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On the Helmholtz operator for Euler morphisms

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Abstract

The variational sequence describes the Helmholtz conditions for local variationality in terms of the Helmholtz map, which is defined on a factor space. We study a tensor modification of this construction and characterize a unique representative, which is called the Helmholtz operator. For the first and the second order cases we prove that, up to a multiplicative constant, the Helmholtz operator is the unique natural operator of the type in question.

1 Introduction

Besides the fundamental analytical aspects, one can observe several geometric phenomena in the contemporary variational calculus. In the classical theory, the differential geometrical methods were mostly related with the symmetries of variational functionals, the Noether theorem being the best known example. However, in the last two decades the global point of view to various concrete problems evoked a new approach, in which the basic objects are sections of a fiber bundle and their variations are generated by suitable vector fields. This approach clarified that several constructions of

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the variational calculus can be treated in a purely geometrical way. First of all, this is true for the Euler equations, which can be characterized in terms of a globally defined operator. A similar problem appears for the Helmholtz conditions that testify whether certain Euler-like equations are really the Euler equations of a Lagrangian. Recently, the variational bicomplex has been invented as a general machinery for such problems [2, 13, 14, 15, 17, 18]. However, the variational bicomplex defines the Helmholtz map on a factor space. On the other hand, in [3, 5, 6, 7] some morphisms of more tensorial character are used instead of exterior forms. In the present paper, we apply the latter approach to the Helmholtz conditions and we characterize a unique representative of the Helmholtz map, which is called the Helmholtz operator.

In the first part of Section 1, we extend the concept of formal exterior differential of some morphisms, which was introduced in [6, 7], to a more general situation. Then we outline the relations to the horizontal differential and the vertical differential of exterior forms on jet bundles [14]. Next, we summarize the basic facts from the theory of variational sequences on finite order jet bundles, as were developed by Krupka in [10]. The concept of formal Euler operator in Section 2 is based on some ideas from [6, 7]. If we compose it with the vertical differential of a Lagrangian, we obtain the standard Euler operator. Then we point out the role of this operator in the canonical representation of the variational sequence, which was studied in [20, 21, 22].

Section 3 starts with the definition of the Helmholtz operator H in a morphism form. In Proposition 4 we give a direct proof of the fact that $H(B) = 0$ is a necessary and sufficient condition for local variationality of an Euler morphism B. Proposition 5 is a kind of generalized second variational formula. It introduces an antisymmetric version $\tilde{H}(B)$ of $H(B)$. Then we compare the morphism approach with some other techniques used in the calculus of variations [1, 2, 3, 4, 5, 10, 11, 13, 15, 16, 17]. Theorem 2 reads that $\tilde{H}(B) = 0$ is equivalent to $H(B) = 0$, so that each of both conditions characterizes local variationality.

In Section 4 we clarify the role of the Helmholtz operator from the purely geometric viewpoint of the theory of natural operators [9]. The basic idea of this theory is that 'natural' is a precisely defined equivalent of the somewhat vague word 'geometric'. The main result of Section 4 is Theorem 3 which reads that every natural operator of the type of the second order Helmholtz operator is a constant multiple of the Helmholtz operator. The second order case is the most interesting one from the viewpoint of applications. Before that, in Proposition 7 we deduce the same result for the first order Helmholtz operator. The main reason is that the proof of the first order case expresses clearly the basic ideas of our procedure. However, we expect that the same result holds even in arbitrary order. But the higher order analogy of the proofs of Proposition 7 and Theorem 3 is technically too complicated to be discussed here.

All manifolds and maps between manifolds are assumed to be infinitely differentiable. All morphisms of fibred manifolds over the same base will be morphisms over the identity of the base manifold, unless otherwise specified. Given two fibred manifolds $Y \to X$ and $Z \to X$ over the same base, we denote by $C^\infty_X(Y, Z)$ the sheaf of all (local) smooth base preserving morphisms of Y into Z. Our main source for the sheaf theory is [23].

2 Formal exterior differential and variational sequences

Our framework is a fibred manifold

$$\pi\colon Y \to X,$$

with $\dim X = m$ and $\dim Y = n + m$. The indices i, j run from 1 to m and label base coordinates, the indices p, q run from 1 to n and label fibre coordinates. Charts on Y adapted to π are denoted by (x^i, y^p). We denote by (∂_i, ∂_p) or (d^i, d^p), respectively, the local bases of vector fields or 1–forms on Y induced by an adapted chart. We set $\omega := m! d^1 \wedge \cdots \wedge d^m$, and $\omega_i := i_{\partial_i}\omega$.

Let $Z \to X$ be a vector bundle and $F\colon Y \to Z$ be a base preserving morphism. Its vertical differential

(1) $$\delta F\colon VY \to Z$$

is defined as the second component of the vertical tangent map $VF\colon VY \to VZ = Z_X Z$. In particular, every function $f\colon Y \to \mathbb{R}$ can be interpreted as a morphism $Y \to X \times \mathbb{R}$, so that we obtain another function $\delta f\colon VY \to \mathbb{R}$. The vertical differential of a fibre coordinate y^p will be denoted by δ^p. A vertical Z–valued k–form on Y is a base preserving linear morphism $F\colon \wedge^k VY \to Z$. We define fibrewise its vertical exterior differential

(2) $$\delta F\colon \wedge^{k+1} VY \to Z.$$

For $0 \le r$, we are concerned with the r–jet space J_rY; in particular, we set $J_0Y = Y$. We recall the canonical fibrings $\pi^r_s\colon J_rY \to J_sY$, $s \le r$, $\pi^r\colon J_rY \to X$. We denote multi-indices of range m by Greek letters such as $\alpha = (\alpha_1, \dots, \alpha_m)$. We identify the index i with the multi-index with $\alpha_i = 1$ and $\alpha_j = 0$ otherwise. We set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $\alpha! := \alpha_1! \dots \alpha_m!$. The induced charts on J_rY are denoted by (x^i, y^p_α), with $0 \le |\alpha| \le r$; in particular, we set $y^p_0 = y^p$. The local vector fields or 1-forms of J_rY induced by the fibre coordinates are denoted by (∂^α_p) or (d^p_α), respectively. A section $s\colon X \to Y$ with coordinate expression $s^p(x)$ is prolonged to a section $j^rs\colon X \to J_rY$ of the coordinate form $(j^rs)^p_\alpha = \partial_\alpha s^p$.

For every function $f\colon J_rY \to \mathbb{R}$, its formal differential $Df\colon J_{r+1}Y \to T^*X$ is defined by

(3) $$Df(j^{r+1}_x s) = d_x(f \circ j^r s), \qquad x \in X,$$

where d_x means the differential at x of the function $f \circ j^r s\colon X \to \mathbb{R}$. If x^i are some local coordinates on X, we have $Df = (D_i f) d^i$, where

(4) $$D_i f = \partial_i f + y^p_{\alpha+i} \partial^\alpha_p f$$

is said to be the formal derivative of f with respect to x^i. By iteration, we set $D_{\alpha+i} f = D_i D_\alpha f$.

Every vertical vector field $u\colon Y \to VY$ induces a vector field $u_r\colon J_rY \to VJ_rY$ by prolongating its flow $\exp tu$ to J_rY, *i.e.*

$$u_r = \frac{\partial}{\partial t}\Big|_0 J_r(\exp tu),$$

[9]. If $u^p\partial_p$ is the coordinate form of u, then

$$u_r = (D_\alpha u^p)\partial_p^\alpha. \tag{5}$$

The concept of formal differential can be generalized as follows. First of all, let $f\colon J_rY \to \wedge^k T^*X$ be a base preserving morphism. Then its formal exterior differential $Df\colon\ J_{r+1}Y \to \wedge^{k+1}T^*X$ is defined analogously to (3) by

$$Df(j_x^{r+1}s) = d_x(f \circ j^r s), \qquad x \in X, \tag{6}$$

where d_x now means the exterior differential of the k–form $f \circ j^r s$ on X.

Further, for every morphism $F\colon J_rY \to \times^l V^*J_sY \times \wedge^k T^*X$ over J_sY, $s \le r$, and every l–tuple of vertical vector fields $u_{(1)}, \dots, u_{(l)}$ on Y, we have the evaluation $F(u_{(1)\,s}, \dots, u_{(l)\,s})\colon J_rY \to \wedge^k T^*X$. One verifies easily in coordinates that there exists a unique morphism $DF\colon J_{r+1}Y \to \otimes^l V^*J_{s+1}Y \otimes \wedge^{k+1}T^*X$ satisfying

$$D(F(u_{(1)\,s}, \dots, u_{(l)\,s})) = (DF)(u_{(1)\,s+1}, \dots, u_{(l)\,s+1}), \tag{7}$$

for all $u_{(1)}, \dots, u_{(l)}$. Even in this case DF will be called the formal exterior differential of F. The coordinate form of DF in the case $l = 1$ can be found in [6, 7]. Write $i_{u_s}F\colon\ J_rY \to \otimes^{l-1}V^*J_sY \otimes \wedge^k T^*X$ for the evaluation at the first factor. In the case F is antisymmetric in V^*J_sY, we obtain the classical inner product fibrewise. Then (7) yields directly

$$D(i_{u_s}F) = i_{u_{s+1}}(DF). \tag{8}$$

The vertical differential δf of a function f on J_rY can be interpreted as a map $J_rY \to V^*J_rY$. This corresponds to the case $r = s$, $k = 0$, $l = 1$. Hence $D(\delta f)\colon J_{r+1}Y \to V^*J_{r+1}Y \otimes T^*X$. On the other hand, $Df\colon J_{r+1}Y \to T^*X$ and its vertical differential in the sense of (1) can be interpreted as a map $\delta(Df)\colon J_{r+1}Y \to V^*J_{r+1}Y \otimes T^*X$. The following assertion can be easily proved.

Lemma 1. *For every $f\colon J_rY \to \mathbb{R}$, $D(\delta f) = \delta(Df)$.*

More generally, every morphism $F\colon J_rY \to \wedge^l V^*J_rY \otimes \wedge^k T^*X$ can be interpreted as a linear morphism $\wedge^l VJ_rY \to \wedge^k T^*X$. Hence we can construct $\delta F\colon J_rY \to \wedge^{l+1}V^*J_rY \otimes \wedge^k T^*X$ and then $D(\delta F)\colon J_{r+1}Y \to \wedge^{l+1}V^*J_{r+1}Y \otimes \wedge^{k+1}T^*X$. On the other hand, $DF\colon J_{r+1}Y \to \wedge^l V^*J_{r+1}Y \otimes \wedge^{k+1}T^*X$ and $\delta(DF)\colon J_{r+1}Y \to \wedge^{l+1}V^*J_{r+1}Y \otimes \wedge^{k+1}T^*X$. In a standard way, Lemma 1 implies the following general formula

$$D(\delta F) = \delta(DF). \tag{9}$$

Let $\vartheta\colon TJ_{r+1}Y \to VJ_rY$ be the contact morphism [12, 14]. The composition of $F\colon J_rY \to \wedge^l V^*J_rY \otimes \wedge^k T^*X$ with ϑ is identified with a $(k+l)$–form $\omega(F)$ on $J_{r+1}Y$, which is k–horizontal and l–contact. Then we can apply the horizontal differential d_h and the vertical differential d_v [14]. These are derivations of degree one along π^{r+2}_{r+1}, so that they can be characterized by the action on functions and 1–forms as follows

$$d_h f = D_i f\, d^i, \qquad d_h d^i = 0, \qquad d_h \vartheta^p_\alpha = -\vartheta^p_{\alpha+i} \wedge d^i,$$
$$d_v f = \partial^\alpha_p f \vartheta^p_\alpha\,, \qquad d_v d^i = 0, \qquad d_v \vartheta^p_\alpha = 0,$$

where $\vartheta^p_\alpha = d^p_\alpha - y^p_{\alpha+i} d^i$, $0 \le |\alpha| \le r+1$. We remark the property $d_h + d_v = (\pi^{r+2}_{r+1})^* \circ d$.

Using Lemma 1 one deduces in the standard way the following assertion, which compares some properties of forms and morphisms on J_rY.

Proposition 1. *We have* $\pi^{r+2*}_{r+1}\omega(\delta F) = d_v\omega(F)$ *and* $\omega(DF) = (-1)^l d_h(\omega(F))$.

Now, we recall the theory of variational sequences on J_rY, as was developed by Krupka in [10]. Let Λ^k_r be the sheaf of (local) k–forms on J_rY. The horizontalization of exterior forms is a map $h\colon \Lambda^k_r \to \Lambda^k_{r+1}$ [10, 22]. We set Θ^k_r to be the sheaf generated by the presheaf $\ker h + d\ker h$ [23]. Hence we have the following subsequence of the de Rham sequence on J_rY

$$0 \longrightarrow \Theta^1_r \xrightarrow{\;d\;} \Theta^2_r \xrightarrow{\;d\;} \dots \xrightarrow{\;d\;} \Theta^I_r \xrightarrow{\;d\;} 0$$

where I depends on the dimension of the fibers of $J_rY \to X$ [10]. It is proved in [10] that the above subsequence is exact, and the sheaves Θ^k_r are soft.

If $0 \le k \le n$, then $d\ker h \subset \ker h$, and

$$\ker h = \{\alpha \in \Lambda^k_r \mid (j^r s)^*\alpha = 0 \text{ for every section } s\colon X \to Y\};$$

this partly shows the connection of Θ^k_r with the calculus of variations [10, 18, 19, 20].

Standard arguments of homological algebra prove that the following diagram is commutative, and its rows and columns are exact.

$$\begin{array}{ccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & \Theta^1_r & \xrightarrow{d} & \Theta^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \Theta^I_r & \xrightarrow{d} & 0 & \to \dots \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{R} & \to & \Lambda^0_r & \xrightarrow{d} & \Lambda^1_r & \xrightarrow{d} & \Lambda^2_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \Lambda^I_r & \xrightarrow{d} & \Lambda^{I+1}_r & \xrightarrow{d} \dots \xrightarrow{d} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{R} & \to & \Lambda^0_r & \xrightarrow{\mathcal{E}_0} & \Lambda^1_r/\Theta^1_r & \xrightarrow{\mathcal{E}_1} & \Lambda^2_r/\Theta^2_r & \xrightarrow{\mathcal{E}_2} & \dots & \xrightarrow{\mathcal{E}_{I-1}} & \Lambda^I_r/\Theta^I_r & \xrightarrow{\mathcal{E}_I} & \Lambda^{I+1}_r & \xrightarrow{d} \dots \xrightarrow{d} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

We say the bottom row of the above diagram to be the r–th order *variational sequence* associated with the fibred manifold $Y \to X$ [10].

We remark that the r–th order variational sequence is included into the $(r+1)$–st variational sequence, and that the morphisms of the sequences commute with the inclusion in an obvious way.

3 The Euler operator

It has been proved [10, 20, 22] that, for $m \le k \le m+2$, the sheaves and the morphisms of the variational sequence are closely related to the calculus of variations. Namely, there is a canonical representation of the variational sequence into an exact sheaf sequence where quotient sheaves are replaced by sheaves of sections of vector bundles [22]. We are going to summarize the steps which yield this canonical representation. We start with the first variation formula for higher order variational calculus [7].

Proposition 2. *For every morphism* $B \colon J_rY \to V^*J_rY \otimes \wedge^m T^*X$ *over* J_rY *there is a unique pair of sheaf morphisms*

$$\boldsymbol{E}(B) \colon J_{2r}Y \to V^*Y \otimes \wedge^m T^*X, \qquad F(B) \colon J_{2r}Y \to V^*J_rY \otimes \wedge^m T^*X,$$

over Y *and* J_rY*, respectively, such that* $(\pi^{2r}_r)^*B = \boldsymbol{E}(B) + F(B)$*, and* $F(B)$ *is locally of the form* $F(B) = DP$*, with* $P \in C^\infty_{J_{r-1}Y}(J_{2r-1}Y, V^*J_{r-1}Y \otimes \wedge^m T^*X)$*.*

Remark 1. The uniqueness of the decomposition in the above theorem implies that both $\boldsymbol{E}(B)$ and $F(B)$ are intrinsic geometric objects. According to [6], it is possible to determine a global section P satisfying $F(B) = DP$; such a P is said to be a *momentum* of B. If $\dim X = 1$ or $r = 1$, then $P = P(B)$ is uniquely determined. If $r = 2$, then a section $P(B)$ with the required properties can be uniquely determined by the additional requirement of *quasisymmetry* (see [7] for a complete discussion).

If B has the coordinate expression $B^\alpha_p \, \delta^p_\alpha \otimes \omega$, then we have

$$E(B) = (-1)^{|\alpha|} D_\alpha B^\alpha_p \, \delta^p \otimes \omega, \qquad 0 \le |\alpha| \le r. \tag{10}$$

Definition 1. A morphism $B \in C^\infty_Y(J_rY, V^*Y \otimes \wedge^m T^*X)$ will be called an Euler morphism. The operator $\boldsymbol{E} \colon C^\infty_{J_rY}(J_rY, V^*J_rY \otimes \wedge^m T^*X) \to C^\infty_Y(J_{2r}Y, V^*Y \otimes \wedge^m T^*X)$ will be called the formal Euler operator.

Let $L \in C^\infty_X(J_rY, \wedge^m T^*X)$ be an r–th order Lagrangian. We have $\delta L \in C^\infty_{J_rY}(J_rY, V^*J_rY \otimes \wedge^m T^*X)$.

Definition 2. The morphism $E(L) := \boldsymbol{E}(\delta L) \colon J_{2r}Y \to V^*Y \otimes \wedge^m T^*X$ is called the Euler morphism of L. This defines an operator

$$E \colon C^\infty_X(J_rY, \wedge^m T^*X) \to C^\infty_Y(J_{2r}Y, V^*Y \otimes \wedge^m T^*X)$$

which is said to be the Euler operator.

We remark that in [8] it is proved that, for $m \geq 2$, all natural operators $C^\infty_X(J_r Y, \wedge^m T^*X) \to C^\infty_Y(J_{2r}Y, V^*Y \otimes \wedge^m T^*X)$ are the constant multiples of the Euler operator. In [1] a similar result is announced under the condition that the operators are assumed to be linear.

A key concept of this paper is expressed in the following definition.

Definition 3. An Euler morphism $B \in C^\infty_Y(J_s Y, V^*Y \otimes \wedge^m T^*X)$ is said to be locally variational if there locally exists a Lagrangian L such that $E(L) = B$.

The canonical representation of the variational sequence [22] is summarized as follows. Let us define the following maps

$$I_m \colon \Lambda^m_r/\Theta^m_r \to C^\infty_X(J_{r+1}Y, \wedge^m T^*X); [\alpha] \mapsto h(\alpha),$$
$$I_{m+1} \colon \Lambda^{m+1}_r/\Theta^{m+1}_r \to C^\infty_Y(J_{2r+2}Y, V^*Y \otimes \wedge^m T^*X); [\alpha] \mapsto \boldsymbol{E}(h(\alpha)).$$

Proposition 3. *The maps I_m and I_{m+1} are injective morphisms, and the following diagram commutes*

$$\begin{array}{ccc}
\Lambda^m_r/\Theta^m_r & \xrightarrow{\quad \mathcal{E}_m \quad} & \Lambda^{m+1}_r/\Theta^{m+1}_r \\
\Big\downarrow{\scriptstyle I_m} & & \Big\downarrow{\scriptstyle I_{m+1}} \\
C^\infty_X(J_{r+1}Y, \wedge^m T^*X) & \xrightarrow{E} & C^\infty_Y(J_{2r+2}Y, V^*Y \otimes \wedge^m T^*X)
\end{array}$$

We observe that the quotient $\Lambda^{m+1}_r/\Theta^{m+1}_r$ annihilates 'local divergencies', *i.e.* morphisms of the type of $F(B)$ of Proposition 2.

4 The Helmholtz operator

In this section we give two intrinsic formulations of the local conditions of local variationality, or *Helmholtz conditions* [1, 2, 3, 5, 10, 11, 16], and we prove their equivalence.

First of all, we prove that for every s–th order Euler morphism B there exists a natural morphism

$$H(B) \colon J_{2s}Y \to V^*J_sY \otimes V^*Y \otimes \wedge^m T^*X$$

over J_sY whose vanishing is equivalent to the local variationality of B. The local components of $H(B)$ are equal to the Helmholtz conditions, as given in [2, 5, 11, 16].

Let $B \in C^\infty_Y(J_sY, V^*Y \otimes \wedge^m T^*X)$ be an Euler morphism. Using the canonical projection $VJ_sY \to VY$, we can interpret B as a vertical $\wedge^m T^*X$–valued 1–form on J_sY. Then δB is a vertical $\wedge^m T^*X$–valued 2–form on J_sY. For every vertical vector field u on Y, we have $i_{u_s}\delta B \colon J_sY \to V^*J_sY \otimes \wedge^m T^*X$. Then we can apply the formal Euler operator and we obtain $\boldsymbol{E}(i_{u_s}\delta B) \colon J_{2s}Y \to V^*Y \otimes \wedge^m T^*X$. Clearly, in the following assertion $H(B)(u_s)$ is defined by the evaluation at the first factor.

Theorem 1. *There exists a unique morphism* $H(B)\colon J_{2s}Y \to V^*J_sY \otimes V^*Y \otimes \wedge^m T^*X$ *over* J_sY *satisfying*

$$\boldsymbol{E}(i_{u_s}\delta B) = H(B)(u_s) \tag{11}$$

for every vertical vector field u *on* Y.

Proof. In the above notation, the coordinate form of $i_{u_s}\delta B$ is

$$[(\partial_q^\alpha B_p)D_\alpha u^q - (\partial_q^\alpha B_p)u^p\delta_\alpha^q\delta^p] \otimes \omega. \tag{12}$$

Write $H(B) = H_{p\,q}^{\alpha}\delta_\alpha^p \otimes \delta^q \otimes \omega$. Then (11) together with Leibnitz' rule for formal derivatives [14] implies the following coordinate formulae

$$H_{p\,q}^{\alpha} = \partial_p^\alpha B_q - \sum_{|\beta|=0}^{s-|\alpha|}(-1)^{|\alpha+\beta|}\frac{(\alpha+\beta)!}{\alpha!\beta!}D_\beta\partial_q^{\alpha+\beta}B_p. \tag{13}$$

□

Definition 4. The morphism $H(B)\colon J_{2s}Y \to V^*J_sY \otimes V^*Y \otimes \wedge^m T^*X$ is called the Helmholtz morphism of B. This defines an operator

$$H\colon C_Y^\infty(J_sY, V^*Y \otimes \wedge^m T^*X) \to C_{J_sY}^\infty(J_{2s}Y, V^*J_sY \otimes V^*Y \otimes \wedge^m T^*X)$$

which is said to be the Helmholtz operator.

We remark that a formal Helmholtz operator can be introduced, in analogy to the case of the formal Euler operator.

Proposition 4. *An* s*–th order Euler morphism* B *is locally variational if and only if* $H(B) = 0$.

Proof. If $\delta L = B + DP$, then $\delta B = -\delta DP$. Using (8) and (1), we obtain $i_{u_s}\delta B = -i_{u_s}D\delta P = -D(i_{u_{s-1}}\delta P)$. But $\boldsymbol{E}(D(i_{u_{s-1}}\delta P)) = 0$ in consequence of the uniqueness in Proposition 2. The condition is also sufficient; the so–called Volterra's (local) Lagrangian [1, 16] is sent by E into B provided that the above condition is fulfilled. □

Now, we are going to the variational sequence formulation. It is known [3, 10] that there exists a locally defined morphism whose vanishing is equivalent to the local conditions of local variationality. We prove that this morphism is intrinsically characterized. In particular, we prove that for every s–th order Euler morphism B there exists a natural morphism

$$\tilde{H}(B) \in C_{J_sY}^\infty(J_{2s}Y, V^*J_sY \wedge V^*Y \otimes \wedge^m T^*X)$$

whose vanishing is equivalent to the local variationality of B. This morphism $\tilde{H}(B)$ coincides locally with the locally defined morphism introduced in [3, 10].

Let $B \in C_Y^\infty(J_sY, V^*Y \otimes \wedge^m T^*X)$. We have $\delta B\colon J_sY \to V^*J_sY \wedge V^*Y \otimes \wedge^m T^*X$.

Proposition 5. *There is a unique pair of sheaf morphisms*

$$\tilde{H}(B) \in C^\infty_{J_sY}(J_{2s}Y, V^*J_sY \wedge V^*Y \otimes \wedge^m T^*X),$$
$$G(B) \in C^\infty_{J_sY}(J_{2s}Y, V^*J_sY \wedge V^*Y \otimes \wedge^m T^*X)$$

such that

i. $\pi^{2s*}_s \delta B = \tilde{H}(B) + G(B)$,

ii. $\tilde{H}(B) = 1/2\, A(H(B))$, *where A is the antisymmetrization map.*

Moreover, $G(B)$ is locally of the form $G(B) = DQ$, where

$$Q \in C^\infty_{J_{s-1}Y}(J_{2s-1}Y, \wedge^2 V^*J_{s-1}Y \otimes \wedge^{m-1}T^*X).$$

Proof. It is clear that $G(B)$ is uniquely determined by δB and the choice $\tilde{H}(B) = 1/2\, A(H(B))$. In particular, its coordinate expression is

$$G(B) = G^\alpha_{p\,q} \delta^p_\alpha \wedge \delta^q \otimes \omega$$

with

$$G^\alpha_{p\,q} = \frac{1}{2}(\partial^\alpha_p B_q + \sum_{|\beta|=0}^{s-|\alpha|} (-1)^{|\alpha+\beta|} \frac{(\alpha+\beta)!}{\alpha!\beta!} D_\beta \partial^{\alpha+\beta}_q B_p).$$

Moreover, it can be easily seen [14] by induction on $|\alpha|$ that, on a coordinate open subset $U \subset Y$, we have

$$\begin{aligned}\delta B &= \partial^\alpha_p B_q \delta^p_\alpha \wedge \delta^q \wedge \omega = \partial^\alpha_p B_q L_\alpha(\vartheta^p) \wedge \vartheta^q \wedge \omega \\ &= (-1)^{|\alpha|} \vartheta^p \wedge L_\alpha(\partial^\alpha_p B_q \vartheta^q) \wedge \omega + 2DQ,\end{aligned}$$

where L_i is the Lie derivative with respect to the coordinate vector field $D_i = \partial_i + y^p_{\alpha+i} \partial^\alpha_p$ and for $|\alpha| \geq 1$ we apply the induction $L_{\alpha+i} = L_i L_\alpha$. This yields the thesis by the Leibnitz' rule. □

In general, the section Q is not uniquely characterized. But, if $\dim X = 1$, then there exists a unique $Q(B)$ fulfilling the conditions of the statement of Proposition 5 [22]. Further, consider $B\colon J_2Y \to V^*Y \otimes \wedge^m T^*X$. Then we are able to characterize a unique $Q(B)$ as follows.

Lemma 2. *There exists a unique $S(B)\colon J_3Y \to \otimes^2 V^*J_1Y \otimes \wedge^m T^*X$ such that, for any vertical vector field $u\colon Y \to VY$, we have $S(B)(u_1) = P(i_{u_2}\delta B)$, where $S(B)(u_1)$ denotes the evaluation at the first factor.*

Proof. In fact, we have the coordinate expression (see Remark 1)

$$P(i_{u_2}\delta B) = (-u^q \partial^i_p B_q + D_j u^q \partial^{i+j}_p B_q + u^q D_j \partial^{i+j}_p B_q)\, \delta^p \otimes \omega_i - u^q \partial^{i+j}_p B_q\, \delta^p_i \otimes \omega_j.$$

So, we have

$$S(B) = (\partial^i_p B_q - D_j \partial^{i+j}_p B_q) \delta^p \otimes \delta^q \otimes \omega_i + (\partial^{i+j}_p B_q + \partial^{i+j}_q B_p) \delta^p_i \otimes \delta^q \otimes \omega_j.$$

□

Proposition 6. *Let $B\colon J_2Y \to V^*Y \otimes \wedge^m T^*X$. Then we have*

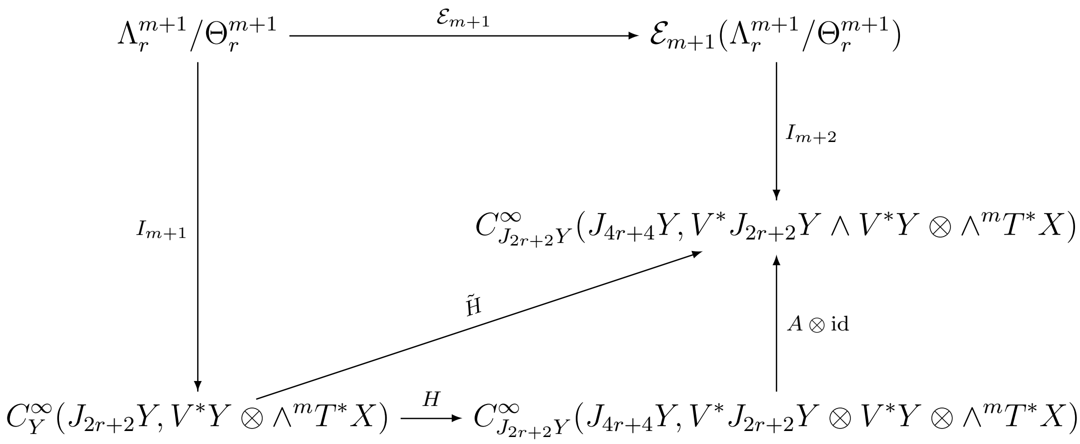
$$Q(B) = \frac{1}{2} A(S(B)).$$

Proof. It is straightforward to check that $D\frac{1}{2}A(S(B)) = G(B)$. □

The canonical representation of the variational sequence [22] is completed as follows. Let us define the following map

$$\begin{aligned} I_{m+2}\colon \mathcal{E}_{m+1}(\Lambda^{m+1}_r/\Theta^{m+1}_r) &\to C^\infty_{J_{2r+2}Y}(J_{4r+4}Y, V^*J_{2r+2} \otimes V^*Y \otimes \wedge^m T^*X), \\ [d\alpha] &\mapsto \tilde{H}(d\boldsymbol{E}(h(\alpha))). \end{aligned}$$

Corollary 1. *[22] The map I_{m+2} is an injection, and the following diagram commutes*



We observe that the quotient $\mathcal{E}_{m+1}(\Lambda^{m+1}_r/\Theta^{m+1}_r)$ annihilates 'local divergencies', *i.e.* morphisms of the type of $G(B)$ of the above Proposition.

Remark 2. The reader should have noticed that, in the case $m+1$, we only provide a representation for the image sheaf

$$\mathcal{E}_{m+1}(\Lambda^{m+1}_r/\Theta^{m+1}_r) \subset \Lambda^{m+2}_r/\Theta^{m+2}_r.$$

This is because the image sheaf is the only part of the quotient sheaf that admit an interpretation in the calculus of variations. Indeed, the image sheaf is made by Helmholtz morphisms corresponding to all Euler morphisms. In order to obtain information about an Euler morphism B (its symmetries, and whether it be locally variational or not) the knowledge of other elements than $H(B)$ in the quotient $\Lambda^{m+2}_r/\Theta^{m+2}_r$ is not essential. A similar argument holds for quotient spaces of degree $m+k$, $k \geq 3$.

Due to the exactness of the variational sequence, if $B \in C^\infty_Y(J_sY, V^*Y \otimes \wedge^m T^*X)$ then B is locally variational if and only if $\tilde{H}(B) = 0$. So, we can summarize the results of this section into the following assertion.

Theorem 2. *Let $B \in C^\infty_Y(J_sY, V^*Y \otimes \wedge^m T^*X)$. Then the following conditions are equivalent.*

i. B is locally variational;

ii. $\tilde{H}(B) = 0$;

iii. $H(B) = 0$.

Remark 3. The equivalence between (ii) and (iii) could also be derived from the coordinate expressions. Namely, in coordinates, (ii) and (iii) are two systems of equations on the components $H^{\alpha}_{p\,q}$, which differ only when $|\alpha| = 0$. In this case, the equation $H_{pq} = 0$ in (ii) splits into its symmetric and antisymmetric part; the former vanishes, provided that (iii) holds.

Remark 4. It follows from the above theorem that the operator $\tilde{H}$, which is different from H, has the same kernel as H. And, of course, the kernel is an essential feature of both $\tilde{H}$ and H.

5 Natural operators

Having in mind the interesting geometric properties of the Helmholtz operator H, we find it attractive to discuss H from the viewpoint of the theory of the natural operators [9]. We are going to determine all first and second order natural operators of the type of H.

The first order Helmholtz operator is an operator

(14) $$H \colon C^\infty_Y(J_1Y, V^*Y \otimes \wedge^m T^*X) \to C^\infty_{J_1Y}(J_2Y, V^*J_1Y \otimes V^*Y \otimes \wedge^m T^*X)$$

of the coordinate form

(15) $$\big[(\partial_p E_{\mu q} - \partial_q E_{\mu p} + D_i \partial^i_q E_{\mu p})\delta^p \otimes \delta^q + (\partial^i_q E_{\mu p} + \partial^i_p E_{\mu q})\delta^p_i \otimes \delta^q\big] \otimes d^\mu,$$

where $\mu = i_1 \dots i_m$ and d^μ is a shorthand for $d^{i_1} \wedge \dots \wedge d^{i_m}$.

Proposition 7. *All natural operators*

$$C^\infty_Y(J_1Y, V^*Y \otimes \wedge^m T^*X) \to C^\infty_{J_1Y}(J_2Y, V^*J_1Y \otimes V^*Y \otimes \wedge^m T^*X)$$

are of the form cH, with $c \in \mathbb{R}$.

Proof. Consider first a second order operator D. By Lemma 1 of [8], the derivatives with respect to x^i can be replaced by formal derivatives. Write

(16) $$D = (H_{\mu pq}\delta^p \otimes \delta^q + H_{\mu pq}{}^{i}\delta^p_i \otimes \delta^q) \otimes d^\mu$$

(17) $$E = E_{\mu p}\delta^p \otimes d^\mu$$

(18) $$E_{\mu p,q} = \partial_q E_{\mu p}, \quad E_{\mu p,}{}^{i}_{q} = \partial^i_q E_{\mu p}, \quad E_{\mu p,i} = D_i E_{\mu p}, \quad E_{\mu p,}{}^{i}_{qj} = D_j E_{\mu p,}{}^{i}_{q},$$

and analogously for higher order derivatives. We are looking for $G^2_{m,n}$–equivariant maps. Using fibre homotheties, we deduce that $H_{\mu pq}$ and $H_{\mu pq}^{\ i}$ cannot depend on any expression with more than 2 fibre subscripts. By the homogeneous function theorem, $H_{\mu pq}$ is linear in $E_{\mu p,q}$, $E_{\mu p,q}^{\ \ \ i}$, $E_{\mu p,qj}^{\ \ \ i}$ and bilinear in $E_{\mu p}$ and $E_{\mu p,i}$. Using base homotheties, we find that $H_{\mu pq}$ is independent of $E_{\mu p}$ and $E_{\mu p,i}$. Using the generalized invariant tensor theorem and the remark on p. 466 of [8] (which says that the contraction in $E_{\mu p,qj}^{\ \ \ i}$ to one factor in μ coincides with the contraction to the last factor), we obtain

$$H_{\mu pq} = k_1 E_{\mu p,q} + k_2 E_{\mu q,p} + k_3 E_{\mu p,qi}^{\ \ \ i} + k_4 E_{\mu q,pi}^{\ \ \ i}. \tag{19}$$

In the next formula, we shall write $\mu = \lambda i_m$, $\lambda = i_1 \dots i_{m-1}$. Analogously to (19), we find

$$H_{\mu pq}^{\ i} = k_5 E_{\mu p,q}^{\ \ \ i} + k_6 E_{\mu q,p}^{\ \ \ i} + k_7 \delta^i_{[i_m} E_{\lambda]jp,q}^{\ \ \ \ j} + k_8 \delta^i_{[i_m} E_{\lambda]jq,p}^{\ \ \ \ j}, \tag{20}$$

where δ^i_j is the Kronecker's delta.

It will suffice to study equivariancy of (19) and (20) with respect to the transformations

$$\bar{x}^i = x^i, \qquad \bar{y}^p = f^p(y). \tag{21}$$

So we consider the canonical injection of G^2_n with coordinates (a^p_q, a^p_{qr}) into $G^2_{m,n}$. We may restrict ourselves to the subgroup $a^p_q = \delta^p_q$. Using direct evaluations, we find the following transformation laws

$$\bar{H}_{\mu pq} = H_{\mu pq} - a^r_{ps} y^s_i H_{\mu rq}^{\ i}, \qquad \bar{H}_{\mu pq}^{\ i} = H_{\mu pq}^{\ i}, \tag{22}$$

$$\begin{cases} \bar{E}_{\mu p,q} = E_{\mu p,q} - a^r_{pq} E_{\mu r} - a^r_{sq} y^s_i E_{\mu p,r}^{\ \ \ i}, \quad \bar{E}_{\mu p,q}^{\ \ \ i} = E_{\mu p,q}^{\ \ \ i}, \\ \bar{E}_{\mu p,qi}^{\ \ \ i} = E_{\mu p,qi}^{\ \ \ i} - a^r_{pt} y^t_i E_{\mu r,q}^{\ \ \ i} - a^r_{sq} y^s_i E_{\mu p,r}^{\ \ \ i}. \end{cases} \tag{23}$$

All terms in (20) are invariant, so that this gives no condition. The equivariancy of (19) reads

(24)
$$\begin{aligned} & k_1 E_{\mu p,q} + k_2 E_{\mu q,p} + k_3 E_{\mu p,qi}^{\ \ \ i} + k_4 E_{\mu q,pi}^{\ \ \ i} \\ & - a^r_{ps} y^s_i (k_5 E_{\mu r,q}^{\ \ \ i} + k_6 E_{\mu q,r}^{\ \ \ i} + k_7 \delta^i_{[i_m} E_{\lambda]jr,q}^{\ \ \ \ j} + k_8 \delta^i_{[i_m} E_{\lambda]jq,r}^{\ \ \ \ j}) = \\ & \qquad k_1 (E_{\mu p,q} - a^r_{pq} E_{\mu r} - a^r_{sq} y^s_i E_{\mu p,r}) + k_2 (E_{\mu q,p} - a^r_{pq} E_{\mu r} - a^r_{sp} y^s_i E_{\mu q,r}) \\ & \qquad + k_3 (E_{\mu p,qi}^{\ \ \ i} - a^r_{pt} y^t_i E_{\mu r,q}^{\ \ \ i} - a^r_{sq} y^s_i E_{\mu p,r}^{\ \ \ i}) + k_4 (E_{\mu q,pi}^{\ \ \ i} - a^r_{qt} y^t_i E_{\mu r,p}^{\ \ \ i} - a^r_{sp} y^s_i E_{\mu q,r}^{\ \ \ i}). \end{aligned}$$

This condition must be satisfied identically. First of all, this yields

$$k_7 = 0, \qquad k_8 = 0. \tag{25}$$

Then the remaining terms imply

$$k_5 = k_3, \quad k_6 = k_2 + k_4, \quad 0 = k_1 + k_2, \quad 0 = k_1 + k_3, \quad 0 = k_4. \tag{26}$$

Clearly, (19) and (20) with (25) and (26) characterize the constant multiples of the Helmholtz operator (15).

Assume further that D is a k–th order operator. Combining fibre and base homotheties we find that no higher order term is natural. Hence D must be a second order operator. Finally, we can transform any finite order jet into any neighbourhood of zero by a suitable combination of fibre and base homotheties. Then the nonlinear Peetre theorem implies the finiteness of the order of D [9]. □

Next we are going to deduce a similar result for the second order case. The second order Helmholtz operator is an operator

$$(27)\qquad H\colon C^\infty_Y(J_2Y, V^*Y\otimes\wedge^m T^*X)\to C^\infty_{J_2Y}(J_4Y, V^*J_2Y\otimes V^*Y\otimes\wedge^m T^*X).$$

If we write the right-hand side of (27) in the form

$$(28)\qquad (H_{\mu pq}\delta^p\otimes\delta^q + H_{\mu p q}^{\ \ i}\delta^p_i\otimes\delta^q + H_{\mu p\ q}^{\ \ ij}\delta^p_{ij}\otimes\delta^q)\otimes d^\mu,$$

then the coordinate expression of H is

$$(29)\qquad \begin{cases} H_{\mu pq} = \partial_p E_{\mu q} - \partial_q E_{\mu p} + D_i\partial^i_q E_{\mu p} - D_{ij}\partial^{ij}_q E_{\mu p}\\ H_{\mu pq}^{\ \ i} = \partial^i_q E_{\mu p} + \partial^i_p E_{\mu q} - 2D_j\partial^{ij}_q E_{\mu p}\\ H_{\mu p\ q}^{\ \ ij} = \partial^{ij}_p E_{\mu q} - \partial^{ij}_q E_{\mu p}\end{cases}$$

Theorem 3. *All natural operators*

$$C^\infty_Y(J_2Y, V^*Y\otimes\wedge^m T^*X)\to C^\infty_{J_2Y}(J_4Y, V^*J_2Y\otimes V^*Y\otimes\wedge^m T^*X)$$

are of the form cH, with $c\in\mathbb{R}$.

Proof. Consider first a third order operator D. Analogously to Proposition 7, the derivatives with respect to x^i can be replaced by the formal derivatives. Consider D in the form (28) and write, in addition to (17) and (18),

$$(30)\qquad E_{\mu p,q}^{\ \ \ \ ij} = \frac{\alpha!}{2}\partial^\alpha_q E_{\mu p}, \alpha = i+j,\quad E_{\mu p,q}^{\ \ \ \ ij}{}_k = D_k E_{\mu p,q}^{\ \ \ \ ij}, E_{\mu p,q}^{\ \ \ \ ij}{}_{kl} = D_l D_k E_{\mu p,q}^{\ \ \ \ ij},$$

and analogously for higher order derivatives. We are looking for $G^3_{m,n}$–maps. Analogously to the proof of Proposition 7, we obtain

$$(31)\qquad H_{\mu pq} = k_1 E_{\mu p,q} + k_2 E_{\mu q,p} + k_3 E_{\mu p,q}^{\ \ \ \ i}{}_i + k_4 E_{\mu q,p}^{\ \ \ \ i}{}_i + k_5 E_{\mu p,q}^{\ \ \ \ ij}{}_{ij} + k_6 E_{\mu q,p}^{\ \ \ \ ij}{}_{ij},$$

$$(32)\qquad H_{\mu pq}^{\ \ i} = k_7 E_{\mu p,q}^{\ \ \ \ i} + k_8 E_{\mu q,p}^{\ \ \ \ i} + k_9 E_{\mu p,q}^{\ \ \ \ ij}{}_j + k_{10} E_{\mu q,p}^{\ \ \ \ ij}{}_j + k_{11}\delta^i_{[i_m} E_{\lambda]jp,q}^{\ \ \ \ \ \ j} + k_{12}\delta^i_{[i_m} E_{\lambda]jq,p}^{\ \ \ \ \ \ j} + k_{13}\delta^i_{[i_m} E_{\lambda]jp,q}^{\ \ \ \ \ \ jk}{}_k + k_{14}\delta^i_{[i_m} E_{\lambda]jq,p}^{\ \ \ \ \ \ jk}{}_k,$$

$$(33)\qquad H_{\mu p\ q}^{\ \ ij} = k_{15} E_{\mu p,q}^{\ \ \ \ ij} + k_{16} E_{\mu q,p}^{\ \ \ \ ij} + k_{17}\delta^{(i}_{[i_m} E_{\lambda]jp,q}^{\ \ \ \ \ \ j)k} + k_{18}\delta^{(i}_{[i_m} E_{\lambda]jq,p}^{\ \ \ \ \ \ j)k}.$$

We study the equivariancy of (31), (32), (33) with respect to the canonical injection of G^3_n into $G^3_{m,n}$ given by (21). We restrict ourselves to the subgroup $a^p_q = \delta^p_q$ and we shall denote by $(\delta^p_q, \tilde a^p_{qr}, \tilde a^p_{qrs})$ the inverse element to $(\delta^p_q, a^p_{qr}, a^p_{qrs})$. Using direct evaluations, we find the following transformation laws

$$
\left\{\begin{array}{l}
\bar H_{\mu pq} = H_{\mu pq} + \tilde a^s_{pr} y^r_i H_{\mu sq}{}^i + \tilde a^t_{prs} y^r_i y^s_j H_{\mu t}{}^{ij}{}_q, \\
\bar H_{\mu pq}{}^i = H_{\mu pq}{}^i + 2\tilde a^s_{pr} y^r_j H_{\mu s}{}^{ij}{}_q, \\
\bar H_{\mu p}{}^{ij}{}_q = H_{\mu p}{}^{ij}{}_q,
\end{array}\right. \tag{34}
$$

$$
\left\{\begin{array}{l}
\bar E_{\mu p,q} = E_{\mu p,q} + \tilde a^r_{pq} E_{\mu r} + \tilde a^r_{qs} y^s_i E_{\mu p,r}{}^i + \tilde a^r_{qs} \bar y^s_{ij} E_{\mu p,r}{}^{ij} + \tilde a^t_{qrs} y^r_i y^s_j E_{\mu p,t}{}^{ij} \\
\bar E_{\mu p,q}{}^i = E_{\mu p,q}{}^i + 2\tilde a^s_{qr} y^r_j E_{\mu p,s}{}^{ij}, \\
\bar E_{\mu p,q}{}^{ij} = E_{\mu p,q}{}^{ij}, \\
\bar E_{\mu p,q}{}^{ij}{}_j = E_{\mu p,q}{}^{ij}{}_j + \tilde a^r_{ps} y^s_j E_{\mu r,q}{}^{ij} + \tilde a^s_{qr} y^r_j E_{\mu p,s}{}^{ij}.
\end{array}\right. \tag{35}
$$

For the remaining two quantities, we shall need the transformation laws with $\tilde a^p_{qr} = 0$ only

$$
\left\{\begin{array}{l}
\bar E_{\mu p,q}{}^i{}_i = E_{\mu p,q}{}^i{}_i + 2\tilde a^s_{qrt} y^r_i y^t_j E_{\mu p,s}{}^{ij}, \\
\bar E_{\mu p,q}{}^{ij}{}_{ij} = E_{\mu p,q}{}^{ij}{}_{ij} + \tilde a^r_{ptv} y^t_i y^v_j E_{\mu r,q}{}^{ij} + \tilde a^s_{qtv} y^t_i y^v_j E_{\mu p,s}{}^{ij}.
\end{array}\right. \tag{36}
$$

Consider first the equivariancy condition for $H_{\mu pq}$. The expressions with δ's are on the left-hand side only. This implies

$$k_{11} = k_{12} = k_{13} = k_{14} = k_{17} = k_{18} = 0. \tag{37}$$

Then the condition of equivariancy of $H_{\mu pq}{}^i$ reads

$$
\begin{aligned}
2\tilde a^s_{pr} y^r_j (k_{15} E_{\mu s,q}{}^{ij} + k_{16} E_{\mu q,s}{}^{ij}) = 2k_7 \tilde a^s_{qr} y^r_j E_{\mu p,s}{}^{ij} + 2k_8 \tilde a^s_{pr} y^r_j E_{\mu q,s}{}^{ij} + \\
k_9(\tilde a^r_{ps} y^s_j E_{\mu r,q}{}^{ij} + \tilde a^s_{qr} y^r_j E_{\mu p,s}{}^{ij}) + k_{10}(\tilde a^r_{qs} y^s_j E_{\mu r,p}{}^{ij} + \tilde a^s_{qr} y^r_j E_{\mu p,s}{}^{ij}).
\end{aligned} \tag{38}
$$

This yields

$$2k_{15} = k_9, \quad 2k_{16} = 2k_8, \quad 0 = 2k_7 + k_9 + k_{10}, \quad 0 = k_{10}. \tag{39}$$

If we put $\tilde a^p_{qr} = 0$ to the equivariancy condition for $H_{\mu pq}$, we find

$$2k_{16} = k_2 + 2k_4 + k_6, \quad k_{15} = k_5, \quad 0 = k_1 + 2k_3 + k_5, \quad 0 = k_6. \tag{40}$$

In the same condition, the terms with $\tilde a^r_{pq} E_{\mu r}$, $\tilde a^r_{qs} y^s_i E_{\mu p,r}{}^i$ and $\tilde a^r_{qs} y^s_i E_{\mu r,p}{}^i$ imply

$$k_1 + k_2 = 0, \quad k_1 + k_3 = 0, k_4 = 0. \tag{41}$$

The only solution of (39), (40), (41) is $k_1 = -c$, $k_2 = c$, $k_3 = c$, $k_4 = 0$, $k_5 = -c$, $k_6 = 0$, $k_7 = c$, $k_8 = c$, $k_9 = -2c$, $k_{10} = 0$, $k_{15} = -c$, $k_{16} = c$. This corresponds to cH.

Assume further that D is a k–th order operator. Combining fibre and base homotheties we find that no higher order term is natural. Hence D must be a third order operator. Finally, one can transform any finite order jet into any neighbourhood of zero by a suitable combination of fibre and base homotheties. Then the nonlinear Peetre theorem implies the finiteness of the order of D. □

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