Absence of inelastic collapse for a 1D gas of grains with an internal degree of freedom

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Abstract

In a cooling gas of rigid particles interacting with a constant coefficient of restitution, groups of particles within the gas may experience an infinite number of collisions in a finite time. This singularity, named inelastic collapse, is a shortcoming of the mathematical model, and it hampers the efforts of simulating a freely evolving, cooling granular system. After a brief review of previous works addressing the problem, we propose a one-dimensional model where a grain is seen as a pair of point masses joined by a massless, dissipative spring. We show that binary interactions of such grains are described as impacts with a constant restitution coefficient, whose expression is given in terms of the spring parameters. However, the impact is not instantaneous, but it requires a finite time. We show that in situations that would lead to inelastic collapse, multiple interactions among grains transfer kinetic energy into potential energy associated with the deformation of the springs, rather than dissipate it. This effectively avoids the collapse. Finally, we discuss the results of the simulations of a cooling granular system in comparison with other models.

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1. Introduction

It is customary to call “granular gas” a set of solid, macroscopic particles (or grains) which are in relative motion with respect to each other. The fundamental difference between an ordinary gas of molecules and a gas of grains is that in the latter interactions are dissipative: pairwise collisions of grains preserve momentum, but do not preserve kinetic energy. If no external energy is provided, a granular gas cools until it turns spontaneously into a “liquid” state, and, in the presence of gravity, it comes eventually to rest as a “granular solid”. A satisfactory mathematical description of the passage from “gas”, to “liquid”, to “solid” is still missing. In fact, it is unclear whether framing the problem as a transition across distinct “phases” is correct or useful. However, here we shall call “granular gas” a system where grains interact by pairwise collisions only. We shall say that a portion of granular media is in liquid state if in that region multiple interactions are not negligible, while granular solids will not be dealt with.
The standard approach in modeling interactions in a granular gas is based on the assumption of instantaneous collisions among grains with energy losses described by a constant restitution coefficient. In one dimension the velocities before and after the impact for identical grains are given by

\[
\begin{align*}
\dot{u}_l &= \frac{1}{2}(1 - r)u_l + \frac{1}{2}(1 + r)u_r, \\
\dot{u}_r &= \frac{1}{2}(1 + r)u_l + \frac{1}{2}(1 - r)u_r,
\end{align*}
\]  

(1)

where subscripts “\(l\)” and “\(r\)” from now on will denote, respectively, the left and the right grain. The coefficient of restitution \(r\) ranges in the interval [0, 1], with \(r = 0\) corresponding to completely inelastic collisions and \(r = 1\) to completely elastic ones.

Although some experiments are compatible with (3) in a reasonable range of impact velocities ([13], see also [14]), the dissipative Hertz model is far from completely satisfactory. To begin with, it is applicable only if the impacting surfaces are smooth, locally approximable by paraboloids (e.g. ch. 4 of [15]). If the impacting surfaces are
both flat, or if they are rough, the validity of the Hertz contact law is questionable. Furthermore, deviations from this law are also expected when grains are not homogeneous (e.g. soft crust model, sec. 2.2.2 of [16]).

An even more important shortcoming of this class of models is the assumption of instantaneous impacts. Whether the expression (3) describes adequately the dynamics of a granular gas is an open question when triple or multiple collisions are expected to occur frequently. This is the case in nearly collapsing configurations, where particles become clustered together and the time interval between impacts becomes so short as to be comparable with the actual duration of an impact.

The simplest model addressing this issue is the so-called TC model [17, 18]. The duration $t_c$ of a contact is assumed to be a basic parameter of the granular material. The restitution coefficient for the $n$-th impact of the $i$-th particle is set as follows

$$r_n^{(i)} = \begin{cases} r & \text{for } t_n^{(i)} > t_c \\ 1 & \text{for } t_n^{(i)} \leq t_c \end{cases}$$

(4)

where $t_n^{(i)}$ is the time interval between impacts $n-1$ and $n$, and $0 < r \leq 1$. The elastic interactions prescribed for $t_n^{(i)} \leq t_c$ should not be viewed as a real collision, but rather as a parameterized description of elastic waves propagating through the interacting bodies, which must be considered as being in contact.

The recognition that internal vibrations could be the main mechanism for subtracting kinetic energy to the grains has led to a more detailed approach in modeling the internal degrees of freedom of colliding grains. Zippelius et al. [19, 20] have studied a set of one-dimensional elastic rods and have derived a stochastic process describing the transfer of translational kinetic energy to vibrational degrees of freedom. The model, however, relies upon several assumptions, mainly on the equipartition of energy among vibrational modes (which are also supposed to be not in equilibrium with the translational one). This assumption is questionable, particularly when dissipation damps some wavelengths faster than others (as, for example, in the Kelvin–Voigt model of continuum mechanics, see [15], ch. 3). In its simplest form the model reduces to (1) where the restitution coefficient $r$ is a random variable sampled from a complicated probability distribution, which depends on the “temperature” of the internal degrees of freedom of each grain [21]. No inelastic collapse is observed in many-body numerical simulations using this class of models. Impacts are binary and instantaneous, under the implicit assumption that changes of the vibrational temperature of the grains take care of multiple interactions.

A different approach for describing internal degrees of freedom is currently under scrutiny: grains are modeled as hollow bodies, having smaller masses enclosed inside the cavity subjected to viscous dissipation. The rich dynamics generated by these complex grains should also be free from inelastic collapse [22].

In this paper we also take into account the presence of internal vibrational degrees of freedom, but we prefer to do so in an explicit way. This choice on the one hand limits severely the number of vibrational degrees of freedom that we can consider (there is only one of them in each grain); on the other hand it leads to a model which is tractable without introducing simplifications and auxiliary assumptions along the road. Our model is inspired by the classical model of diatomic molecules proposed by Jeans and studied by Landau and Teller (see [23] and references therein), but where they had extremely stiff internal degrees of freedom, and soft, long-range potentials for inter-molecular interactions, we must consider compressible grains which interact with extremely stiff and short-range potentials. Among our guiding principles is that collisions, by themselves, do not dissipate energy. They merely transfer it between translational and vibrational degrees of freedom. Dissipation of energy takes place inside each single grain and acts on vibrational motions only.

The paper is organized as follows. In Section 2 the mathematical model is described in detail and the analogies with the standard model are discussed. In Section 3 we show that our model is free from inelastic collapse. In Section 4 we offer a brief comparison of the dynamics of one-dimensional granular gases as described by our model, and by the TC and the dissipative Hertz models. Section 5 summarizes our main findings.

2. A simple model for vibrating viscoelastic grains

We define as a “grain” a mechanical system made of a pair of identical point masses connected by a linear, dissipative, massless spring (Fig. 1). We denote with $x_1$ and $x_2$ the coordinates of the point masses, and with $m$
their mass. The masses of an isolated grain are subject only to the force exerted on them by the spring, thus the equations of motions are

\[ \begin{align*}
m\ddot{x}_1 &= k(x_2 - x_1 - L) + \nu(\dot{x}_2 - \dot{x}_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1 - L) - \nu(\dot{x}_2 - \dot{x}_1)
\end{align*} \]  

(5)

where \( L \) is the length at rest of the spring, \( k \) is the elastic constant, and \( \nu \) is a linear damping coefficient. To make the problem non-dimensional, we adopt \( L \) as a unit of length, and \( \sqrt{m/2k} \) as a unit of time. We introduce new non-dimensional variables \( \psi = (x_2 + x_1 - 1)/2 \) and \( \xi = (x_2 - x_1 - 1)/2 \) to separate the motion of the center of mass (described by \( \psi \)) from the vibrational motion (described by \( \xi \)). Then the equations read

\[ \begin{align*}
\ddot{\psi} &= 0 \\
\ddot{\xi} &= -\xi - 2\varepsilon \dot{\xi}
\end{align*} \]  

(6)

where \( \varepsilon = \nu/\sqrt{2km} \) is the only non-dimensional parameter needed to characterize these idealized grains. The solutions of (6) are easily found, and for \( \varepsilon \in [0, 1) \) we have

\[ \begin{align*}
\psi(t) &= Ut + C \\
\xi(t) &= Ae^{-\varepsilon t} \sin(\omega t + \phi)
\end{align*} \]  

(7)

where \( \omega = \sqrt{1 - \varepsilon^2} \).

Two adjacent grains, marked, respectively, with the subscripts “\( l \)” and “\( r \)”, collide when \( x_{2l} = x_{1r} \). We assume that the impact is instantaneous, and that total energy and momentum are preserved in the process. Thus the collision rule for the point masses of coordinates \( x_{2l} \) and \( x_{1r} \) is the same as that for purely elastic, free point masses, that is \( \dot{x}_{2l} \rightarrow \dot{x}_{1r} \) and \( \dot{x}_{1r} \rightarrow \dot{x}_{2l} \). The points in \( x_{1l} \) and \( x_{2l} \) are left unaffected by the impact. In terms of the variables \( \psi \) and \( \xi \) the collision rule translates into

\[ \begin{align*}
&\frac{1}{2} \left[ \dot{\psi}_l + \dot{\psi}_r - \dot{\xi}_l - \dot{\xi}_r \right] \rightarrow \dot{\psi}_l \\
&\frac{1}{2} \left[ \dot{\psi}_l + \dot{\psi}_r + \dot{\xi}_l + \dot{\xi}_r \right] \rightarrow \dot{\psi}_r \\
&\frac{1}{2} \left[ \dot{\psi}_r - \dot{\psi}_l - \dot{\xi}_r + \dot{\xi}_l \right] \rightarrow \dot{\xi}_l \\
&\frac{1}{2} \left[ \dot{\psi}_r - \dot{\psi}_l + \dot{\xi}_r - \dot{\xi}_l \right] \rightarrow \dot{\xi}_r.
\end{align*} \]  

(8)

Collisions with a rigid, elastic wall follow the same principles as collisions between grains: the impact is instantaneous, and preserves energy and momentum. Thus, if the wall stands to the left of the grain, the rule is \( \dot{x}_1 \rightarrow -\dot{x}_1, \dot{x}_2 \rightarrow \dot{x}_2 \), which implies \( \dot{\psi} \rightarrow \dot{\xi}, \dot{\xi} \rightarrow \dot{\psi} \). If the wall stands to the right of the grain, the rule is \( \dot{x}_1 \rightarrow \dot{x}_1, \dot{x}_2 \rightarrow -\dot{x}_2 \), or \( \dot{\psi} \rightarrow -\dot{\xi}, \dot{\xi} \rightarrow -\dot{\psi} \). We notice that a collision with a wall is equivalent to a collision with a “mirror image” of the grain, with point masses moving at velocities equal in magnitude and opposite in sign. The instant of collision is determined by solving for \( t \) the equation \( x_{2l}(t) - x_{1r}(t) = 0 \). By using (7) and the relationships \( x_{2l} = \psi_l + \xi_l + 1, x_{1r} = \psi_2 - \xi_2 \), we find that the time of contact is the first positive time (if any) such that

\[ F(t) = Be^{-\varepsilon t} \sin(\omega t + \phi) - Vt - D = 0 \]  

(9)

where \( V = U_r - U_l, D = C_r - C_l - 1, B = \sqrt{A_r^2 + A_l^2 + 2A_rA_l \cos(\phi_r - \phi_l)}, \tan(\theta) = (A_r \sin(\phi_r) + A_l \sin(\phi_l))/(A_r \cos(\phi_r) + A_l \cos(\phi_l)) \). Details on the numerical algorithm used to find the time of contact are given in the Appendix.
Fig. 2. Impact of two grains initially moving rigidly. The impact takes a finite time and involves two consecutive collisions of the inner point masses. Here and in the following figures the edges of the shaded bands mark the positions of the point masses of the grains. Shade differences are intended to help the eye and have no physical meaning. (For color images, the reader is referred to the web version of this article.)

Because the vibrational degree of freedom of each grain is damped, one expects that colliding grains are moving almost rigidly in rarefied gases where collision frequencies are much lower than $\varepsilon$. Then it is interesting to study impacts where initially $\xi_l = \dot{\xi}_l = \xi_r = \dot{\xi}_r = 0$. According to (8), after the collision, both grains have the same translational velocity $\psi_0 = \frac{1}{2}[\psi_l + \psi_r]$, and the same negative vibrational velocity $\dot{\xi}_0 = \frac{1}{2}[\dot{\psi}_r - \dot{\psi}_l]$, corresponding to a compression of the spring. This fact implies that a second collision is going to occur after a time $\tau = \pi/\omega$, which is one half of the proper period of the spring. At that time the translational and vibrational velocities of the grains are, respectively, $\dot{\psi}_0$ and $-\dot{\xi}_0 e^{-\varepsilon \tau}$. Applying once again (8) one finds that translational velocities after the second collision are linked to those before the first collision by the linear combination

$$
\dot{\psi}_{\text{end}}^l = \frac{1}{2} \left( 1 - e^{-\varepsilon \tau} \right) \dot{\psi}_l + \frac{1}{2} \left( 1 + e^{-\varepsilon \tau} \right) \dot{\psi}_r \\
\dot{\psi}_{\text{end}}^r = \frac{1}{2} \left( 1 + e^{-\varepsilon \tau} \right) \dot{\psi}_l + \frac{1}{2} \left( 1 - e^{-\varepsilon \tau} \right) \dot{\psi}_r
$$

(10)

and that both grains are moving rigidly again. The entire “double-bounce” interaction is illustrated in Fig. 2. Eq. (10) is formally identical to the standard model (1), if the coefficient of restitution is defined by

$$r(\varepsilon) = e^{-\varepsilon \tau}.
$$

(11)

It is straightforward to check that interactions with a wall also lead to a double-bounce, with an effective coefficient of restitution given by (11). The only difference with the standard model is that impacts are not instantaneous: Eq. (10) links events separated by a time $\tau$. The two collisions of the point masses $x_{2l}$ and $x_{1r}$ should not be regarded as separate impacts of the left and the right grain, but as the beginning and the end of a single impact which lasts for a time $\tau$, during which a fraction of the vibrational energy is dissipated. This interpretation is consistent with the dynamics of two colliding elastic rods with the same density, length and Young modulus (cfr. [24]). Furthermore, the present model avoids the complications arising when the surfaces of colliding grains maintain contact for a finite time, and allows for simulations with event-driven algorithms.

In this sense our model allows for multiple interactions. Fig. 3 shows an example of triple impact occurring when a pair of grains (on the left) is unable to complete the double-bounce before experiencing a collision with a third particle (coming from the right). The whole interaction is accomplished with four collisions (different initial conditions would lead to a different number of collisions): during the interval between the second and the third one all three grains should be seen as being in contact.

The limit $\varepsilon \rightarrow 0$ corresponds to purely elastic impacts. For $\varepsilon \geq 1$ Eq. (6) have overdamped solutions: the time required to complete the double-bounce is infinite, but after the first collision the center of mass of both grains moves with the same velocity. The dynamics then becomes that of sticky grains, and we believe this model may be able to reproduce the results of [25], although this is outside the scope of the present paper.

In real granular systems an upper limit exists for the energy of impacts, beyond which plastic deformations take place, or grain fragmentation occurs. Similarly, the present model loses validity if the length of the internal spring
shrinks to zero or to negative values. To avoid meaningless dynamics, initial conditions have to be selected with energy as low as to avoid crushing of the grains.

3. Absence of inelastic collapse

Two classes of collapsing solutions are well-studied in the literature: the cushion and the self-similar set-up (see, respectively, Refs. [1] and [26]). In the first case $N$ grains are arranged to the right of a wall. The positions of the centers of mass are initially $x_i = x_1 + (i - 1)\Delta, \ i = 1 \ldots N$, and the velocities are $\dot{x}_1 = \cdots = \dot{x}_{N - 1} = 0, \dot{x}_N = -U$. The constants $x_1, \Delta$ and $U > 0$ are free parameters, and $\Delta$ must be larger than the diameter of the grains. In the second case there is a time $\bar{t}$ such that $x_i(t + \bar{t}) = \alpha x_i(t), \dot{x}_i(t + \bar{t}) = \beta \dot{x}_i(t)$. The parameters $\alpha, \beta \in (0, 1)$ satisfy $\alpha < \beta$. An explicit expression for the initial conditions leading to this kind of behaviour, and for the values of $\alpha$ and $\beta$ as functions of $r$ and $N$, is given in [26].

We simulated the dynamics, as described in Section 2, starting from the cushion and the self-similar initial conditions, with vibrational degrees of freedom initially at rest, and we compared the results with those coming out from the standard model (1). In Figs. 4 and 5 results are shown for systems of twenty particles, with a coefficient of restitution $r = 0.7$ (equivalent, according to (11), to a dissipation parameter $\varepsilon = 0.1128 \ldots$).

Initially the dynamics appears model independent, and the only quantitative difference is a slight delay in our model, due to the finite time required by the impacts. (Note that at the scale of the figures the double bounces are not resolved.) Qualitative features, such as the shape of the collision waves in Fig. 4 are the same in both cases.

The differences become remarkable when the time interval between impacts approaches the contact duration $\tau$. In our model most of the kinetic energy changes into potential energy stored in the compressed springs (see Fig. 6). Part of this energy is then dissipated and the remaining one is used to break-up the cluster of grains. The standard model does not allow any storage of potential energy, and it dissipates all the kinetic energy in the process of inelastic collapse.

The results of numerical simulations lead us to state the following:

**Proposition 3.1.** In a system of grains modeled as described in Section 2, each grain is subject to a finite number of collisions per unit time.

To give theoretical support to the proposition let us assume that a cluster of $N$ grains undergoes inelastic collapse at time $t^*$. We choose the reference frame in which the center of mass of the cluster is at rest. According to the model of Section 2, the total mechanical energy of the cluster is

$$E(t) = \sum_{i=1}^{N} \left( \dot{\psi}^2_i + \dot{\xi}^2_i + \xi^2_i \right) / 2,$$

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Fig. 4. Panel A: Cushion collapse of 20 grains, where impacts are modeled as instantaneous and with a constant coefficient of restitution $r = 0.7$. Panel B: Non-collapsing solution starting from the same initial conditions where impacts are modeled as described in Section 2 with $\varepsilon = 0.1128 \ldots$

Fig. 5. Panel A: Self-similar collapse of 20 grains, where impacts are modeled as instantaneous and with a constant coefficient of restitution $r = 0.7$. Panel B: Non-collapsing solution starting from the same initial conditions where impacts are modeled as described in Section 2 with $\varepsilon = 0.1128 \ldots$

Fig. 6. Time evolution of mechanical energies for the simulation of Fig. 5B. The thick line is the kinetic energy of the translational modes (T.K.E.); the dotted line is the kinetic energy of the vibrational modes (V.K.E.); the thin line is the potential energy of the vibrational modes (V.P.E.).

with $t < t^*$, $E(t) > 0$ and $\dot{E}(t) \leq 0$. Using (6) we compute the power dissipated by the spring, which is

$$-P(t) = 2\varepsilon \sum_{i=1}^{N} \dot{\xi}_i^2 \leq 4\varepsilon E(t),$$

and the energy dissipated in the time interval $(t, t^*)$ is

$$E_D(t) = -\int_{t}^{t^*} P(t') \, dt' \leq 4\varepsilon E(t)(t^*-t).$$
Fig. 7. Time evolution of the mechanical energy in simulations comparing the quasielastic limit of the model of Section 2, of the TC model, and of the dissipative Hertz model. The first two curves are perfectly superimposed. Reference lines scaling as the theoretical limits are also shown.

If we assume that \( E(t) \to 0 \) as \( t \to t^* \), then it has to be \( E_D(t) = E(t) \), implying \( 4\epsilon(t^* - t) \geq 1 \), which is false for \( t \) sufficiently close to \( t^* \). Thus, if collapse were to happen, the residual mechanical energy should be \( E(t^*) > 0 \). However, if we approximate the positions of the point masses of a grain with Taylor series, we obtain

\[
x_{1i}(t) = x_{1i}(\bar{t}) + \dot{x}_{1i}(\bar{t})(t - \bar{t}) - \ddot{x}_{1i}(\bar{t})(t - \bar{t})^2/2 + \cdots,
\]

and

\[
x_{2i}(t) = x_{2i}(\bar{t}) + \dot{x}_{2i}(\bar{t})(t - \bar{t}) + \ddot{x}_{2i}(\bar{t})(t - \bar{t})^2/2 + \cdots;
\]

simple geometrical considerations suggest that two consecutive collisions of the same point mass must be separated by a time of the order

\[
\Delta t \approx \left| 2\dot{x}_{ji}(\bar{t})/\ddot{x}_{ji}(\bar{t}) \right|,
\]

\( j = 1, 2 \). But \( \ddot{x}_{ji} \) are bounded functions of time (cfr. Eq. (7)), hence \( \Delta t \) vanishes only for vanishing velocities of the point masses, and this appears to be in contradiction with the existence of a residual mechanical energy.

4. Comparison with other models

In this section we offer a brief and non-exhaustive comparison between our model, the dissipative Hertz model (Eq. (3)), and the TC model (Eq. (4)). In all cases, grains are initially randomly and uniformly distributed in the whole domain, taking care to avoid overlaps. Initial velocities, expressed in non-dimensional units, are randomly and uniformly distributed in \((-0.01, 0.01)\). When using our model, vibrational degrees of freedom initially are not excited.

In order to make the simulations comparable, we choose the various parameters of the models as follows. Each group of simulations is characterized by the same restitution coefficient \( \epsilon \). Then, when using our model we solve for \( \epsilon \) Eq. (11); when using the dissipative Hertz model we solve for \( U_0 \) Eq. (3), having set \( U \) equal to the root mean squared velocity of the initial conditions; when using the TC model we set \( t_c = \pi/\sqrt{1 - \epsilon^2} \) in Eq. (4), which is, according to our model, the duration of the impact between two grains.

In the first set of simulations we check the models against quasielastic theories. We use 200 grains with \( \epsilon = 0.99 \), which does not satisfy the collapse condition (2). The domain is confined between rigid walls, and has a length of 2000 non-dimensional units. The time evolution of mechanical energy is shown in Fig. 7. For our model only the kinetic energy of translational modes is shown. We checked that the amount of energy associated with vibrational modes is negligible through the whole simulation. The results of our model are indistinguishable from those of the TC model (this is confirmed also by the collision count: our model performs almost exactly twice as many collisions as the TC model, which is expected when grains have enough time to perform double-bounce impacts, like that of Fig. 2). In both cases the energy decays exactly in the same way with an asymptotic scaling proportional to \( t^{-2} \), in agreement with the theory (cfr. e.g. [27]). The dissipative Hertz model behaves differently, and in this case the theory predicts an
asymptotic energy decay proportional to $t^{-5/3}$ (cfr. e.g. ch. 11 of [28]), which is recovered by our simulation. For the whole length of the simulations no signs of clustering can be seen in the spatial distribution of the grains.

The second set of simulations uses the same number of grains, initial conditions and domain length as the first one, but we set $r = 0.9$, to satisfy the collapse condition (2). The time evolution of mechanical energy is shown in Fig. 8. Up to about $t = 10^6$ the energy for all three models decays in the same way. During this time a dense cluster of particles forms roughly in the center of the domain. All particles in the cluster have about the same velocity, which is approximately $-10^{-4}$ nondimensional units. The sharp drop in energy visible just after $t = 10^7$ marks the impact of this cluster with the left wall. In the simulations where we use the TC and the dissipative Hertz model the cluster sticks to the wall; gradually, all grains left isolated on the right side of the domain fall into the cluster, triggering further small drops of energy. When we use our model, the drop in energy is less broad, and the cluster rebounds on the left wall. The second sharp drop in energy marks the impact with the right wall. Vibrational modes are strongly excited every time the translational kinetic energy drops, and in particular upon impacts with the walls. Repeating this simulations with the same parameters, but with different realizations of the random initial conditions, we have observed an ample variability in the dynamics ensuing after the formation of the cluster. Further work is needed to clearly separate the contribution of the model and that of the initial conditions.

To explore the dynamics of closely packed configurations, for the third set of simulations we choose $r = 0.9$, $N = 199$, with a domain length of only 200 non-dimensional units. The time evolution of mechanical energy is shown in Fig. 9. A noteworthy feature is the initial constancy of energy in the TC model. This happens because the ratio between the mean free path and the r.m.s. velocity of the grains is smaller than the time of contact $t_c$. Then, according to (4) almost all collisions are expected to be nondissipative. Only when statistical fluctuations allow a grain to remain isolated for a time greater than $t_c$ a dissipative collision may take place. This slight drop in the energy

Fig. 8. Time evolution of the mechanical energy in simulations with $r = 0.9$. For discussion see the text.

Fig. 9. Time evolution of the mechanical energy in simulations with $r = 0.9$, $N = 199$ and length of the domain equal to 200 non-dimensional units.
of the system, in turn, increases the probability of another grain remaining isolated for a time larger than $t_c$. Cooling apparently increases in an exponential way, until about $t = 3 \times 10^3$. After this time most collisions are dissipative, and energy decays following roughly a $t^{-2}$ law. The dynamics of the dissipative Hertz model is completely different, and qualitatively the same as in the previous set of simulations. A cluster of particles, all with very similar velocity, condensates in the center of the domain and grows at the expense of isolated particles. The sharp drops in energy happening at about $t = 10^4$ and $t = 5 \times 10^4$ mark the impact of the cluster with the left and the right wall of the domain, respectively. In our model, for the initial phases of the simulation, energy decays roughly as in the dissipative Hertz model, but some differences are evident by looking at the system in the (position, velocity) space (Fig. 10). Clustering is less marked, and a few sharp jumps in velocity propagate through the medium. At later times the energy decay is very close to that of the TC model.

This brief survey shows that ours is a viable model, with similarities and interesting differences with the TC and the dissipative Hertz model, but we defer to a future work the task of fully clarifying the issues arisen in this comparison.

5. Conclusions

We have presented a simple mechanical model for one-dimensional, viscoelastic grains. Each grain is constituted by pairs of identical point masses connected by linear, massless, dissipative springs. The model, cast in non-dimensional form, is constituted by two uncoupled equations for the translational and the vibrational modes, and by a collision rule. The grains are characterized by a single non-dimensional parameter $\varepsilon$. The collision rule is conservative; energy losses are caused by damping of internal vibrations. As a first result we have shown that this model reproduces the standard model of collisions characterized by a constant restitution coefficient in the case of binary impact. However, a time constant $\tau$ naturally arises, which we interpret, macroscopically, as a time of contact between the impacting grains. From a microscopic point of view, collisions between point masses are instantaneous, and this property allows for event-driven simulations.

The absence of inelastic collapse in this model is demonstrated by numerical simulations and by a theoretical argument showing that dissipation of energy is not fast enough to allow for collapse. In configurations leading to
collapse according to the standard model (1), our model stores potential energy in the inner springs, which is used to break-up clusterization.

We have extended the comparison to two other popular models of granular media which are free from inelastic collapse, namely the TC model and the dissipative Hertz model. Our model is indistinguishable from the TC one in the quasielastic limit, and all three models show a roughly similar dynamics with low volume fraction of grains. Strong differences among models are evident when the volume fraction is high. In this situation our model still resembles the TC one at low energies, but allows for a gradual cooling of the granular medium at high energies. Further work is needed to reach a complete understanding of the differences.

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Appendix. Finding the first positive zero of the collision function

The time of collision between two grains is given by the smallest positive zero (if it exists) of Eq. (9). Bracketing the zero is the main difficulty in order to solve the problem numerically. In our code we consider the first and the second derivative of the function $F$:

$$\dot{F}(t) = Be^{-\epsilon t} \cos(\omega t + \theta_1) - V,$$

$$\ddot{F}(t) = Be^{-\epsilon t} \sin(\omega t + \theta_2),$$

where the $\theta_1$ and $\theta_2$ are defined by the relationships

$$\tan(\theta_1) = (\omega \sin(\theta) + \epsilon \cos(\theta))/ (\omega \cos(\theta) - \epsilon \sin(\theta))$$

and

$$\tan(\theta_2) = (\omega \sin(\theta_1) + \epsilon \cos(\theta_1))/ (\omega \cos(\theta_1) - \epsilon \sin(\theta_1)).$$

The zeros of the second derivative, found at $t = (n\pi - \theta_2)/\omega$ for integer $n$, single out the position of the extrema of $\dot{F}$. The zeros of $\dot{F}$ are bracketed by consecutive pairs of extrema, and are found numerically (for ease of implementation we use Ridder’s method), in correspondence of which we find the extrema of $F$. We examine the sign of consecutive extremal pairs of $F$ for progressively large times until we find a pair of extrema that brackets a zero, or until $F$ becomes monotonic. $F$ has a zero in its monotonic region iff its linear part has a zero; in this case, the consecutive pair of times of the form $t = (n\pi - \theta)/\omega$, with integer $n$, that brackets the zero of the linear part, also brackets the zero of $F$.

Round-off errors may cause tiny overlaps of the grains. In the presence of overlaps one finds zeros of $F$ corresponding to point masses which are actually moving away from each other (i.e. de-overlapping) rather than colliding. To avoid this ambiguity we accept a zero of $F$ only if in that point $\dot{F}$ is positive. Finally, we remark that we have no claim of optimality on the algorithm sketched here, which is reported for the sake of completeness.

References


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