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Some Valid Inequalities for the Probabilistic Minimum Power Multicasting Problem

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Abstract

In this paper we describe some results on the linear integer programming formulation of the Probabilistic Minimum Power Multicast (PMPM) problem for wireless networks. The PMPM problem consists in optimally assigning transmission powers to the nodes of a given network in order to establish a multihop connection between a source node and a set of destination nodes. The nodes are subject to failure with some probability, however the assignment should be made so that the reliability of the connection is above a given threshold level. This model reflects the necessity of taking into account the uncertainty of hosts' availability in a telecommunication network.

Keywords: Minimum Power Multicasting, Probabilistic Mathematical Models, Multihop Networks, Integer Programming

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1 Introduction and problem definition

A multihop wireless network is a collection of devices that can communicate without using any wired infrastructure. Even though each device has a limited transmission range, global connectivity may be ensured by using multihop links. This paper deals with the problem of minimizing the power required to connect a given source device with a set of target hosts. This optimization problem, which is at present a topic of intensive study (see, for instance, [2,3,4,5,6,7]), is known as the Minimum Power Multicast (MPM) problem and customarily assumes a deterministic behavior of the transmitting devices. In reality, it has to be expected that the terminals can be affected by temporary or permanent failure, therefore it is reasonable to consider a probabilistic formulation that takes into account the uncertain nature of node availability. One such formulation has been presented in [1], along with an exact algorithm for solving the problem and some computational experiments. Here we focus on the theoretical properties of the problem, in particular describing some results about its polyhedral structure. In the rest of this section we give the necessary definitions.

A network of wireless devices can be modelled mathematically as a directed graph $G = (V, A)$, where the elements of the set V are the devices and those of A are all the possible connections between pairs of devices. Let $|V| = n$ and assume $n \geq 3$. We select a node s to be the source of the communication and a subset R of nodes to be the destinations for the signal generated in s, with $|R| = r$. Each node $i \in V$ can receive data from other nodes of the network and send data to any node in its transmission range and is available with a given probability $q_i \in]0, 1]$. We assume that $q_i = 1$ for $i \in R \cup \{s\}$. For each arc $(i, j) \in A$, p_{ij} is the minimum amount of power that must be assigned to node i in order to establish a direct communication with node j. The PMPM problem consists in assigning to each node $i \in V$ a transmitting power $\rho(i)$, minimizing the total transmission power, so that a connection between s and each destination $d \in R$ is established with probability at least equal to a given reliability threshold $\alpha \in]0, 1[$. For each node $i \in V$, we assume that the arcs $(i, j) \in A$ outgoing from i are ordered so that the associated p_{ij} values are nondecreasing. For each subset K of the set of nodes V, we denote by $b(i, K)$ the first arc in the ordering relative to i that is incoming in a node of K. Let B be the set of arcs defined by $B := \{b(i, V) \in A : \ i \in V\}$. For every arc $(i, j) \in A \setminus B$ we denote by $a(i, j)$ the arc that immediately precedes (i, j) in the ordering with respect to i. The incremental cost c_{ij} associated with each arc $(i, j) \in A \setminus B$ is defined as $c_{ij} := p_{ij} - p_{a(i,j)}$, while we set $c_{ij} := p_{ij}$ for each arc $(i, j) \in B$. The Wireless Multicast Advantage (WMA) (see [8]) consists in the following property: all the nodes that are within the transmission range of a transmitting node receive its signal. Therefore, several nodes can be reached at the same time without increasing the transmission power. For each arc $(i, j) \in A$ let y_{ij} be a binary variable which assumes value 1 if and only if $\rho(i) \geq p_{ij}$. In this case we say that the arc (i, j) is active. Hence, the objective function that we want to minimize is $\sum_{(i,j)\in A} c_{ij} y_{ij} = \min \sum_{i\in V} \rho(i)$. Observe that, due to the WMA property, if an arc $(i, j) \in A \setminus B$ is active then $a(i, j)$ is active too. Since any node but the source and the destinations may be, with some probability, not available for the communication, several network scenarios are possible, each one of them characterized by a different set C_l of "working" nodes. The number of such scenarios is $N := 2^{n-r-1}$. We will refer to the N sets of nodes as *configurations*. It is easy to see that the probability Q_l that a given configuration C_l is realized is given by $Q_l := \prod_{i\in C_l} q_i \cdot \prod_{i\notin C_l} (1-q_i)$. We say that a given solution y is *connective* on configuration l, if the set of active arcs outgoing from the nodes of C_l contains an arborescence rooted at s and spanning all the destinations in R. For each configuration C_l, with $l \in \{1, \ldots, N\}$, let v_l be a binary variable which is zero if and only if y is connective on configuration C_l. To ensure that solution y is feasible in every subgraph induced by the nodes of a connective configuration, we require the fulfillment of suitable connectivity constraint, strengthened by exploiting the WMA advantage. Moreover, we express the reliability requirement by means of a further constraint on the v_l variables.

To summarize, we can formulate the PMPM problem as an integer linear programming problem as follows:

$$\min \sum_{(i,j)\in A} c_{ij} y_{ij} \qquad \text{subject to}$$

$$y_{ij} \leq y_{a(i,j)} \qquad \forall (i,j) \in A \setminus B \tag{1}$$

$$\sum_{i\in S} y_{b(i,C_l\setminus S)} + v_l \geq 1 \qquad \forall l \in \{1, \ldots, N\}, \quad \forall S \subseteq C_l : s \in S, (C_l \setminus S) \cap R \neq \emptyset \tag{2}$$

$$\sum_{l=1}^{N} v_l Q_l \leq 1 - \alpha \tag{3}$$

$$y_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A \tag{4}$$

$$v_l \in \{0, 1\} \qquad \forall l \in \{1, ..., N\}\,. \tag{5}$$

2 Polyhedral structure of the problem

In this section we focus on the convex hull P of all the integer feasible solutions of formulation (1)-(5). Suppose, without loss of generality, that for each node i there is not any pair of nodes j and h such that $p_{ij} = p_{ih}$. First, it is easy to observe that P lies in the hyperplane defined by $y_{b(s,V)} = 1$ and, thus, the variable relative to the arc $b(s, V)$ can be eliminated. Furthermore, if a configuration l is such that $Q_l > 1-\alpha$ then, in view of constraint (3), v_l is forced to assume value 0. All the variables corresponding to such configurations may be fixed to zero and, thus, eliminated from the problem. Moreover, since the costs are always nonnegative, all the arcs incoming in s and their associated variables can be eliminated (it is easy to see that they will never be in any optimal solution). With these modifications, let M be the final number of variables relative to the configurations, then the total number H of variables in the reduced formulation is $H := n(n-2) + M$. Does the reduced formulation induce a full-dimensional polytope P'? In the following proposition we give a sufficient condition under which the convex hull P' of the feasible solutions of the reduced formulation (1)-(5) is full-dimensional.

Proposition 2.1 *Let i^* be the head node of the arc $b(s, V)$. If $q_{i^*} \geq \alpha$, then $dim(P') = H$.*

Proof. Let $q_{i^*} \geq \alpha$, we show that there are $H+1$ affinely independent feasible solutions of the problem constructing a non singular square block matrix W of dimension H whose rows are incidence vectors of feasible solutions of the problem. The first block is composed by $n-2$ columns relative to the arcs outgoing from node s which are different from the arc $b(s, V)$ and ordered in an increasing way with respect to the powers. Then the following $n-1$ blocks are relative to the other nodes of the network. Each of these blocks is constituted by $n-2$ columns relative to the arcs outgoing from the corresponding node ordered again in function of the powers. The last block has M columns associated with the configurations. The matrix W is thus the following

$$W := \begin{pmatrix} E_{M\times t} & E_{M\times t} & E_{M\times t} & \cdots & E_{M\times t} & I_M \\ T_t & E_{t\times t} & E_{t\times t} & \cdots & E_{t\times t} & F_{t\times M} \\ E_{t\times t} & T_t & 0_{t\times t} & \cdots & 0_{t\times t} & 0_{t\times M} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ E_{t\times t} & 0_{t\times t} & 0_{t\times t} & \cdots & T_t & 0_{t\times M} \end{pmatrix}, \tag{6}$$

where, for sake of simplicity of notation, t is equal to $n-2$. The blocks have the following structure: $E_{k\times h}$ is the $k \times h$ matrix with all the elements equal to one, I_k is the identity matrix of dimension k, $0_{k\times h}$ is the $k \times h$ zero matrix and T_k is the square matrix of dimension k of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{7}$$

Each row of the $t \times M$ block $F_{t\times M}$ is associated with one arc outgoing from node s excluded arc $b(s, V)$. The rows are ordered according to the ordering of the arcs. For each arc (s, i) preceding $b(s, R)$, the corresponding row of F has entry 1 on each column corresponding to a configuration C_l such that for each $j \in C_l$ it holds: $p_{sj} \geq p_{si}$, zero otherwise. It is possible to verify that W is non singular and, because of the condition $q_{i*} \geq \alpha$, all of its rows are incidence vectors of feasible solutions of the problem. Furthermore, the vector having the first $n(n-2)$ components equal to 1 and the following M components equal to zero is also the incidence vector of a feasible solution of the problem, which is affinely independent with respect to the rows of W. Thus, there are $H + 1$ affinely independent feasible solutions and hence, the thesis follows. □

2.1 Valid inequalities

In this section we provide a family of valid inequalities for the polyhedron defined by formulation (1)-(5). For this purpose, we start by an example.

Example 2.2 Consider the complete directed graph with source node s, destination nodes $R = \{d_1, d_2\}$ and intermediate nodes i, j, k. Moreover, consider the configurations $C_1 = \{s, i, j, d_1, d_2\}$, $C_2 = \{s, i, k, d_1, d_2\}$ and $C_3 = \{s, j, k, d_1, d_2\}$. These configurations define several inequalities of type (2). For each $l \in \{1, 2, 3\}$ we select the subsets $S_l = C_l \setminus R$. The inequalities (2) relative to these sets are

$$y_{b(s,R)} + y_{b(i,R)} + y_{b(j,R)} + v_1 \geq 1 \tag{8}$$
$$y_{b(s,R)} + y_{b(i,R)} + y_{b(k,R)} + v_2 \geq 1 \tag{9}$$
$$y_{b(s,R)} + y_{b(j,R)} + y_{b(k,R)} + v_3 \geq 1 \tag{10}$$

respectively. Summing these inequalities, dividing by 2 and rounding up all the coefficients to the nearest integer, we obtain that the inequality

$$2y_{b(s,R)} + (y_{b(i,R)} + y_{b(j,R)} + y_{b(k,R)}) + v_1 + v_2 + v_3 \geq 2 \tag{11}$$

is valid. Considering an additional node h, we obtain the following configurations: $C_1 = \{s, i, j, d_1, d_2\}$, $C_2 = \{s, i, k, d_1, d_2\}$, $C_3 = \{s, j, k, d_1, d_2\}$, $C_4 = \{s, i, h, d_1, d_2\}$, $C_5 = \{s, j, h, d_1, d_2\}$ and $C_6 = \{s, k, h, d_1, d_2\}$. Select the subsets $S_l = C_l \setminus R$ for each $l \in \{1, 2, \cdots, 6\}$. Regrouping the configurations in a suitable way, we obtain the following inequalities of type (11):

$$2y_{b(s,R)} + (y_{b(i,R)} + y_{b(j,R)} + y_{b(k,R)}) + v_1 + v_2 + v_3 \geq 2; \tag{12}$$
$$2y_{b(s,R)} + (y_{b(i,R)} + y_{b(j,R)} + y_{b(h,R)}) + v_1 + v_4 + v_5 \geq 2; \tag{13}$$
$$2y_{b(s,R)} + (y_{b(i,R)} + y_{b(k,R)} + y_{b(h,R)}) + v_2 + v_4 + v_6 \geq 2; \tag{14}$$
$$2y_{b(s,R)} + (y_{b(j,R)} + y_{b(k,R)} + y_{b(h,R)}) + v_3 + v_5 + v_6 \geq 2. \tag{15}$$

If we sum these inequalities, divide by 3 and round all the coefficients up, then we obtain the valid inequality:

$$3y_{b(s,R)} + (y_{b(i,R)} + y_{b(j,R)} + y_{b(k,R)} + y_{b(h,R)}) + \sum_{l=1}^{6} v_l \geq 3. \tag{16}$$

Generalizing the results obtained in Example 2.2, we can prove the following proposition:

Proposition 2.3 *For any $m \in \mathbb{N}$ such that $2 \leq m \leq n - r - 2$ and for any subset $U \subseteq V \setminus (R \cup \{s\})$ such that $|U| = m + 1$, let $\mathcal{C}(U)$ be the set of configurations*

$$\mathcal{C}(U) := \{l \in \{1, .., N\} : |C_l| = r + 3,\ |C_l \cap U| = 2\}, \tag{17}$$

then the inequality

$$my_{b(s,R)} + \sum_{i \in U} y_{b(i,R)} + \sum_{l \in \mathcal{C}(U)} v_l \geq m \tag{18}$$

is valid for formulation (1)-(5).

Proof. Let (y^*, v^*) be an integer feasible solution of formulation (1)-(5). We show by induction on m that (18) is a valid inequality. For $m = 2$, let $\overline{U}$ be a proper subset of V such that $|\overline{U}| = 3$, and $\overline{U} \cap (R \cup \{s\}) = \emptyset$. For all the $l \in \mathcal{C}(\overline{U})$, consider the configurations C_l and their relative subsets $S_l = C_l \setminus R$. We observe that the number of configurations of the set $\mathcal{C}(\overline{U})$ is 3. The three inequalities of type (2), relative to the configurations C_l and to the subsets S_l, is satisfied by the optimal solution and hence, summing these inequalities, dividing the resulting inequality by 2 and rounding up all the coefficients to the nearest integer, we have:

$$2y^*_{b(s,R)} + \sum_{i \in \overline{U}} y^*_{b(i,R)} + \sum_{l \in \mathcal{C}(\overline{U})} v^*_l \geq 2 \tag{19}$$

and therefore (18) is satisfied for $m = 2$. For the induction step, suppose that the inequality (18) is satisfied for $m = k - 1$ with $k - 1 < n - r - 2$ and for every subset U with $|U| = k$ and $U \cap (R \cup \{s\}) = \emptyset$. We prove the thesis for $m = k$.

Let $\overline{U}$ be a proper subset of V such that $|\overline{U}| = k+1$ and $\overline{U} \cap (R \cup \{s\}) = \emptyset$. Consider all the configurations C_l with $l \in \mathcal{C}(\overline{U})$ and the subsets $S_l = C_l \setminus R$. Consider now all the subsets U_j of the set $\overline{U}$ of cardinality k and the corresponding sets $\mathcal{C}(U_j)$.

For each of the $\binom{k+1}{k} = k + 1$ subsets U_j, for the inductive hypothesis, the inequality (18) relative to $U = U_j$ and $m = k - 1$ is valid. We observe that every element of $\overline{U}$ is contained in $\binom{k+1}{k} - 1 = k$ subsets U_j and that each pair of elements of $\overline{U}$ is contained in $\binom{k+1}{k} - 2 = k - 1$ subsets U_j. Furthermore, we easily have that the union of the sets $\mathcal{C}(U_j)$ for $j = 1, \ldots, k+1$ is equal to $\mathcal{C}(\overline{U})$. If we sum all the inequalities (18) relative to all the subsets U_j of $\overline{U}$, we have that the feasible solution verifies

$$(k^2 - 1)y^*_{b(s,R)} + k \sum_{i \in \overline{U}} y^*_{b(i,R)} + (k-1) \sum_{i \in \mathcal{C}(\overline{U})} v^*_i \geq (k^2 - 1). \tag{20}$$

Dividing by k and rounding all the coefficients up, we have that the inequality

$$k y_{b(s,R)} + \sum_{i \in \overline{U}} y_{b(i,R)} + \sum_{l \in \mathcal{C}(\overline{U})} v_l \geq k \tag{21}$$

is satisfied by the integer feasible solution (y^*, v^*). For the arbitrariness of the set $\overline{U}$, by induction, the proposition follows. □

In Example 2.2 the sets S_l do not contain any destination. We can generalize the proposition by including at most one destination in the sets S_l. We start with the following example:

Example 2.4 With reference to the graph of Example 2.2, consider the configurations $C_1 = \{s, i, j, d_1, d_2\}$, $C_7 = \{s, i, d_1, d_2\}$ and $C_8 = \{s, j, d_1, d_2\}$, and the sets $S_1 = \{s, i, j, d_1\}$, $S_7 = \{s, i, d_1\}$, $S_8 = \{s, j, d_1\}$, then the inequality

$$2(y_{sd_1} + y_{d_1d_2}) + y_{id_2} + y_{jd_2} + v_1 + v_7 + v_8 \geq 2, \tag{22}$$

constructed in the same way as (11), is valid.

Proposition 2.5 *For any $m \in \mathbb{N}$ such that $2 \leq m \leq n - r - 1$ and for any $U \subset V$ such that $|U| = m + 1$, $s \notin U$, and $U \cap R = \{d\}$, let $\mathcal{C}'(U)$ be the set $\mathcal{C}'(U) := \{l \in \{1, .., N\} : C_l \setminus (\{s\} \cup R) \subseteq U, 2 \leq |C_l \cap U| \leq 3\}$ then the inequality*

$$m(y_{b(s,R\setminus\{d\})} + y_{b(d,R\setminus\{d\})}) + \sum_{i \in U\setminus\{d\}} y_{b(i,R)} + \sum_{l \in \mathcal{C}'(U)} v_l \geq m$$

is valid for formulation (1)-(5).

Proof. The proof is similar to that of Proposition 2.3. □

References

[1] J. Barta, R. Montemanni, V. Leggieri, P. Nobili, and C. Triki. Minimum power multicasting in wireless networks under probabilistic node failures. *Computational Optimization and Applications*, to appear.

[2] A.K. Das, R.J. Marks, M. El-Sharkawi, P. Arabshahi, and A. Gray. Minimum power broadcast trees for wireless networks: integer programming formulations. In *Proceedings of the IEEE Infocom 2003 Conference, San Francisco, CA, March 30 - April 3*, 2003.

[3] S. Guo and S. Yang. Energy-aware multicasting in wireless ad hoc networks: a survey and discussion. *Computer Communications*, 30:2129–2148, 2007.

[4] V. Leggieri, P. Nobili, and C. Triki. Minimum power multicasting problem in wireless networks. *Mathematical Methods of Operations Research*, 68:295–311, 2008.

[5] M. Min and A. Chinchuluun. Optimization in wireless networks. In *Handbook of Optimization in Telecommunication* (M.G.C. Resende and P.M. Pardalos eds.), pages 891–915. Springer, 2006.

[6] M. Min, O. Prokopyev, and P.M. Pardalos. Optimal solutions to minimum total energy broadcasting problem in wireless ad hoc networks. *Journal of Combinatorial Optimization*, 11:59–69, 2006.

[7] R. Montemanni and L.M. Gambardella. Exact algorithms for the minimum power symmetric connectivity problem in wireless networks. *Computers and Operations Research*, 32(11):2891–2904, 2005.

[8] J. Wieselthier, G. Nguyen, and A. Ephremides. On the construction of energy-efficient broadcast and multicast trees in wireless networks. In *IEEE Infocom Proceedings*, pages 585–594, 2000.