Asymptotic Behavior of a Bouncing Ball

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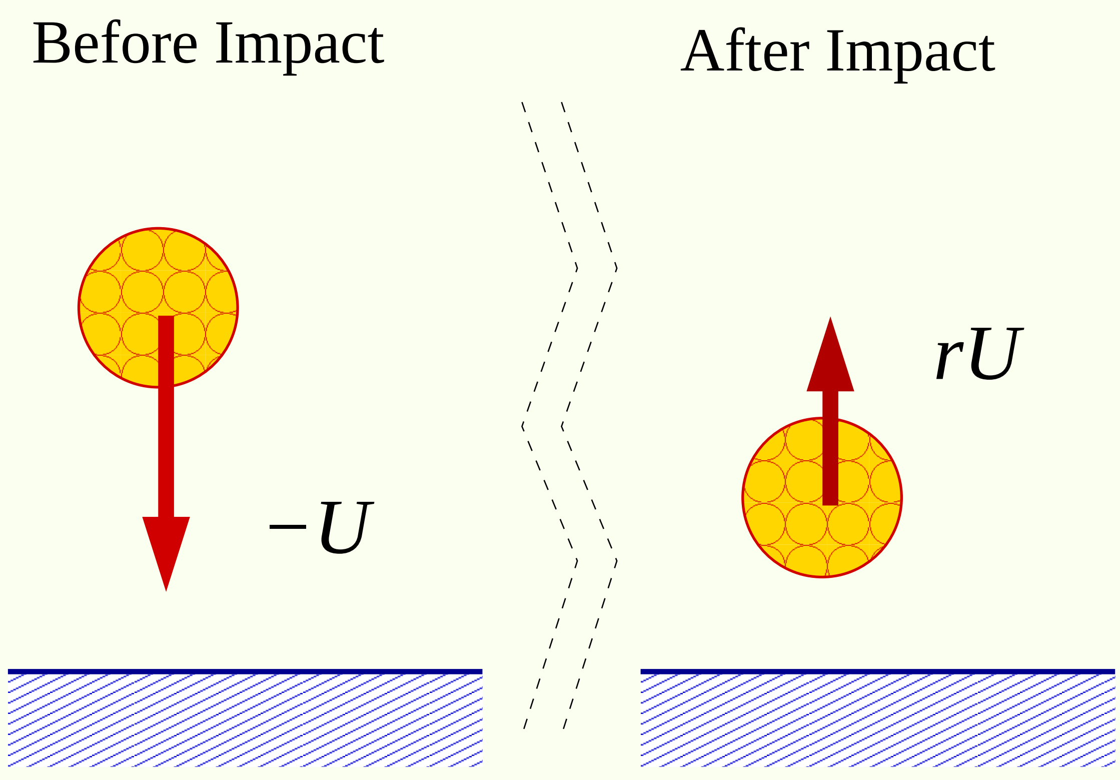
Constant Coefficient of Restitution: the "Standard Model"



$0 \leq r \leq 1$

$r = 0$ totally inelastic impact

$r = 1$ perfectly elastic impact

The Simplest Example of Inelastic Collapse

A ball repeatedly bouncing off the floor (neglect air friction):

$$\begin{aligned} \text{Time of flight :} \quad & \tau_n = \frac{2U_n}{g} \\ \text{Velocity map :} \quad & U_{n+1} = rU_n \\ \text{Sum of the times of flight :} \quad & \sum_{n=0}^{\infty} \tau_n = \frac{2U_0}{g} \sum_{n=0}^{\infty} r^n \end{aligned}$$

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\end{aligned}$$

Convergent!

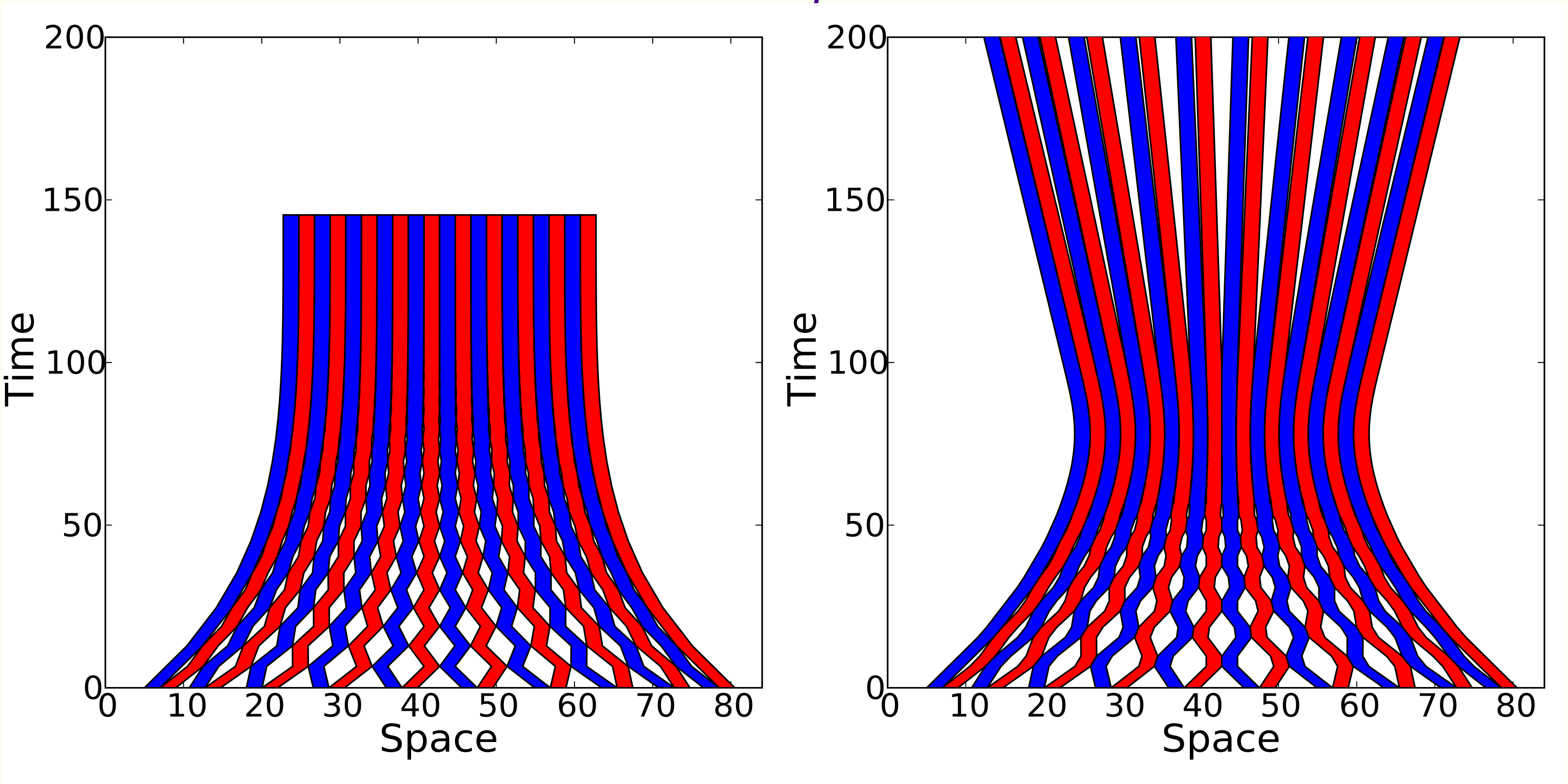
What happens after the time

$$t_\infty = \frac{2U_0}{g\,(1-r)} \quad ?$$

Inelastic Collapse

A set of grains described by the above model may be subject to the **inelastic collapse**:

"*at least one grain experiences*
an infinite number of impacts in a finite time"



Left panel: the self-similar collapse of Benedetto & Caglioti, Physica D 132 (1999), for 20 grains.

A Variable r Is Not the Cure...

It has been suggested (spherical geometry, homogeneous material, Hertz's contact law) that it should be:

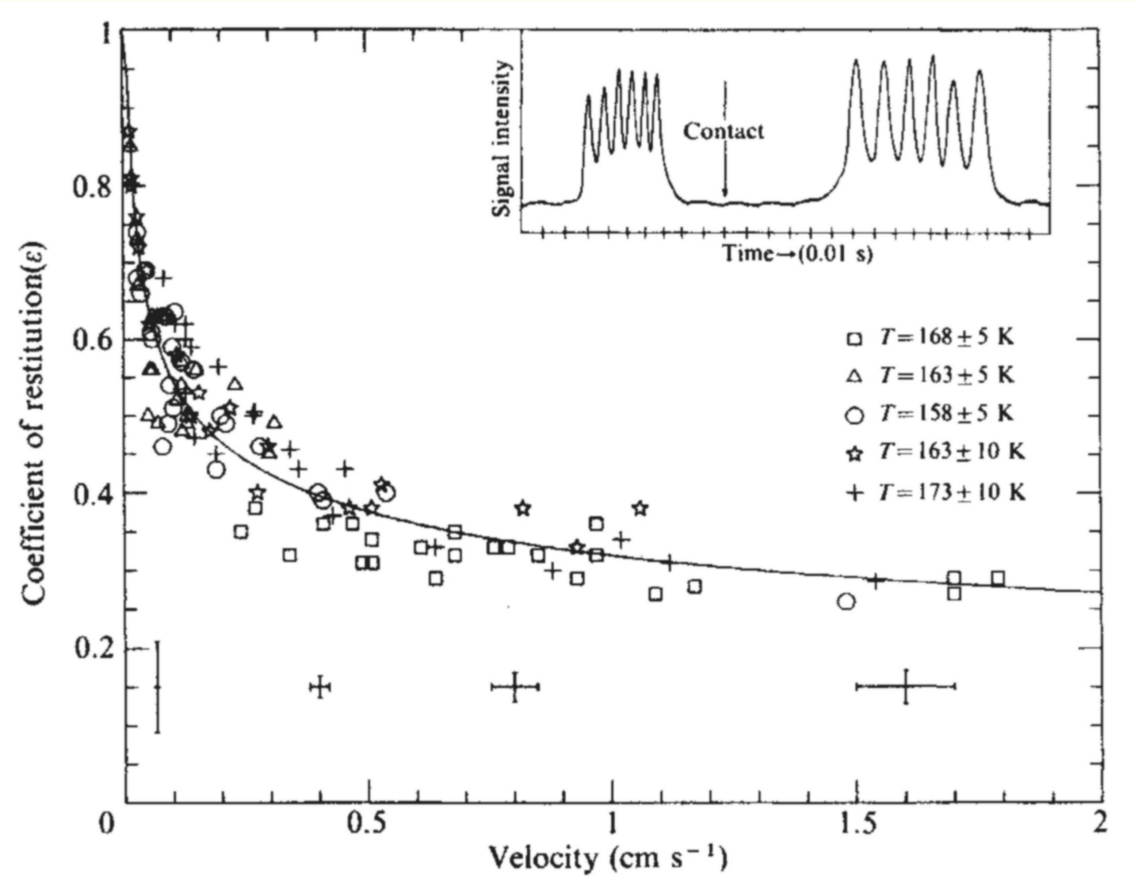
$$r(U) = 1 - \left(\frac{U}{U_c}\right)^{1/5} + O\left(\left(\frac{U}{U_c}\right)^{2/5}\right)$$

It can be proved that with this model:

- Three spheres without gravity are not affected by inelastic collapse.
- But a bouncing ball is still subject to inelastic collapse.

...But It Appears to Agree with Experiments!

From Briggs et al., Nature, **309** (1984):



Beware: these are **single impacts**, they are **not repeated bounces**!

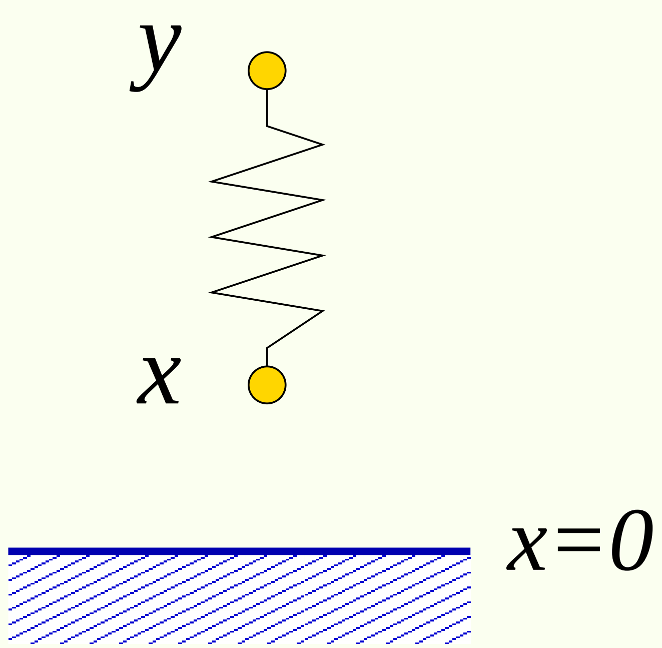
Our Model: Deformability is the Key

Non-dimensional equations of motion:

$$\begin{aligned} \ddot{x} + \mu\dot{x} + \frac{1}{2}x &= -\gamma - \frac{1}{2} + \frac{1}{2}y + \mu\dot{y} \\ \ddot{y} + \mu\dot{y} + \frac{1}{2}y &= -\gamma + \frac{1}{2} + \frac{1}{2}x + \mu\dot{x} \end{aligned}$$

or

$$\begin{aligned} \ddot{\psi} &= -\gamma \\ \ddot{\xi} &= -\xi - 2\mu\dot{\xi} \end{aligned}$$



where $\psi = \frac{1}{2}(y + x - 1)$, $\xi = \frac{1}{2}(y - x - 1)$

and the non-dimensional parameters are

$\gamma = gm/(2Lk), \qquad \mu = \nu/\sqrt{2km}$

L length at rest of the spring,

k elastic constant of the spring

ν dissipativity of the spring

m mass of each material point

Impact Rule

Let $t_0, t_1, \ldots, t_n, \ldots$ be the times such that $x(t_n) = 0$ (impact times). Then we impose

$$\dot{x}(t_n^+) = -\dot{x}(t_n^-)$$

The interaction with the floor is **instantaneous** and **conservative** (but the spring is dissipative...).

This is a *piecewise smooth* dynamical system.

Main Results

1. Except for a set of zero Lebesgue measure of initial conditions, the asymptotic time of flight is

$$\tau_n \sim \frac{3}{\mu} \cdot \frac{1}{n}$$

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$$x = \dot{x} = \dot{y} = 0, \quad y = 1 - 2\gamma$$

3. An initial condition that does not obey to 1. leads to a sticky contact: the lower point mass hits the floor with zero velocity, and maintains the contact forever, while the upper point mass performs exponentially damped oscillations.

No Inelastic Collapse!

Energy of the System

The energy is a non-increasing function of time:

$$E = \frac{\dot{\xi}^2}{2} + \frac{\xi^2}{2} + \frac{\dot{\psi}^2}{2} + \gamma\psi \qquad \frac{dE}{dt} = -2\mu\dot{\xi}^2$$

The energy has a minimum

$$E_m = -\gamma^2/2$$

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If I take an initial condition sufficiently close to the static equilibrium, then (from the equations of motion and their derivatives) it can be seen that

$$\ddot{X}_n < 0, \qquad \dddot{X}_n > 0$$

where $\ddot{X}_n = x(t_n^+)$. More precisely, they are confined in a neighborhood of -2γ and $2\mu\gamma$ respectively.

Sketch of a Proof Close to Equilibrium I

Taylor expansion of the trajectory of the lower point mass

$$0 \equiv x(t_{n+1}) = \dot{X}_n \tau_n + \ddot{X}_n \frac{\tau_n^2}{2} + \dddot{X}_n \frac{\tau_n^3}{6} + O(\tau_n^4)$$

$$-\dot{X}_{n+1} \equiv \dot{x}(t_{n+1}^-) = \dot{X}_n + \ddot{X}_n \tau_n + \dddot{X}_n \frac{\tau_n^2}{2} + O(\tau_n^3)$$

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Eliminating $\ddot{X}_n \tau_n$, we find a map for $\dot{X}_n$

$$\dot{X}_{n+1} = \dot{X}_n - \frac{\dddot{X}_n}{6} \tau_n^2$$

Sketch of a Proof Close to Equilibrium II

From

$$\dot{X}_n \tau_n + \ddot{X}_n \frac{\tau_n^2}{2} + \dddot{X}_n \frac{\tau_n^3}{6} = 0$$

we find the following approximation to the time of flight (further expanded for small $\dot{X}_n$):

$$\tau_n = -\frac{3}{2}\frac{\ddot{X}_n}{\dddot{X}_n}\left(1 - \sqrt{1 - \frac{8}{3}\frac{\dddot{X}_n \dot{X}_n}{\ddot{X}_n^2}}\right) = -2\frac{\dot{X}_n}{\ddot{X}_n}$$

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Inserting this into the map for $\dot{X}_n$ we get

$$\dot{X}_{n+1} = \dot{X}_n - \frac{2}{3}\frac{\dddot{X}_n}{\ddot{X}_n^2}\dot{X}_n^2$$

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It can be proved that this map implies

$$\dot{X}_n \sim \frac{3}{2}\frac{\ddot{X}_n^2}{\dddot{X}_n} \cdot \frac{1}{n}$$

Sketch of a Proof Close to Equilibrium III

Then

$$\tau_n \sim -3\frac{\ddot{X}_n}{\dddot{X}_n} \cdot \frac{1}{n}$$

But $\ddot{X}_n,\ \dddot{X}_n$ are in an arbitrarily small neighbour of -2γ and $2\mu\gamma$, so

$$\tau_n \sim \frac{3}{\mu} \cdot \frac{1}{n}$$

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- $x(t)$, $y(t)$ and all their derivatives are bounded. They all have a finite limit for $t \longrightarrow t_\infty$.
- The jumps across the discontinuities at the impacts are all functions of $\dot{X}_n$:

$$[\dot{x}(t_n)] = 2\dot{X}_n, \quad [\ddot{y}(t_n)] = 2\mu\dot{X}_n, \quad [\dddot{y}(t_n)] = (1 - 4\mu^2)\dot{X}_n, \quad etc.$$

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- Integrate $(k+1)-$th derivative of y from time t to collapse t_∞

$$Y_\infty^{(k)} - y^{(k)}(t) = \int_t^{t_\infty} y^{(k+1)}(s)ds + \sum_{i=n+1}^{\infty} [y^{(k)}(t_i)]$$

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- Get an estimate for the k-th derivative of y

$$y^{(k)}(t) = Y_\infty^{(k)} + Y_\infty^{(k+1)}(t-t_\infty) + O((t-t_\infty)^2) + O\left(\sum_{i=n+1}^{\infty} \dot{X}_i\right)$$

Highlights of a General Proof II

- Use the estimates in the equation of motion for x and find

$$\ddot{x}+\mu\dot{x}+\frac{1}{2}x = P-Q(t_\infty-t)+R(t_\infty-t)^2+O((t_\infty-t)^3)+O\left((t_\infty - t)\sum_{i=n+1}^{\infty}\dot{X}_i\right)$$

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- Now (using Lagrange's average function thm.) the map becomes

$$\left\{\begin{array}{lcl} \dot{X}_{n+1} & = & \dot{X}_n - \Delta_n \tau_n^2 \\ T_{n+1} & = & T_n - \tau_n \end{array}\right.$$

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- Study separately the three cases $\ddot{X}_\infty \neq 0$, $\ddot{X}_\infty = 0, \dddot{X}_\infty \neq 0$, $\ddot{X}_\infty = 0, \dddot{X}_\infty = 0$ and find a (different) contradiction in each case.

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So the assumption of inelastic collapse always leads to contradiction.

Highlights of a General Proof III

Having established that there is no inelastic collapse we may prove that there is only one asymptotic state (which is the static equilibrium), corresponding to the minimum of the mechanical energy.

First we show that

$$\lim_{t\to\infty} \dot{\xi}(t) = 0$$

(otherwise the mechanical energy would decrease without bound)

Then we show that

$$\lim_{t\to\infty} \xi(t) = -\gamma$$

using the custom Lyapunov function

$$\mathcal{L}(t) = \frac{1}{2}[(\dot{\xi} + 2\mu(\xi+\gamma))^2 + (\dot{\psi} + 2\mu(\xi+\gamma))^2 + \xi^2] + \gamma\psi$$

In practice, there are a lot of technicalities due to the presence of impacts.

Anomalous Contacts

An **anomalous contact** occurs if $\dot{x}(t_n) = 0$.
Anomalous contacts are

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$$E < \frac{3 - 2\mu^2}{2(1 + 2\mu^2)}\gamma^2$$

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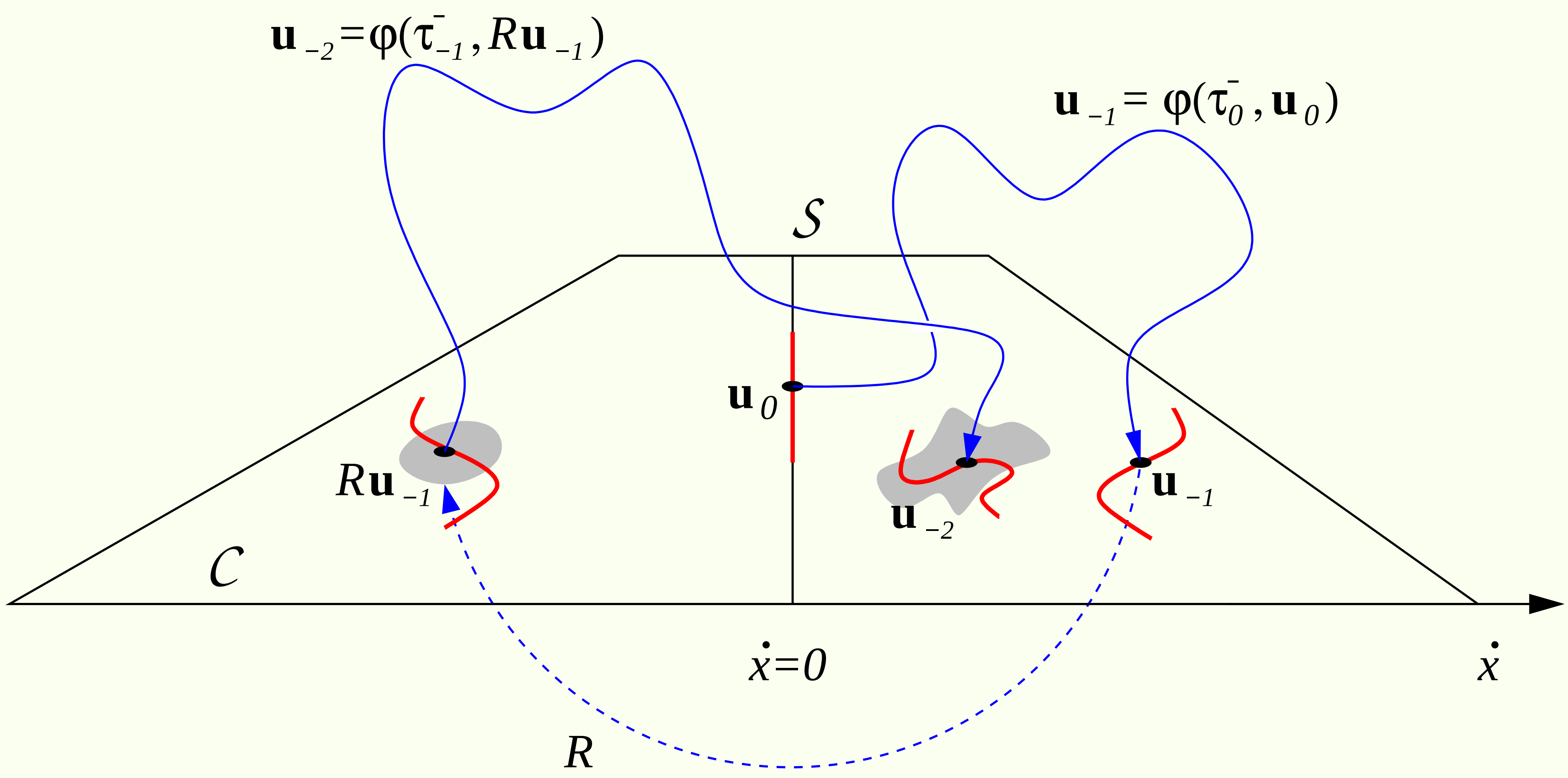
- At most a finite number of them may occur, starting from any initial condition.
- They come in two flavours:
  - **Grazings**: the trajectory of the x point mass is locally parabolic, with the minimum tangent to the floor.
  - **Sticky events**: for a finite time interval the x point mass is in contact with the floor, which reacts with an upward force.

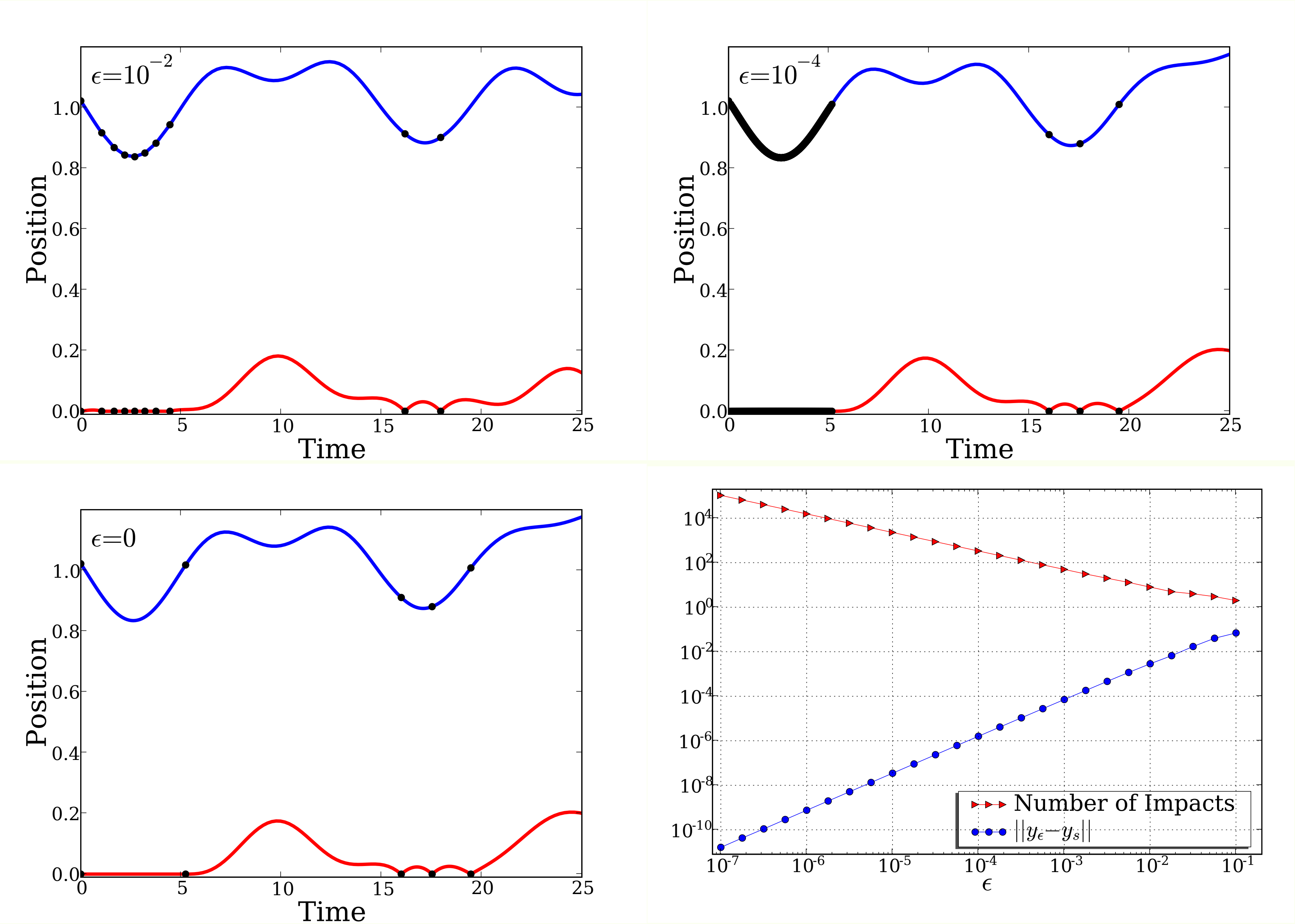
Anomalous Contacts Are Non-Generic

Anomalous contacts come from a set of initial conditions having zero measure. Here is a graphical 'proof'



A Numerically Supported Conjecture

Any sticky event may be approximated to arbitrarily high accuracy by a non-sticky trajectory.



Does the Ball Ever Stop? I

Three characteristic times:

Gravity Time

$$T_\psi = \sqrt{\frac{1}{2\gamma}}$$

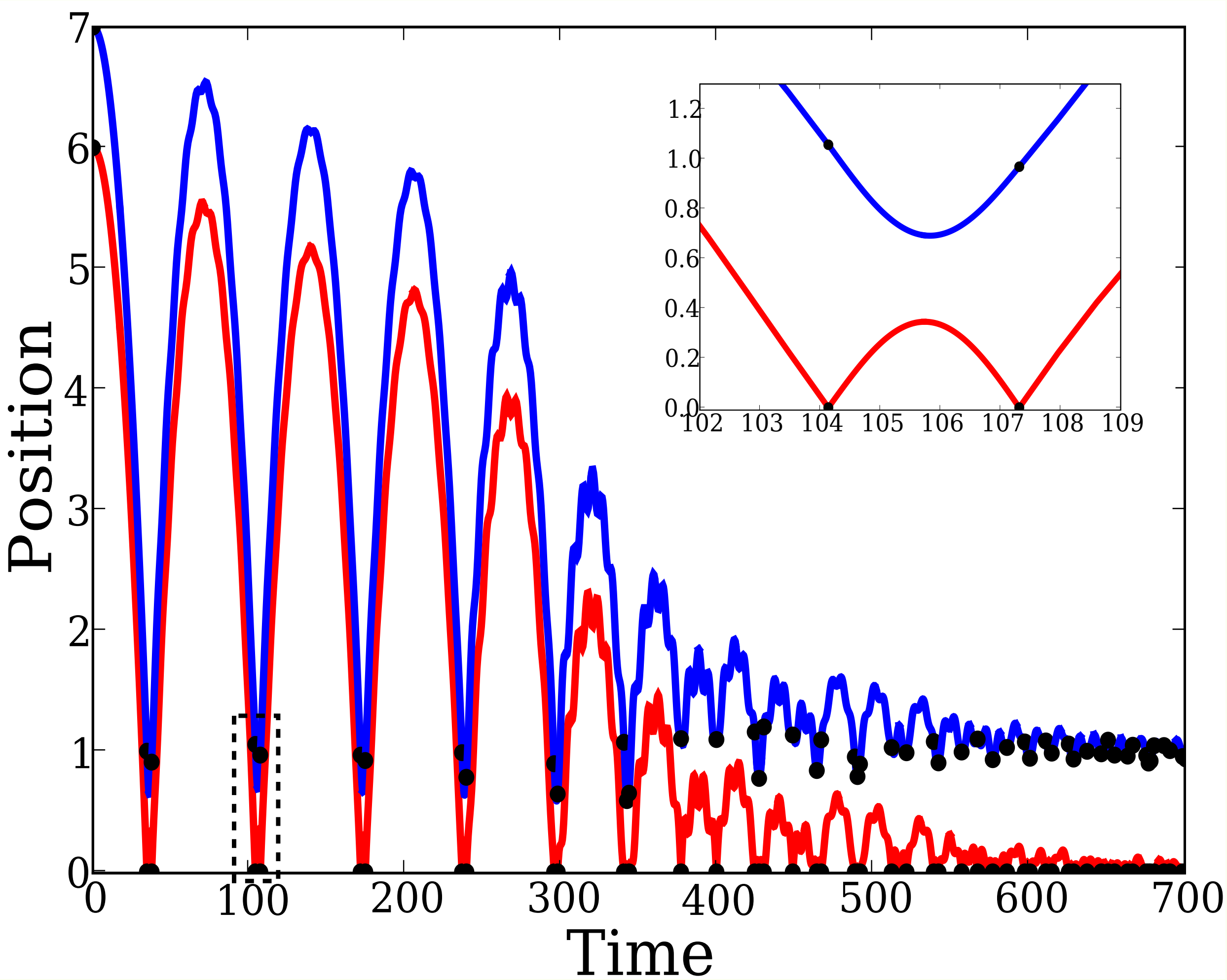
Proper Period of the Spring

$$T_\xi = \frac{\pi}{\sqrt{1-\mu^2}}$$

Damping Time

$$T_d = \frac{1}{\mu}$$

Does the Ball Ever Stop? II



"emergent" restitution coefficient:

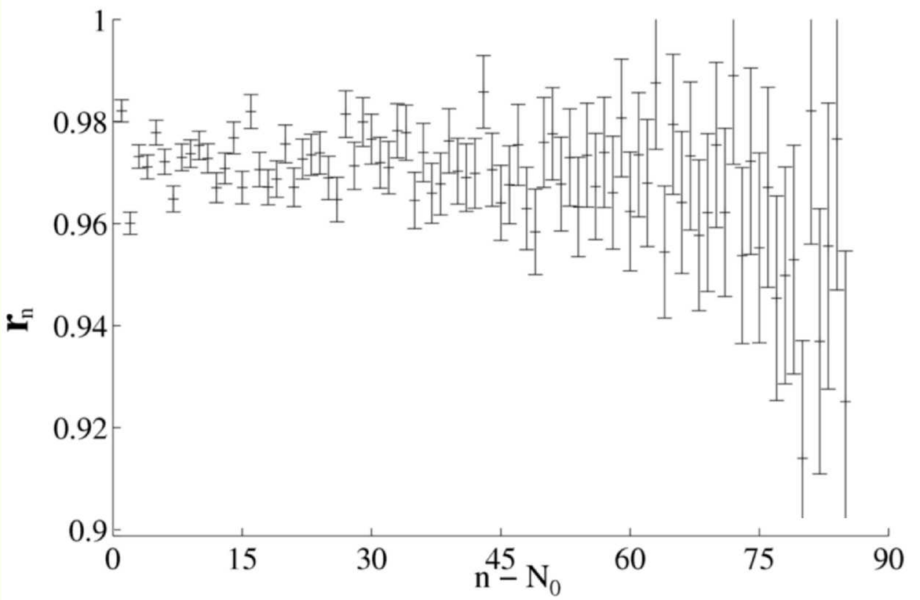
$$r = e^{-\mu T_\xi}$$

When the time of fligh is equal or less than T_ξ, the ball must be seen as mantaining "macroscopic" contact with the floor. The microscopic rebounds are interpreted as internal vibrations of the ball.

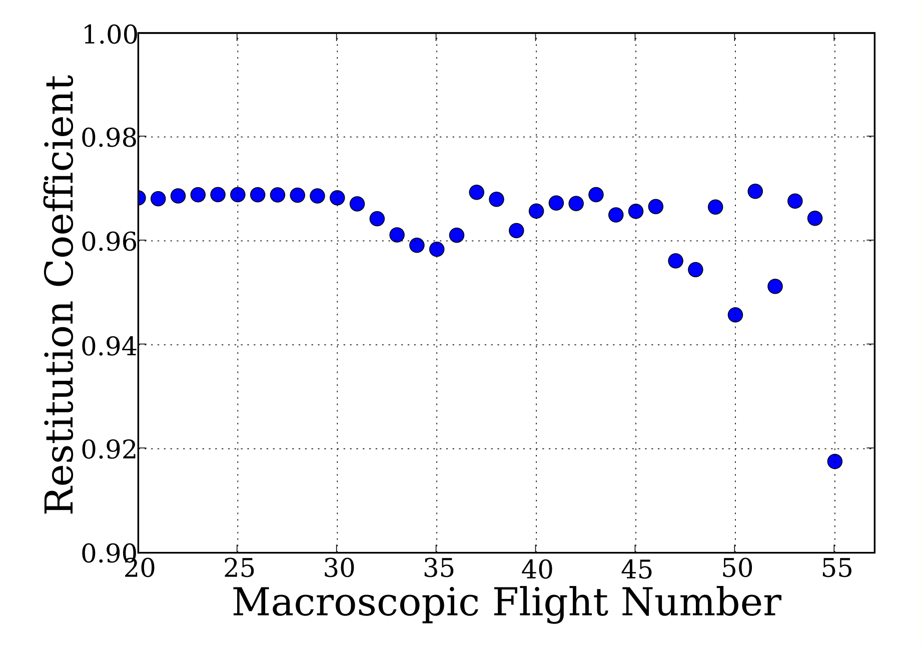
Comparison with the Experiments

Restitution coefficient measured as the ratio of length of two consecutive flights of a bouncing ball.

Tungsten carbide bead
(Falcon et. al. Europ. Phys. J. B, **3** (1998))



Our model
with $\gamma = 10^{-5}$ and $\mu = 10^{-2}$.



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- Even for very rigid materials, deformability of the ball is crucial in order to model the last rebounds of a sequence.
- The restitution coefficient is a meaningful quantity only if the time of flight is larger than the characteristic damping time.