and *restricted recourse* [15]. The first allows to control both *model* and *solution* robustness by minimizing violations of recourse constraints under different scenarios and higher moment of the distribution of the cost outcomes, respectively. The second develops a recursive approach which controls the dispersion of the recourse decisions by generating a family of solutions that are progressively less sensitive to data differences in the scenarios.

In this paper emphasis is put on *both* first and second-stage decisions, rather than recourse solutions only. Thus we extend the restricted recourse approach towards applications where robustness is required between first and second-stage decisions and not among second-stage decisions under different scenarios. This is indeed a more practical modelling framework. Since only one of the N scenarios will be realized only one of the second-stage optimal decisions will be implemented in practice. Any differences between the second-stage optimal decisions under different scenarios will not be perceived by the user. However, both first-stage and one of the N second-stage decisions will be implemented in practice, and that is where robustness may be required by the user.

The approach we propose, referred to as *limited recourse*, aims at stabilizing simultaneously design and recourse actions. Low variability represents a desirable property in many real-world applications, where the two stage solutions (or some component of them) represent identical variables of the real-world problem. For example, in pricing problems, a corrective price at the second-stage which is much higher than the first-stage originally proposed may define an operational plan inapplicable because of restrictive market laws.

The rest of the paper is organized as follows. Section 2 defines the limited recourse model and describes the general solution scheme. Enforcing stability in the solution causes an increase of the objective function value and Section 3 analyzes the effect of the limiting constraints on a set of randomly generated test problems. Section 4 illustrates the advantage of the limited recourse formulation for two applications from electric power planning and portfolio management, respectively. Conclusions and open issues are discussed in Section 5.

2 Limited Recourse Model and Solution Procedure

Limited recourse offers a modelling framework to represent two-stage problems where variability between first and second-stage decisions should be controlled. The approach is seen as an extension of restricted recourse [15] where restrictions are imposed on recourse actions only.

The limited recourse model is obtained from (SLP_2S) by adding range constraints that limit the difference between the first and second-stage decisions. Such constraints will involve all ($\tilde{n} = n_1 = n_2$) or a subset $\tilde{n} < n_1$ of the vector components on which restriction has to be imposed. Without loss of generality we assume that $n_1 = n_2$ and formulate the model with restriction on all the components of the decision vectors as follows:

$$
\begin{aligned}
(\mathsf{SLP\_LR}) \qquad \min \quad & c_0^T x + \sum_{l=1}^{N} \pi_l c_l^T y_l && && (5)\\
\text{s.t.} \quad & A_0 x = b_0, && && (6)\\
& T_l x + W_l y_l = h_l, && l = 1, \ldots, N && (7)\\
& -u \leq x - y_l \leq u, && l = 1, \ldots, N && (8)\\
& x \geq 0, \;\; y_l \geq 0. && l = 1, \ldots, N && (9)
\end{aligned}
$$

Here u represents a limiting vector bound that allows to control the variability. The determination of u can be carried out by using a recursive approach, whose details are introduced below.

Let x^* and y_l^*, $l \in \Omega$, denote the optimal solution of the (SLP_2S) model. On the basis of these values, we can compute the *virtual upper bound vector* $\tilde{u}$ and the *virtual lower bound vector* $\tilde{l}$ as follows:

$$
\begin{aligned}
\tilde{u}_i &= \max_{l=1,\ldots,N} |x_i^* - (y_l^*)_i|, \qquad i = 1, \ldots, \tilde{n},\\
\tilde{l}_i &= \min_{l=1,\ldots,N} |x_i^* - (y_l^*)_i|, \qquad i = 1, \ldots, \tilde{n}.
\end{aligned}
$$

The difference between the two bounds defines the *virtual range vector*

$$\tilde{\omega} = \tilde{u} - \tilde{l},$$

whose maximum component

$$\tilde{\delta} = \max_{i=1,\ldots,\tilde{n}} \{\tilde{w}_i\},$$

represents the size of the smallest $\tilde{n}$-dimensional hypercube including all the difference vectors between the first and second-stage decisions. Such hypercube can be characterized by the $\tilde{n}$-dimensional vector u

$$u = \tilde{\delta}\,\mathbf{1}.$$

which appears in the range constraints. In order to limit recourse, the hypercube shrinks by progressively restrincting the components of u which are updated iteratively as explained below.

The dispersion of the two stage solutions can be evaluated by using several measures. We have used both the *mean* distance of the recourse decisions from the first-stage ones, defined as $\rho = \sum_{l=1}^{N} \pi_l ||y_l - x||^2$, and the ℓ_2 norm, Δ, of the virtual range $\tilde{w}$, $\Delta = ||\tilde{w}||$.

The idea underlying the solution procedure is simple: by iteratively reducing the size of the hypercube, we enforce tighter limiting conditions on the two stage variables until the dispersion measure matches a user-defined tolerance denoted by ϵ. The iterative scheme is shown below.

```
/* Solve the original Two-Stage problem (SLP_2S) */

solve_problem(solution, is_feasible, restriction_level);

/* Add the range constraints (8) */

expand_problem();

/* Start iterative solution procedure */

while (restriction_level > ε && is_feasible)
do
 {

 /* Reduce the hypercube dimension */

 limit_variability(lower_bounds, upper_bounds);

 /* Solve the Limited Recourse problem (SLP_LR) */

 solve_problem(solution, is_feasible, restriction_level);

 }
```

The procedure starts by solving the (SLP_2S) problem and computing a variability measure. If this value is less than a user-defined tolerance, the procedure is stopped because the solution is both optimal and has limited recourse. Otherwise the limited recourse problem is solved for decreasing values of the hypercube vector u until either the problem becomes infeasible (i.e. the restriction is too tight) or the tolerance is reached.

2.1 Computational Issues

The core of the procedure is the solution of the (SLP_LR) problem that is performed at each iteration of the limited recourse scheme. Such solution process can be efficiently carried out by observing the following facts. First, during successive iterations the only change concerns the difference vectors $x - y_l$, whereas the constraints matrix remains unaltered. This allows to take advantage from warm starting mechanisms, typically employed in interior point solvers. The same symbolic factorization of the projection matrix is used and only the numerical factorization is computed at each iteration.

Secondly, specialized interior point methods for two-stage stochastic programs can be applied. All these methods rely on the particular block-angular structure of the constraint matrix. Such a structure is not immediately recognized in the limited recourse model, but it can be regained by appropriately re-organizing the range constraints.

More specifically, by introducing the slack variables $s_l = x - y_l$, $l \in \Omega$, constraints (6)-(8) can be rewritten in the following compact form:

$$\tilde{A}t = \tilde{h} \quad -u \leq s_l \leq u \quad l \in \Omega$$

where the constraint matrix $\tilde{A} \in \Re^{(m_1+2Nm_2)\times(N+1)\tilde{n}}$ is given as:

$$\begin{pmatrix}
A_0 & & & & & & & \\
T_1 & W_1 & & & & & & \\
-I & I & -I & & & & & \\
T_2 & & & W_2 & & & & \\
-I & & & I & -I & & & \\
\vdots & & & & & \ddots & & \\
T_N & & & & & & W_N & \\
-I & & & & & & I & -I
\end{pmatrix}$$

and t, $\tilde{h}$ are defined as:

$$t = (\, x \quad y_1 \quad s_1 \quad y_2 \quad s_2 \quad \cdots \quad y_N \quad s_N \,)^T$$

$$\tilde{h} = (\, b_0 \quad h_1 \quad 0 \quad h_2 \quad 0 \quad \cdots \quad h_N \quad 0 \,)^T$$

Efficient solution methods can be designed by exploiting in addition to the block-angular structure of $\tilde{A}$, the special structure of the matrices

$$\tilde{T}_l = \begin{pmatrix} T_l \\ -I \end{pmatrix} \quad \tilde{W}_l = \begin{pmatrix} W_l & 0 \\ I & -I \end{pmatrix}$$

corresponding to each subblock $l \in \Omega$.

A detailed description of the solution procedure for restricted recourse problems is reported in [1]. Here we use a slight modification of that algorithmic scheme. The computational advantages deriving from the use of a such specialized solution procedure with respect to general purpose interior point solvers (see, for example, [7, 17] is highlighted by the numerical results reported in [1].

3 Effect of the Limited Recourse

The introduction of the range constraints produces an increase of the objective function value. In order to analyze this effect, we first generated four random test problems. Their characteristics are reported in Tables 1 and 2. In all the problems c, T, and W are scenario independent and the uncertainty is only incorporated in the right-hand-side vectors h_l. In Table 1 we report the dimensions of the first and the second stage problems for both the original model (SLP_2S) and the limited recourse model (SLP_LR). Table 2 reports the size of the deterministic equivalent problems with increasing number of scenarios.

Table 1: Size of Randomly Generated Test Problems

Test Problem	*SLP_2S model*			*SLP_LR model*		
	$n_1 = n_2$	m_1	m_2	$n_1 = n_2$	m_1	m_2
P1	4	3	2	4	5	4
P2	37	28	28	37	56	56
P3	13	9	7	13	16	14
P4	11	2	7	11	9	14

Our objective here is to illustrate how the objective function value increases when a progressively tighter tolerance is imposed on the model. We choose a very small value of ϵ and we execute the procedure up to 50 iterations. In this way, the iterative process stops either when the problem becomes infeasible or when the maximum number of iterations is reached. In Table 3 we report the value of the variability measure Δ and the value of the objective function before

Table 2: Size of the Deterministic Equivalent Problems

Test problem	*# of scenarios*	*SLP_2S model* Constraints	*SLP_2S model* Variables	*SLP_LR model* Constraints	*SLP_LR model* Variables
P1.4	4	11	20	27	40
P1.8	8	19	36	51	72
P2.8	8	252	333	548	666
P2.16	16	476	629	1068	1258
P2.32	32	924	1221	2108	2442
P2.64	64	1820	2405	4188	4810
P3.4	4	37	65	89	130
P3.8	8	65	117	169	234
P3.16	16	121	221	329	442
P3.32	32	233	429	649	858
P4.4	4	30	55	74	110
P4.8	8	58	99	146	198
P4.16	16	114	187	290	374
P4.32	32	226	363	578	726
P4.64	64	450	715	1154	1430

and after imposing the limiting constraints on the two-stage model. Moreover we report the maximum relative increase in the objective function ϵ_f computed as:

$$\epsilon_f = \frac{f_{(SLP\_LR)} - f_{(SLP\_2S)}}{f_{(SLP\_2S)}}.$$

Table 3 shows that the limitation of the variability and the resulting increase in the objective function value depend on the problem under consideration. Both the problem's structure and its size (number of scenarios) affect the process of the limited recourse. Some problems can reach lower variability than others. In effect, in the solution of all versions of problems P2 and P3, for which the solution procedure reached the maximum number of iterations, the reduction of the variability is followed by a substantial increase in the objective function. In the case of the other test problems the solution procedure is stopped because infeasibility occurs after a certain number of iterations (7 iterations for P1.4, 11 for P1.8 and 39 for all versions of problem P4).

For some test problems we can reach a (reasonable) limited solution without sacrificing too much the objective function value. This is shown in Figure 3 which depicts the increase of the objective function that corresponds to the variability decreases at each iteration in the solution of the problem P3.32. This

Table 3: Effect of the Limited Recourse Procedure on the Two-Stage Model

Test problem	*Solution of SLP_2S* Δ	f	*Solution of SLP_RL* Δ	f	ϵ_f (%)
P1.4	2.00	39.43	1.49	41.54	5.35
P1.8	3.60	36.65	1.77	40.50	10.50
P2.8	141.60	1122.97	4.40	1674.46	49.10
P2.16	141.67	879.22	4.40	1669.81	89.91
P2.32	141.68	834.61	4.41	1597.57	91.41
P2.64	141.70	669.54	4.39	1437.63	114.71
P3.4	2430.81	$3.41e{+}05$	24.53	$6.23e{+}05$	82.69
P3.8	2167.75	$3.48e{+}05$	26.12	$8.05e{+}05$	131.32
P3.16	4267.31	$3.50e{+}05$	51.53	$1.00e{+}06$	187.71
P3.32	4631.29	$3.04e{+}05$	52.24	$1.24e{+}06$	309.50
P4.4	3.50	433.00	0.035	536.89	23.99
P4.8	4.29	433.00	0.056	542.94	25.39
P4.16	7.28	433.50	0.104	649.66	49.86
P4.32	9.07	433.56	0.104	715.85	65.10
P4.64	10.43	434.01	0.105	726.92	67.48

figure shows that after 8 iterations we can reach a variability decreases of 42% with respect to the initial value while the resulting increase in the objective function remains under 6%. A remarkable increase in the objective function occurs (over 309%) only when tighter limitation on the variability is required. Similar behaviour can be observed in the solution of the problem P4.32 (Figure 4) for which the reduction in the variability reaches, after 10 iterations, 48% while the corresponding increase of the objective function is only 7.8%.

We note that comparable behaviour was noted in the experiments performed by Vladimirou and Zenios in [15] using some real-world problems. However, none of their problems are suitable for limited recourse formulation so here we restrict our attention to randomly generated problems. The effect of the limited recourse on two real-world applications is investigated in Section 4.

4 Applications of the Limited Recourse Approach

Limited recourse represents a suitable approach to model problems arising in a broad spectrum of real-world applications. In this section, we present two problems in the electrical and financial domains, respectively, and we show how the introduction of constraints limiting variability between the two stage decisions can provide more appropriate decision plans.

Figure 1: Effect of Limited Recourse on the Problem P1 (8 scenarios)

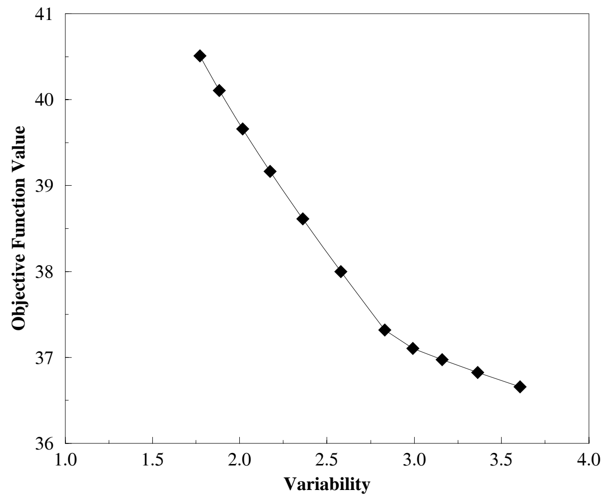


Figure 2: Effect of Limited Recourse on the Problem P2 (8 scenarios)

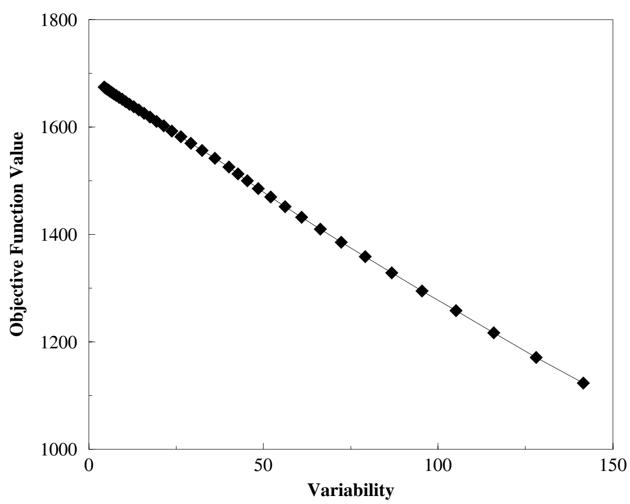


Figure 3: Effect of Limited Recourse on the Problem P3 (32 scenarios)

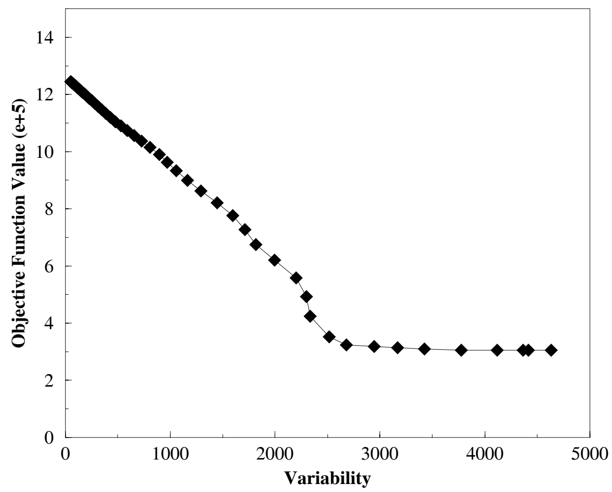
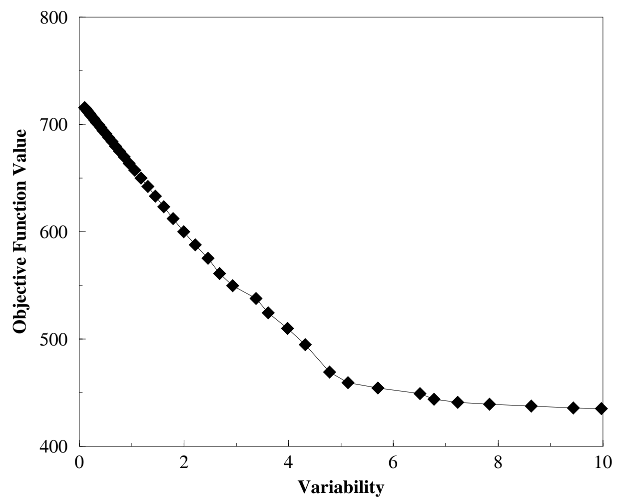


Figure 4: Effect of Limited Recourse on the Problem P4 (32 scenarios)



4.1 Capacity Expansion Planning for Electricity Generation

The problem consists of choosing to invest among different power generation technologies in order to meet the demand and to minimize investment and operation costs. The model we present here is a two-stage program and represents a modification of the one developed by Louveaux and Smeers [10] (see also [11]). In the first-stage we decide investments in the generation plant, i.e. type of technology and capacity of the plant, to meet a peak demand and satisfy budget limits. Second stage decisions represent the amount of power effectively generated by each plant in order to satisfy the particular realization of the random demand data. The two stage decisions are related by the trivial constraints ensuring that the effective output of each plant does not exceed its capacity.

Each generation plant is characterized by investment and operation costs. Even though these costs are generally random, we will assume them for simplicity as deterministic. The main source of uncertainty in our model is the load demand whose values varies with the temperature, the population behaviour, the activities in particular events, and so on.

We concentrate our attention on the peak-load demand and we represent it as a discrete random variable. To each possible realization d_l is assigned a probability value that measures its likelihood of occurrence.

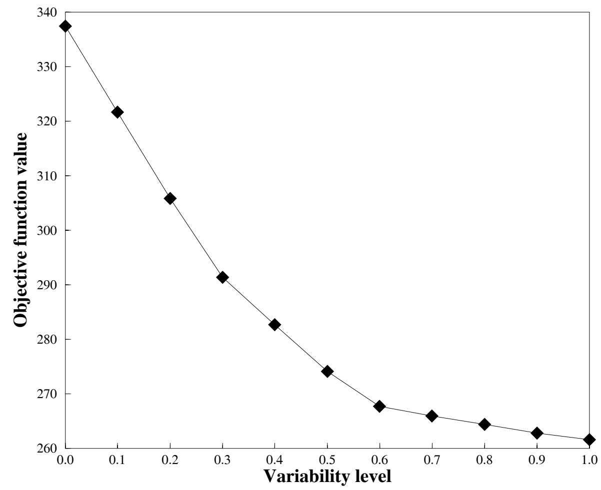
In the model that we consider here we suppose that the peak-demand d_l can assume only 3 different values: 3, 5 and 7 units with a probability of 0.3, 0.4 and 0.3, respectively. We also suppose that we can choose to invest in 4 available technologies with a budget limit of 120. The unit investment cost and unit operating cost vectors are $(10, 7, 16, 6)$ and $(40, 45, 32, 55)$, respectively. The resulting two-stage model is the following:

$$
\begin{aligned}
\min \quad & 10x_1 + 7x_2 + 16x_3 + 6x_4 + \sum_{l=1}^{3} \pi_l (40y_{l1} + 45y_{l2} + 32y_{l3} + 55y_{l4}) && (10)\\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \geq 7 && (11)\\
& 10x_1 + 7x_2 + 16x_3 + 6x_4 \leq 120 && (12)\\
& y_{li} \leq x_i \qquad i = 1, \ldots, 4 \;\; l = 1, \ldots, 3 && (13)\\
& \sum_{i=1}^{4} y_{li} \geq d_l \qquad l = 1, \ldots, 3 && (14)\\
& x_i \geq 0, \;\; y_{li} \geq 0 \qquad i = 1, \ldots, 4, \;\; l = 1, \ldots, 3. && (15)
\end{aligned}
$$

The model minimizes the sum of the investment cost and the demand operation cost subject to the following constraints:

- the total installed capacity should be higher than the maximum value of the load demand (11);

Figure 5: Cost-Variability Trade-Off



- the budget available for the generation investment can not be exceeded (12);
- the first-stage capacity installed for each technology represents an upper bound on the effective output generated (13);
- the total power effectively generated should satisfy the stochastic demand (14);
- non-negativity of the decision variables (15).

In addition to the above requirements we imposed constraints that limit the variability between first and second-stage decision variables. With these constraints we impose that the gap between the capacities of the installed plans and the power effectively used is minimal. The objective is to avoid the installation of too many different types of technologies in order to make easier the management and maintenance of the production system. This can not be paid, however, by an excessive increase of the objective function. For this reason a trade-off cost-variability should be evaluated in order to identify an acceptable solution. Therefore, we solve a limited recourse problem for different values of variability level.

Another way of imposing the limited recourse in the model introduced above, is to represent the range constraints in the following equivalent form:

$$x_i - y_{li} \leq k\, x_i, \quad i = 1, \ldots, 4, and\; l = 1, \ldots, 3.$$

In this way the difference between the variables x_i and y_{li} is a fraction of the installed capacity for each technology. Figure 5 depicts ten different combinations of cost and variability values which were obtained at successive iterations of the limited recourse model. Originally no limitation on variability is imposed, and the last iteration means that no variability is admitted, and in this case x coincides with y_l for all scenarios. Of course neither of these two extreme values are of interest for our application and it is up to the decisions makers to choose a robust solution without sacrificing a lot in terms of objective function value.

For example, it is possible to accept a solution with 60% of variability and an increase of only 2.35% of the objective function with respect to the original (SLP_2S) solution (after 5 iterations). Further limitation on variability will cause an increase of approximately 30% of the objective function (last iteration).

It is instructive to observe how the decision variables vary with the decrease of variability:

(i) First iteration

$$x = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \; y_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \; y_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \; y_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

(ii) Intermediate iteration

$$x = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \; y_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; y_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \; y_3 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

(iii) Last iteration

$$x = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \; y_1 = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \; y_2 = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \; y_3 = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

These results show that the initial solution, corresponding to the (SLP_2S) model, utilizes three of the four different technologies. Even though all three plants are used only in the third scenario, they should be switched on several hours in advance to be able to deliver when is required. This means that we have additional start-up costs that are justified only in one realization of the three scenarios. Two of these plants will be used during the operation time if the second scenario occurs, and only one plant is used if the first scenario is observed. When variability limitation is imposed, the number of technologies needed is reduced to only two plants and both are used whatever realization of the random demand is observed. At the end of the limited recourse process no variability is admitted and all the vectors are identical. In this iteration the objective function could be equivalently expressed as:

$$\min \quad (10+40)x_1 + (7+45)x_2 + (16+32)x_3 + (6+55)x_4.$$

It is clear that the minimum (investment plus operation) cost correspond to the third technology and this explains the advantage to invest there.

4.2 A Limited Recourse Mean-Variance Model

Mean-variance models represent the milestone reference for financial planning decisions (see Markowitz's seminal paper [12]). Expected return and risk are the two main criteria used to define the portfolio's composition: investor chooses different securities in such a way that the variance (overall risk) of portfolio is minimized subject to the constraint that expected portfolio return should achieve a predefined level.

The mathematical formulation of the basic mean-variance model (see Zenios [18] for a detailed survey) is as follows:

$$\begin{aligned}
\min \quad & x^T Q x && (16)\\
\text{s.t.} \quad & \sum_{j\in J} \mu_j x_j \geq \mu_p, && (17)\\
& \sum_{j\in J} x_j = 1, && (18)\\
& x_j \geq 0. && (19)
\end{aligned}$$

Here J denotes the universe of securities, x_j is the fraction of portfolio invested in the security $j \in J$; Q indicates the covariance matrix of the returns of the securities, whereas μ_j is the expected return of security $j \in J$, and μ_p is the target expected return of portfolio.

Model (16–19) is a static myopic model: x represent the design decisions used to define the portfolio's composition and, in addition, uncertainty affecting the financial instrument returns, is not explicitly introduced into the model.

A two-stage version of the mean-variance model can be obtained by allowing control decisions to rebalance portfolio composition once a particular scenario of the uncertain returns is observed. For each scenario $l \in \Omega$, we denote by r_{lj} the return of security j and by y_{lj} the second-stage variable representing the fraction of the portfolio devoted to security j. In addition, we denote by Q_l the covariance matrix under scenario l.

The two-stage mean-variance model can be mathematically formulated as follows:

$$\min \quad \sum_{l=1}^{N} \pi_l y_l^T Q_l y_l \tag{20}$$

$$\text{s.t.} \quad \sum_{j \in J} x_j = 1, \tag{21}$$

$$\sum_{j \in J} y_{lj} = 1, \qquad l = 1, \ldots, N \tag{22}$$

$$\sum_{j \in J} (\mu_j x_j + r_{lj} y_{lj}) \geq \mu_p, \quad l = 1, \ldots, N \tag{23}$$

$$x_j \geq 0. \tag{24}$$

An important issue concerning the portfolio management model introduced above is the "frequency" of rebalancing. If model (20-24) is applied every time new data become available (e.g. simulated scenarios and estimated returns are referred, for example, to a weekly horizon) we would constantly modify our portfolio. This can lead to an increase of transaction costs, making happy our broker only.

The limited recourse modelling approach allows to avoid changes in portfolio composition by including additional constraints limiting the variability between the initial and the rebalanced investment decisions. More specifically, for each scenario $l \in \Omega$, we include into the model the following range constraints:

$$-u \leq x - y_l \leq u.$$

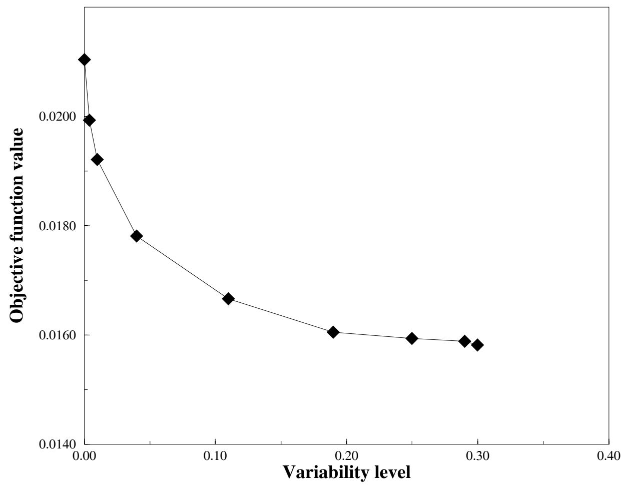
By progressively reducing the magnitude of the variability, it is possible to trace the solution variability versus the cost increase, leaving to the decision maker the choice of the most appropriate compromise between the two conflicting factors. (We note that even if the cost function of the limited recourse model is quadratic the solution procedure described in Section 2 can be applied.)

The relevance of the limited recourse paradigm for the two-stage mean-variance model is shown by a simple example. We consider three financial instruments and we fix the target expected value to 1.30. We assume the expected return vector μ:

$$\mu = \begin{bmatrix} 1.0890 \\ 1.2136 \\ 1.2345 \end{bmatrix},$$

and we represent the uncertainty related to the return of the available instru-

Figure 6: Cost-variability trade-off



ments by considering three equally likely scenarios:

$$r_1 = \begin{bmatrix} 0.954 \\ 0.728 \\ 0.918 \end{bmatrix} \qquad r_2 = \begin{bmatrix} 1.300 \\ 1.225 \\ 1.908 \end{bmatrix} \qquad r_3 = \begin{bmatrix} 1.035 \\ 0.928 \\ 1.260 \end{bmatrix}.$$

All the input data are taken from an example reported in [9]. On the basis of these data, we simulated the scenarios covariance matrices Q_l.

By using the solution procedure introduced in Section 2, we generate a family of solutions with a decreasing level of variability. Figure 6 shows the trade-off between the objective function value and the variability measure ρ.

The initial value of ρ corresponds to the solution of the (SLP_2S) problem. By imposing a tighter limitation on the variability, the cost function is increased. We observe, however, that such increase is not so noticeable at least in the first iterations. For example, a reduction of 63% in the value of ρ causes an objective value increase of 5% only.

We report below the values of the decision variables at three iterations of the solution procedure.

(i) First iteration

$$x = \begin{bmatrix} 0.43 \\ 0.34 \\ 0.23 \end{bmatrix}, \; y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; y_2 = \begin{bmatrix} 0.26 \\ 0.74 \\ 0 \end{bmatrix}, \; y_3 = \begin{bmatrix} 0.11 \\ 0.62 \\ 0.27 \end{bmatrix}$$

(ii) Intermediate iteration

$$x = \begin{bmatrix} 0.49 \\ 0.49 \\ 0.02 \end{bmatrix}, \; y_1 = \begin{bmatrix} 0.74 \\ 0.26 \\ 0 \end{bmatrix}, \; y_2 = \begin{bmatrix} 0.26 \\ 0.74 \\ 0 \end{bmatrix}, \; y_3 = \begin{bmatrix} 0.24 \\ 0.48 \\ 0.28 \end{bmatrix}$$

(iii) Last iteration

$$x = \begin{bmatrix} 0.30 \\ 0.54 \\ 0.16 \end{bmatrix}, \; y_1 = \begin{bmatrix} 0.30 \\ 0.54 \\ 0.16 \end{bmatrix}, \; y_2 = \begin{bmatrix} 0.30 \\ 0.54 \\ 0.16 \end{bmatrix}, \; y_3 = \begin{bmatrix} 0.30 \\ 0.54 \\ 0.16 \end{bmatrix}$$

The first iteration solution represents the portfolio's composition obtained by solving the problem without any restriction. The investments are distributed among three securities in different proportions and the same is for the third scenario's solution. In the first scenario only the first security is considered, whereas in the second the first and the second securities are included.

At an intermediate iteration both first and second-stage decisions are changed with respect to the first iteration ones. At this iteration, the limitation imposed on the difference between rebalanced investment decisions and the initial portfolio composition is reduced to 35%.

Finally, the last iteration produces the same portfolio for all scenarios (highest limitation level). The resulting portfolio is "static" and the increase in the objective function value is around 32%.

As mentioned above, cost function represents the risk of the portfolio, thus higher levels of limitation produce more risky portfolios. Such a risk increase has to be carefully evaluated and compared with the transaction cost when choosing portfolios. This trade-off is the cornerstone with the original Markowitz model, except that in the original model transaction costs are not included.

The savings in the transaction costs for the three stategies are illustrated below. They are computed by measuring the changes of the second-stage investment decisions from the first-stage ones. More specifically, for the first strategy (first iteration) transaction costs are:

$$\frac{1}{3}(0.57 + 0.34 + 0.23) + \frac{1}{3}(0.17 + 0.40 + 0.23) + \frac{1}{3}(0.32 + 0.28 + 0.04) = 0.86.$$

For the second stategy, they are reduced to 0.42 as results from the calculation:

$$\frac{1}{3}(0.25 + 0.23 + 0.02) + \frac{1}{3}(0.23 + 0.25 + 0.02) + \frac{1}{3}(0.25 + 0.01 + 0.26) = 0.42$$

and they are 0 for the strategy corresponding to the last iteration.

5 Conclusion

In this paper we propose an approach to enforce and control variability between design and corrective actions in two-stage stochastic programs. The need to have limited solutions is recognized is some real-world applications, where solutions characterized by a high level of variability produce contingency plans hardly implementable. We propose a parametric approach for the solution of the limited recourse model and we discuss the effect of the range constraints on the objective function value by considering a set of randomly generated test problem. Finally, we introduce two applications from power generation and financial planning domains and show that the limited recourse approach can provide strategies with several alternatives.

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