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On differential equations characterized by their Lie point symmetries[1]

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Abstract

We study the geometry of differential equations determined uniquely by their point symmetries, that we call *Lie remarkable*. We determine necessary and sufficient conditions for a differential equation to be Lie remarkable. Furthermore, we see how, in some cases, Lie remarkability is related to the existence of invariant solutions. We apply our results to minimal submanifold equations and to Monge-Ampère equations in two independent variables of various orders.

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1 Introduction

One of the most successful achievements in the geometric theory of differential equations (DEs), either ordinary or partial, is the theory of symmetries [5, 6, 18, 32, 33, 34]. Symmetries of DEs are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former, with the further property of sending solutions into solutions. The knowledge of the symmetries of

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a DE may lead to compute some of its solutions, or to transform it in a more convenient form; in the case of an ordinary differential equation (ODE) it may allow to reduce the order, determine integrating factors, etc.

Point symmetries are the main object of study in this paper. Any transformation of the independent and dependent variables induces a transformation of the derivatives, which is said to be a point transformation. Point symmetries of a DE are symmetries defined by point transformations.

The problem of finding the symmetries of a DE has a natural "inverse" problem associated, namely, the problem of finding the most general form of a DE admitting a given Lie algebra of infinitesimal point symmetries. To the authors' knowledge, the inverse problem has been considered for the first time in [4] from an algorithmic viewpoint in order to characterize all DEs admitting a given group. An interesting contribution has been given by Rosenhaus in 1982 [38], who posed the problem of the unique determination of a DE by its group; in fact, in [38] the author considered the projective algebra of $\mathbb{R}^3$ and its subalgebras, and was able to prove that the equation of vanishing Gaussian curvature of surfaces in $\mathbb{R}^3$ (which is a Monge-Ampère-type equation) is uniquely determined by its Lie point symmetries.

One possible approach to this problem is to classify all possible realizations of the given Lie algebra as algebra of vector fields on the base manifold [37]. Then, the theory of differential invariants of such realizations allows to find the most general form of the equation [17, 37]. In fact, it is known (see [33]) that, under suitable hypotheses of regularity, the most general DE admitting a given Lie algebra of point symmetries is locally given by $\Delta_\mu(I_1, I_2, \ldots, I_k) = 0$ where Δ_μ are general smooth functions and I_i $(i = 1, \ldots, k)$ are the differential invariants of the realization under consideration. However, it could be hard to compute the differential invariants, though it amounts to solve a first order differential system. Then, in some cases, it is necessary to confine the investigation to a specified class of DEs, *i.e.*, to impose additional constraints to the form of Δ_μ; in this context interesting questions may arise. For instance, how to derive the functional form of a quasilinear first order system of DEs which are invariant with respect to the Galilean group [28, 42, 44], or to a scaling group [12, 13]. Moreover, in [38, 39, 40] it was considered the problem of finding the minimal subalgebra of the algebra of point symmetries of the equation of vanishing Gaussian curvature of surfaces in $\mathbb{R}^3$ which uniquely determines it. A 6-dimensional Lie subalgebra characterizes this equation provided attention is restricted to fully symmetric systems (see also remark 20).

In this paper we face a problem which is related to that described above, *i.e.*, to *find under which conditions a given DE is uniquely determined by the Lie algebra of its infinitesimal point symmetries*. By following the terminology already used in [26, 30, 31], we call such DE *Lie remarkable*.

The plan of the paper is the following. In section 2, we introduce a DE of order r as a submanifold of a suitable jet space (of order r), which is a manifold whose coordinate functions of a chart can be interpreted as "independent" and "dependent" variables, and by the derivatives of the latter with respect to the former up to the order r. For the dimension of a DE we mean its dimension as submanifold. Symmetries of a given DE will be interpreted as particular vector fields on the jet space tangent to the DE.

In section 3, we introduce two distinguished types of Lie remarkable equations: *strongly* and *weakly* Lie remarkable equations. Strongly Lie remarkable equations are uniquely determined by their point symmetries in the whole jet space; weakly Lie remarkable equations are equations which do not intersect other equations admitting the same symmetries. An interesting question concerns with the construction of solutions of a DE which is uniquely determined by its point symmetries. This problem was posed in [40], where the author proved that the solutions of the equation of vanishing Gaussian curvature of surfaces in $\mathbb{R}^3$ can be characterized as invariant solutions with respect to some of its point symmetries. In the present paper, we show that the Lie remarkability, for DEs which are 'generalized sprays' in the sense of [36], implies the existence of invariant solutions.

Then, we find necessary and sufficient conditions for a given DE to be (strongly or weakly) Lie remarkable by analyzing the dimension of the Lie algebra of point symmetries and the regularity of the local action that these symmetries induce on the jet space where the DE is immersed. Our viewpoint reverses and generalizes the Lie determinant method [33]. Such a method, together with the method of differential invariants, aims at finding the most general form of a scalar ODE which is invariant with respect to a r-dimensional Lie algebra of point symmetries. In this case, for dimensional reasons, the components of the symmetries form a square matrix whose determinant is the Lie determinant.

We stress that, by using our method, we do not need to find differential invariants of symmetries in order to determine if a given DE is Lie remarkable or not, as in [30, 38, 39, 40].

Of course, many DEs are not Lie remarkable for lack of point symmetries (see theorem 5). Among them we recall KdV equation, Burgers' equation, Kepler's equations. Several authors studied the problem of finding, for a given equation, an extension of the algebra of point symmetries for which the equation at hand is determined. For instance, in [1, 19] such a construction is performed in the case of ODEs by considering a non-local extension of the algebra of point symmetries. Also, in [38] it is shown that the algebra of contact symmetries characterizes the equation of minimal surfaces rather than the point ones.

In the remaining sections we give various examples of DEs that are strongly or weakly Lie remarkable.

In section 4, we consider minimal submanifold equations, and prove that minimal surface equation in $\mathbb{R}^k$ is weakly Lie remarkable if $k = 3, 6$, but it is not Lie remarkable if $k = 4, 5$. We also show how strong Lie remarkability of the equation of unparametrized geodesics in a Riemannian surface is related to Gaussian curvature. Its invariant solutions behave as predicted by theorem 12.

Furthermore, in section 5 we consider Monge-Ampère equations in two independent variables of various orders. The computations of Lie algebras of point symmetries of the equations considered are performed through the use of computer algebra packages (mainly Relie [29] and MathLie [3]); the (strong or weak) Lie remarkability is proved by calculating the rank of the distributions determined by prolongations of the Lie algebra and determining the submanifolds where the rank decreases. The differential invariants of the Lie algebras of point symmetries of the examples of weak Lie remarkable equations are determined as a by-product.

2 Preliminaries

Here we recall some basic notions of the theory of jet spaces. We start to define jets of submanifolds, and as by-product we derive the definition of jets of fibrations. Our main sources are [6, 24, 32, 33, 43].

In this paper manifolds and maps are C^∞. If E is a manifold then we denote by $\chi(E)$ the Lie algebra of vector fields on E. Also, for the sake of simplicity, all submanifolds of E are *embedded* submanifolds.

Let E be an $(n+m)$-dimensional smooth manifold and L an n-dimensional embedded submanifold of E. Let (V, y^A) be a local chart on E. The coordinates (y^A) can be divided in two sets, $(y^A) = (x^\lambda, u^i)$, $\lambda = 1 \dots n$ and $i = 1 \dots m$, such that the submanifold L is locally described as the graph of a vector function $u^i = f^i(x^1, \dots, x^n)$.

The chart (x^λ, u^i) is said to be a *divided chart* which is *concordant* to L. Here, and in what follows, Greek indices run from 1 to n and Latin indices run from 1 to m unless otherwise specified.

Let $\iota\colon L \hookrightarrow E$ and $\iota'\colon L' \hookrightarrow E$ be two submanifolds, and $p \in L \cap L'$. We say that L and L' have a *contact of order* r at p if ι and ι' have a contact of order r at p. Locally, this means that the Taylor expansion of $(\iota - \iota')$ around p in a chart which is concordant with respect to both L and L' vanishes up to the order r. This property is invariant by coordinate transformations.

The above relation is an equivalence relation; an equivalence class is denoted by $[L]^r_p$. The set of such classes is said to be the *r-jet of n-dimensional submanifolds of E* and it is denoted by $J^r(E, n)$.

The set $J^r(E, n)$ has a natural manifold structure. Namely, let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_k)$, with $1 \leq \sigma_1 \leq \dots \leq \sigma_k \leq n$ and $r \in \mathbb{N}$, be a multi-index, and $|\boldsymbol{\sigma}| \overset{\text{def}}{=} k$. Any divided chart (x^λ, u^i) at $p \in E$ induces the local chart $\left(x^\lambda, u^i_{\boldsymbol{\sigma}}\right)$ at $[L]^r_p \in J^r(E, n)$, where $|\boldsymbol{\sigma}| \leq r$ and the functions $u^j_{\boldsymbol{\sigma}}$ are determined by $u^i_{\boldsymbol{\sigma}} \circ j_r L = \partial^{|\boldsymbol{\sigma}|} \iota^i \big/ \partial x^{\boldsymbol{\sigma}}$. The dimension of $J^r(E, n)$ is readily computed:

$$\dim J^r(E, n) = n + m \sum_{h=0}^{r} \binom{n+h-1}{n-1} = n + m \binom{n+r}{r}. \tag{1}$$

We have the following natural maps:

1. the embedding $j_r L\colon L \to J^r(E, n)$, $p \mapsto [L]^r_p$,

2. the projection $\pi_{k,h}\colon J^k(E, n) \to J^h(E, n)$, $[L]^k_p \mapsto [L]^h_p \quad k \geq h$.

We denote by $L^{(r)}$ the image of $j_r L$. We call the tangent plane $T_{\theta_r} L^{(r)}$ at $\theta_r = [L]^r_p$ an *R-plane*. It is easy to realize that to each point $\theta_{r+1} \in J^{r+1}(E,n)$ such that $\pi_{r+1,r}(\theta_{r+1}) = \theta_r$ there corresponds the R-plane $R_{\theta_{r+1}} = T_{\theta_r} L^{(r)}$.

The span $\mathcal{C}^r_{\theta_r}$ of all R-planes at θ_r is said to be the *contact plane* (or Cartan plane). The correspondence $\theta_r \mapsto \mathcal{C}^r_{\theta_r}$ is said to be the *contact distribution*. The contact distribution on $J^r(E,n)$ is generated by the vectors

$$D_\lambda \overset{\text{def}}{=} \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \quad \text{and} \quad \frac{\partial}{\partial u^j_{\boldsymbol{\tau}}},$$

where $0 \leq |\boldsymbol{\sigma}| \leq r-1$, $|\boldsymbol{\tau}| = r$ and $\boldsymbol{\sigma}\lambda$ denotes the multi-index $(\sigma_1, \ldots, \sigma_{r-1}, \lambda)$. The vector fields D_λ are said to be *total derivatives*, and are tangent to any prolonged n-dimensional submanifold of E.

A diffeomorphism of $J^r(E,n)$ preserving the contact distribution is called a *contact transformation*. Analogously, a vector field $\Xi \in \chi(J^r(E,n))$ is said to be a *contact vector field* if $[\Xi, Y]$ is a vector field lying in $\mathcal{C}^r$ whenever Y is a vector field lying in $\mathcal{C}^r$.

We can lift a contact transformation $G : J^r(E,n) \to J^r(E,n)$ to a contact transformation $G^{(1)} : J^{r+1}(E,n) \to J^{r+1}(E,n)$ in the following way: take an element $\theta_{r+1} \in J^{r+1}(E,n)$ and identify it with $R_{\theta_{r+1}}$. Then define $G^{(1)}$ as the map that takes $R_{\theta_{r+1}}$ to $T_{\theta_r} G(R_{\theta_{r+1}})$. We call such an operation the *1-lift*. By induction, we can define the *k-lift* of a contact transformation. By using this reasoning, we can lift contact fields by lifting the corresponding one-parameter group of contact transformations. In coordinates, if $\Xi = \Xi^\lambda \partial/\partial x^\lambda + \Xi^i \partial/\partial u^i$ is a vector field on E, then its k-lift $\Xi^{(k)}$ has the coordinate expression

$$\Xi^{(k)} = \Xi^\lambda \frac{\partial}{\partial x^\lambda} + \Xi^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \tag{2}$$

where $\Xi^j_{\boldsymbol{\tau},\lambda} = D_\lambda(\Xi^j_{\boldsymbol{\tau}}) - u^j_{\boldsymbol{\tau},\beta} D_\lambda(\Xi^\beta)$ with $|\boldsymbol{\tau}| < k$.

According to a classical result by Lie and Bäcklund, any contact transformation on $J^r(E,n)$ is the lifting: 1) of a contact transformation of $J^1(E,n)$ if $m = 1$; 2) of a diffeomorphism of $J^0(E,n) = E$ (which we call *point transformation*) if $m > 1$. An analogous result holds for contact vector fields.

1 Remark. If E is endowed with a fibring $\pi : E \to M$ where $\dim M = n$, then we can reproduce all the above constructions (see [6, 32, 33, 43]). The space $J^r\pi$ of r-th jets of (local) sections $s \colon M \to E$ of π is an open dense subset of $J^r(E,n)$. In fact, it coincides with the subset of r-th jets of submanifolds of the type $s(M)$, which are transverse to the fibring. Jets of maps $f \colon N \to P$ between two manifolds N and P are easily shown to coincide with jets of the trivial fibring $N \times P \to N$.

A *differential equation $\mathcal{E}$ of order r on n-dimensional submanifolds of a manifold E* is a submanifold of $J^r(E,n)$. The manifold $J^r(E,n)$ is called the *trivial equation*.

The *1-prolongation* $\mathcal{E}^1$ of the equation $\mathcal{E}$ is the set of first order "differential consequences" of $\mathcal{E}$. Geometrically:

$$\mathcal{E}^1 = \{\theta_{r+1} \in J^{r+1}(E,n) \,|\, \theta_r \in \mathcal{E},\ R_{\theta_{r+1}} \subset T_{\theta_r}\mathcal{E}\},$$

with $\pi_{r+1,r}(\theta_{r+1}) = \theta_r$. By iteration, we can define the k-prolongation $\mathcal{E}^k$. The equation $\mathcal{E}$ is said to be *formally integrable* if $\mathcal{E}^{k+1} \to \mathcal{E}^k$ are smooth fibre bundles. Locally, if the equation $\mathcal{E}$ is described by $\{F^i = 0\}$, with $F^i \in C^\infty(J^r(E,n))$, then $\mathcal{E}^k$ is described by $\{D_{\boldsymbol{\sigma}}(F^i) = 0\}$ with $0 \le |\boldsymbol{\sigma}| \le k$, where $D_{\boldsymbol{\sigma}} = D_{\sigma_1} \circ D_{\sigma_2} \circ \cdots \circ D_{\sigma_n}$.

An *infinitesimal classical (external) symmetry* of the equation $\mathcal{E} \subset J^r(E,n)$ is a contact vector field on $J^r(E,n)$ which is tangent to $\mathcal{E}$. An *infinitesimal point symmetry* of $\mathcal{E}$ is an infinitesimal classical external symmetry which is the prolongation on $J^r(E,n)$ of a vector field on E.

Let $\mathcal{E}$ be locally described by $\{F^i = 0\}$, $i = 1 \dots k$ with $k < \dim J^r(E,n)$. Then finding point symmetries amounts to solve the system

$$\Xi^{(r)}\left(F^i\right) = 0 \quad \text{whenever} \quad F^i = 0$$

for some $\Xi \in \chi(E)$.

We denote by $\mathrm{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$.

By an *r-th order differential invariant* of a Lie subalgebra $\mathfrak{s}$ of $\chi(E)$ we mean a smooth function $F\colon J^r(E,n) \to \mathbb{R}$ such that for all $\Xi \in \mathfrak{s}$ we have $\Xi^{(r)}(F) = 0$.

The problem of determining the Lie algebra $\mathrm{sym}(\mathcal{E})$ is said to be *direct Lie problem.* Conversely, given a Lie subalgebra $\mathfrak{s} \subset \chi(J^r(E,n))$ of contact vector fields, we consider the *inverse Lie problem*, *i.e.*, the problem of classifying the equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathrm{sym}(\mathcal{E}) = \mathfrak{s}$ [2, 16].

Within the context of inverse Lie problem an interesting question may arise whether there exist non-trivial equations which are in one-to-one correspondence with their algebra of point symmetries. A detailed analysis of this problem is the content of next section.

3 Lie remarkable equations

Here we give the definition of *Lie remarkable equations* in the framework of jets of submanifolds. The same construction holds in an obvious way in the case of jets of fibrings.

2 Definition. Let E be a manifold, $\dim E = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An l-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be

- *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ passing at any $\theta \in \mathcal{E}$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

- *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Of course, a strongly Lie remarkable equation is also weakly Lie remarkable.

We assume throughout this section an $(n+m)$-dimensional manifold E and an l-dimensional differential equation $\mathcal{E} \subset J^r(E,n)$, with $l < \dim J^r(E,n)$.

Let us analyze some direct consequences of our definitions. For each $\theta \in J^r(E,n)$ denote by $S_\theta(\mathcal{E}) \subset T_\theta J^r(E,n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. In particular, if $\theta \in \mathcal{E}$, then $S_\theta(\mathcal{E}) \subseteq T_\theta \mathcal{E}$. Let us set

$$S(\mathcal{E}) \stackrel{\text{def}}{=} \bigcup_{\theta \in J^r(E,n)} S_\theta(\mathcal{E}).$$

In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E,n)$. The action on $J^r(E,n)$ of the algebra of point symmetries is called *regular* if $S(\mathcal{E})$ is a distribution.

The following inequality holds:

$$\dim \operatorname{sym}(\mathcal{E}) \geq S_\theta(\mathcal{E}), \ \forall\, \theta \in J^r(E,n), \tag{3}$$

where $\dim \operatorname{sym}(\mathcal{E})$ is the dimension, as real vector space, of the Lie algebra of infinitesimal point symmetries $\operatorname{sym}(\mathcal{E})$ of $\mathcal{E}$. If the rank of $S(\mathcal{E})$ at each $\theta \in J^r(E,n)$ is the same, then $S(\mathcal{E})$ is an involutive (smooth) distribution.

A submanifold N of $J^r(E,n)$ is an *integral submanifold* of $S(\mathcal{E})$ if $T_\theta N = S_\theta(\mathcal{E})$ for each $\theta \in N$. Of course, an integral submanifold of $S(\mathcal{E})$ is an equation in $J^r(E,n)$ which admits all elements in $\operatorname{sym}(\mathcal{E})$ as infinitesimal point symmetries. Moreover, due to the fact that point symmetries of $\mathcal{E}$ are tangent to $\mathcal{E}$, we have $\dim S_\theta(\mathcal{E}) \leq l$.

3 Proposition. *The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$.*

Proof. Let us consider a chart U at a point θ of maximal rank. Let us consider a finite number of symmetries $\{\Xi_j\}$ spanning $S_\theta(\mathcal{E})$ at θ. In the chart U let us consider the matrix (Ξ^i_j), where Ξ^i_j is the i-th component of the j-th point symmetry of $\mathcal{E}$. The rank of such a matrix decreases if the determinant of some submatrix vanishes. The points which satisfy such a condition form a closed set of U which does not contain θ. If we remove it from U, we obtain a neighborhood of p of maximal rank. □

4 Corollary. *The equation $\mathcal{E}$ can not coincide with the set of points of maximal rank of $S(\mathcal{E})$.*

Proof. It follows from the condition $\dim \mathcal{E} < \dim J^r(E,n)$. □

5 Theorem. *1. A necessary condition for $\mathcal{E}$ to be strongly Lie remarkable is that* $\dim \operatorname{sym}(\mathcal{E}) > \dim \mathcal{E}$.

2. A necessary condition for $\mathcal{E}$ to be weakly Lie remarkable is that $\dim \operatorname{sym}(\mathcal{E}) \geq \dim \mathcal{E}$.

Proof. 1. Let $U \subset J^r(E,n)$ be an open neighborhood at $\theta \notin \mathcal{E}$ where the rank of $S(\mathcal{E})$ is maximal (see the proposition and corollary above). Let q be this rank. By contradiction, let $q \leq \dim \operatorname{sym}(\mathcal{E})$. Then consider an l-dimensional submanifold of $J^r(E,n)$ passing at θ made up by q-dimensional leaves of $S(\mathcal{E})|_U$. This implies that the equation $\mathcal{E}$ is not strongly Lie remarkable.

2. If $\dim \mathrm{sym}(\mathcal{E}) < \dim \mathcal{E}$, then there would be several submanifolds of $J^r(E,n)$ passing at a point $\theta \in \mathcal{E}$ which are tangent to vector fields in $\mathrm{sym}(\mathcal{E})$, which contradicts the weak Lie remarkability.

□

6 Theorem. *If $S(\mathcal{E})|_{\mathcal{E}}$ is an l-dimensional distribution on $\mathcal{E} \subset J^r(E,n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.*

Proof. Of course, an equation $\tilde{\mathcal{E}}$ passing at $\theta \in \mathcal{E}$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of point symmetries and such that $T_\theta\tilde{\mathcal{E}} \neq T_\theta\mathcal{E}$ can not exist. So, suppose that there is an l-dimensional equation $\tilde{\mathcal{E}}$ passing at $\theta \in \mathcal{E}$ with $T_\theta\tilde{\mathcal{E}} = T_\theta\mathcal{E}$. A symmetry $X \in \mathrm{sym}(\mathcal{E})$ would be tangent to both $\tilde{\mathcal{E}}$ and $\mathcal{E}$, and, in particular, $X_\theta \in T_\theta\tilde{\mathcal{E}} = T_\theta\mathcal{E}$. Then, X would have two distinct integral curves passing at θ, leading to a contradiction. □

7 Theorem. *Let $S(\mathcal{E})$ be such that for any $\theta \notin \mathcal{E}$ we have $\dim S_\theta(\mathcal{E}) > l$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.*

Proof. By contradiction, let $\tilde{\mathcal{E}}$ be another l-dimensional equation passing at $\theta \notin \mathcal{E}$ and admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of point symmetries. Then $T_\theta\tilde{\mathcal{E}} \subsetneq S_\theta(\mathcal{E})$. This would imply that $\tilde{\mathcal{E}}$ would not have $\mathrm{sym}(\mathcal{E})$ as a subalgebra of the algebra of its point symmetries. □

8 Remark. In the proofs of previous theorems we construct some submanifold of the jet space without caring if this submanifold is a formally integrable equation. However, we would like to stress that in the scalar case any submanifold of codimension 1 is always a formally integrable equation. In our examples we will mainly be concerned with this last case.

The fact that an equation $\mathcal{E}$ is uniquely determined by its point symmetry algebra has consequences on the local, and in some cases global, topological structure of $\mathcal{E}$ as a manifold.

We recall (see, *e.g.*, [35]) that a *(left) action* of a Lie group G on a manifold M is a smooth map $a\colon G \times M \to M$ such that $a(g, a(h, x)) = a(gh, x)$ and $a(e, x) = x$ for all g, $h \in G$, $x \in M$, e being the identity in G. This concept has a local analogue (see, *e.g.*, [32]). From theorems XI, p. 58 and IV, p. 98 of [35] we obtain the following theorem.

9 Theorem. *Let $\mathfrak{s}$ be a l-dimensional subalgebra of $\mathrm{sym}(\mathcal{E})$. Let $\{\Xi_i^{(r)}\}_{1\leq i\leq l}$ be a basis of $\mathfrak{s}$. Let us suppose that $\mathrm{span}\{\Xi_i^{(r)}(\theta)\}_{1\leq i\leq l} = S_\theta(\mathcal{E}) = T_\theta\mathcal{E}$ for any $\theta \in \mathcal{E}$. Then $\mathcal{E}$ has the structure of a local Lie group.*

More precisely, there exists a Lie group S whose Lie algebra is $\mathfrak{s}$, and a local differentiable action $\varphi\colon S \times \mathcal{E} \to \mathcal{E}$ by point symmetries of $\mathcal{E}$ whose tangent map is the inclusion of $\mathfrak{s}$ into $\mathrm{sym}(\mathcal{E})$. If the vector fields Ξ_i are complete (in particular if $\mathcal{E}$ is a compact manifold), then the action is global.

Note that if the hypotheses of the above corollary are satisfied, then, following the terminology of [35], the Lie algebra $\mathfrak{s}$ is a *Kobayashi Lie algebra*. For this reason, if the action of S on $\mathcal{E}$ is global, then a necessary and sufficient condition that the map $\varphi(\cdot,\theta)\colon S \to \mathcal{E}$ be a diffeomorphism for $\theta \in \mathcal{E}$ is that the isotropy group of the action at θ is trivial [35, p. 105].

The previous theorem deals with topological aspects of the equation. In this paper we will not study topological aspects of Lie remarkability, even if they are quite interesting (see for instance proposition 18). We will postpone this aspect for a future investigation.

Many differential equations are given as the zero set of just one differentiable function. Indeed, the former definition and examples of Lie remarkable equations have been formulated by one of us [30] in this case (see also [26]). We may rediscover the former definition as a particular case of our present more comprehensive theory.

10 Proposition. *Let $\mathfrak{s}$ be a Lie subalgebra of $\chi(J^r(E,n))$. Let us suppose that the r-prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E,n)$ and that the set of r-th order functionally independent differential invariants of $\mathfrak{s}$ reduces to a unique element $I \in C^\infty(J^r(E,n))$. Then the submanifold of $J^r(E,n)$ described by $\Delta(I)=0$ (in particular $I=k$ for any $k\in\mathbb{R}$), with Δ an arbitrary smooth function, is a weakly Lie remarkable equation.*

Proof. It follows in view of the fact that $\Delta(I)=0$ is the most general equation admitting $\mathfrak{s}$ as a Lie subalgebra of point symmetries [32, 33]. □

When one speaks about Lie remarkable equations, a natural question arises. Namely, about Lie remarkability of the first prolongation $\mathcal{E}^1$ of $\mathcal{E}$ whenever $\mathcal{E}$ is Lie remarkable. Then we immediately realize (see theorem 5) that Lie remarkability is a property that, for dimensional reasons, generally does not hold on prolongations. But there is a special class of differential equations for which this is true. For instance, let an equation $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n) \to J^r(E,n)$. Such equations can be thought as 'generalized sprays' [36]. Examples of such equations are, for instance, the totally geodesic submanifold equation. With such an equation it is possible to associate the n-dimensional distribution $R\circ\nabla$ on $J^{r-1}(E,n)$ defined by $\theta \mapsto R_{\nabla(\theta)}$ (re recall that $R_{\nabla(\theta)}$ is a R-plane, see section 2). We call such equation *integrable* if this distribution is integrable.

11 Theorem. *Let $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n)\to J^r(E,n)$. Let $\mathcal{E}$ be integrable. If $\mathcal{E}$ is weakly Lie remarkable, then also $\mathcal{E}^k$ is weakly Lie remarkable.*

Proof. Of course, it is sufficient to prove the proposition for $k=1$. Following the same reasoning as in [25], we see that the prolongation $(\nabla(\theta))^{(1)}$ of a point $\nabla(\theta)$ is the point represented by the pair $\big(\nabla(\theta), T_\theta\nabla(R_{\nabla(\theta)})\big)$. Let us note that $T_\theta\nabla(R_{\nabla(\theta)})$ is a R-plane in view of the integrability. If we define $\nabla^{(1)}$ by $\nabla^{(1)}(\theta) = (\nabla(\theta))^{(1)}$, then

$$\mathcal{C}_{\nabla^{(1)}(\theta)}\left(\mathcal{E}^1\right) \simeq T_\theta\nabla^{(1)}\left(R_{\nabla(\theta)}\right) \simeq R_{\nabla(\theta)}.$$

Therefore, $\mathcal{E}^1$ is completely identified with $\mathcal{E}$, and point symmetries of both equations are vector fields on E whose $(r-1)$-prolongations are symmetries of the distribution $R\circ\nabla$. □

We have the following interesting result.

12 Theorem. *Let $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n) \to J^r(E,n)$. Let $\mathcal{E}$ be integrable. If it is weakly Lie remarkable, then any solution is invariant with respect to a 1-dimensional subalgebra of its algebra of point symmetries.*

Proof. In view of our previous discussion, we can identify $\mathcal{E}$ with $J^{r-1}(E,n)$, and a solution with an integrable submanifold of the distribution $R \circ \nabla$. Let $\theta \in J^{r-1}(E,n)$ and L be the unique solution passing at θ. Let $v_\theta \in T_\theta L$. By hypothesis, $v_\theta = a^i \Xi_i^{(r)}(\theta)$ with $a^i \in \mathbb{R}$ and $\Xi_i \in \chi(E)$. Let us consider the vector field $\Xi^{(r)} = a^i \Xi_i^{(r)}$. In view of the fact that $\Xi^{(r)}$ is still a point symmetry, and then sends solutions into solutions, we obtain that $\Xi^{(r)}$ is tangent to L. □

4 Lie remarkability of minimal submanifold equations

Let E be a Riemannian manifold with metric g. There are several interesting differential equations which are formulated on submanifolds of E (see, for example, [11]). Here we mostly deal with local geometric aspects of such equations (see [27] for an intrinsic treatment).

We show that the minimal surface equation in $\mathbb{R}^3$ and $\mathbb{R}^6$ are weakly Lie remarkable (provided we exclude from them the differential equation of planes), whereas in $\mathbb{R}^4$ and $\mathbb{R}^5$ they are not. Moreover, we will show how the Gaussian curvature of a surface is related with the strong Lie remarkability of the geodesic equation.

Now, we recall the main geometric concepts that we will use in this paper.

If $L \subset E$ is a submanifold of E, then we can consider the restriction of the metric g to the tangent space TL, which is a subspace of the tangent space of E. Such a restriction, denoted by g_L, is said to be the first fundamental form. The first fundamental form depends only on the first-order jet of L. So, there exists a tensor on $J^1(E,n)$, the *universal first fundamental form* g^H, which has the coordinate expression

$$g^H = (g_{\lambda\mu} + g_{\lambda j} u^j_\mu + g_{i\mu} u^i_\lambda + g_{ij} u^i_\lambda u^j_\mu) \overline{dx}^\lambda \otimes \overline{dx}^\mu. \tag{4}$$

Here $\overline{dx}^\lambda$ is a one-form induced on $J^1(E,n)$ by dx^λ by restricting it to total derivatives D_μ. Of course, for any submanifold $\iota\colon L \hookrightarrow E$ we have $g_L = \iota^* g^H$; this is the reason of the term 'universal'.

Now, on one hand, g_L produces a Levi-Civita connection on L, on the other hand the Levi-Civita connection of the environment space can be restricted to act on vector fields tangent to L. The difference between the two is a bilinear form II on TL with values in the normal space NL to L. The form II depends only on second-order jets of L. So, there exists a symmetric bilinear form on $J^2(E,n)$, the *universal second fundamental* *form* II^H, which has the coordinate expression

$$(5)\quad II^H = \Big(u^i_{\rho\nu} + \Gamma_\nu{}^i{}_\rho + \Gamma_k{}^i{}_\rho u^k_\nu + \Gamma_\nu{}^i{}_k u^k_\rho + \Gamma_j{}^i{}_k u^j_\nu u^k_\rho - u^i_\sigma(\Gamma_\nu{}^\sigma{}_\rho + \Gamma_k{}^\sigma{}_\rho u^k_\nu + \Gamma_\nu{}^\sigma{}_k u^k_\rho + \Gamma_j{}^\sigma{}_k u^j_\nu u^k_\rho)\Big)\overline{dx}^\rho \otimes \overline{dx}^\nu \otimes N_i.$$

Here the Γ's are the standard Christoffel symbols, and for each function $(u^i(x^\lambda))$ the vectors $N_i = \partial/\partial u^i - (g_{\mu i} + g_{ij}u^j_\mu)((g^H)^{-1})^{\mu\lambda}D_\lambda$ are a basis of the normal space to the graph $(x^\lambda, u^i(x^\lambda))$. From the above map we obtain the *mean curvature vector* by a metric contraction

$$(6)\quad H^H = \frac{1}{n}(g^H)^{-1}\lrcorner(II^H) = ((g^H)^{-1})^{\nu\rho}\Big(u^i_{\rho\nu} + \Gamma_\nu{}^i{}_\rho + \Gamma_k{}^i{}_\rho u^k_\nu + \Gamma_\nu{}^i{}_k u^k_\rho + \Gamma_j{}^i{}_k u^j_\nu u^k_\rho - u^i_\sigma(\Gamma_\nu{}^\sigma{}_\rho + \Gamma_k{}^\sigma{}_\rho u^k_\nu + \Gamma_\nu{}^\sigma{}_k u^k_\rho + \Gamma_j{}^\sigma{}_k u^j_\nu u^k_\rho)\Big)N_i.$$

In the case $m = 1$ (L hypersurface) and if L is orientable, then a unit normal vector allows us to convert II into a $(1,1)$ tensor field, and H into a function, the mean curvature scalar. Then, it turns out that the invariants of II are its eigenvalues, which are said to be the principal curvatures of L, and its determinant, which is said to be the Gaussian curvature G. In a way analogous to eqs. (5) and (6) we obtain a function G^H on $J^2(E,n)$ whose coordinate expression is

$$(7)\qquad G^H = \det\left(((g^H)^{-1})^{\sigma\nu} II^H{}_\nu{}^1{}_\rho \, \|N_1\|\right).$$

The submanifold $(II^H)^{-1}(0) \subset J^2(E,n)$ is the *equation of totally geodesic n-dimensional submanifolds.*

The submanifold $(H^H)^{-1}(0) \subset J^2(E,n)$ is the *equation of minimal n-dimensional submanifolds.*

In the case $n = 1$ the equation of minimal 1-dimensional submanifolds and the equation of totally geodesic 1-dimensional submanifolds reduce to the *equation of unparametrized geodesics* (this means that its solutions are geodesics as 1-dimensional submanifolds, *i.e.*, with no distinguished parametrization).

4.1 Minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$.

In our formalism, minimal surfaces in $\mathbb{R}^{n+m}$ are n-dimensional submanifolds L in $\mathbb{R}^{n+m}$ which are solutions of the equation $H^H = 0$, *i.e.* $H^H|_{L^{(2)}} = 0$. The Euclidean metric of $\mathbb{R}^{n+m}$ is δ_{ij} with respect to cartesian coordinates, hence all Γ's are zero. The local form of the equation of minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$ can be obtained by (6). For instance, in the case $n = 2$ and $m = 1$ we have

$$g^H = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix},$$

and $H^H = h^H N_1/\|N_1\|$, where h^H is the scalar mean curvature whose coordinate expression is

$$h^H = \frac{1}{2}\frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{(1+u_x^2+u_y^2)^{3/2}}.$$

Then the local representation of the equation of minimal surfaces of $\mathbb{R}^3$ is

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0. \tag{8}$$

The equation of totally geodesic submanifolds of $\mathbb{R}^3$ is

$$u_{xx} = u_{xy} = u_{yy} = 0. \tag{9}$$

Also, we note that the Gaussian curvature G^H has the coordinate expression

$$G^H = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^2}. \tag{10}$$

Eq. (8) is globally the union of the 3 zero sets of H (or, equivalently, h) in $J^2(\mathbb{R}^3, 2)$ each of which is obtained in a cartesian coordinate set by cycling the dependent variable. This means that one should consider the zero sets also in the charts (x, u, y) and (u, x, y) and glue them into the submanifold $(H^H)^{-1}(0)$.

To compute the expression of the equation of minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$ we need $(g^H)^{-1}$, for which we do not have a general expression. Anyway, it seems to us to be reasonable to conjecture that the only point symmetries of the minimal submanifold equation in $\mathbb{R}^{n+m}$ are isometries and homotheties (or dilatations, *i.e.*, homogeneous scalings of the coordinates). In view of theorem 5, we infer that *possible candidates of Lie remarkable minimal submanifold equations are characterized by having dimension less than or equal to the dimension of the Lie algebra generated by infinitesimal isometries and homotheties.*

13 Lemma. *Let d be the dimension of the equation of minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$, with $n \geq 2$, $m \geq 1$. Let i be the dimension of the Lie algebra generated by infinitesimal isometries and homotheties in $\mathbb{R}^{n+m}$. Then,*

1. *the only integer solutions[2] of the equation $d = i$ are $(n, m) = (2, 1)$ and $(n, m) = (2, 4)$;*
2. *if $n = 2$ the inequality $d > i$ holds for $m = 2$, $m = 3$.*
3. *if $m = 1$ the inequality $d > i$ holds for $n > 2$.*

Proof. From (1) we have $d = \dim J^2(E, n) - m = n + m\binom{n+2}{2} - m$, where m is the number of dependent variables and coincides with the number of components of H^H. Moreover, $i = 1/2\,(n+m)(n+m+1) + 1$. A direct computation proves the statements. □

[2] G. Lo Faro, private communication.

14 Theorem. *The equation of minimal surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ is nor strongly neither weakly Lie remarkable, whereas it is weakly Lie remarkable in $\mathbb{R}^3$ and $\mathbb{R}^6$, provided we remove a singular equation.*

Proof. The point symmetries of the equation of minimal surfaces in $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^6$ are the isometries and homotheties. Then the first part of the theorem follows in view of the above lemma and of theorem 5. In fact, we have that if $n = m = 2$ then $d = 12$ and $i = 11$, while if $n = 2$, $m = 3$ then $d = 17$ and $i = 16$. Now, let us consider the case of $\mathbb{R}^3$. Eq. (8) admits a 7–parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \frac{\partial}{\partial y}, \qquad \Xi_3 = \frac{\partial}{\partial u}, \tag{11a}$$

$$\Xi_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad \Xi_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \quad \Xi_6 = u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u}. \tag{11b}$$

$$\Xi_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \tag{11c}$$

The second order prolonged vector fields give rise to a distribution of rank 7 on the whole jet space provided that we exclude the 6–dimensional submanifold of $J^2(\mathbb{R}^3, 2)$ locally described by the system

$$\begin{cases} (1 + u_y^2)u_{xx} - (1 + u_x^2)u_{yy} = 0, \\ (1 + u_y^2)u_{xy} - u_x u_y u_{yy} = 0, \end{cases} \tag{12}$$

where the rank is less or equal to 6.

The above submanifold intersects (8) in (9). We note that on (9) the rank reduces to 5. Furthermore, the equation of vanishing Gaussian curvature (see (10))

$$u_{xx}u_{yy} - u_{xy}^2 = 0 \tag{13}$$

admits symmetries (11) and intersects Eq. (8) in Eq. (9).

Thus, according to theorem 6, the theorem follows in the case of $\mathbb{R}^3$. Similar reasonings and computations lead to the proof in the case of $\mathbb{R}^6$, then we omit it. □

15 Remark. The unique second order differential invariant of (11) is

$$I = \frac{\big((1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}\big)^2}{(1 + u_x^2 + u_y^2)(u_{xx}u_{yy} - u_{xy}^2)},$$

which can be expressed as the ratio $I = (2h^H)^2/G^H$.

Therefore, it follows that the most general second order scalar partial DE invariant by (11) must be given as $\Delta(I) = 0$.

We note that:

- the system $\{h^H = G^H = 0\}$ reduces to Eq. (9).

- the system $\{h^H = 0,\, G^H \neq 0\}$, that is $I = 0$, is Eq. (8).

- the system $\{h^H \neq 0,\, G^H = 0\}$, that is $1/I = 0$, is Eq. (13).

According to proposition 10, the above equations result weakly Lie remarkable. Also, we note that Eq. (13) is of Monge-Ampère type. Its Lie remarkability will be considered in theorem 16.

4.2 Geodesic equation on surfaces

As a particular case of equation of totally geodesic submanifolds, we have the equation of geodesics in a $(m+1)$-dimensional Riemannian manifolds E. In a local chart (x, u^i_x, u^i_{xx}) of $J^2(E, 1)$ it has the form

$$u^h_{xx} + \Gamma_0{}^h{}_0 + 2u^j_x\Gamma_0{}^h{}_j + u^i_x u^j_x \Gamma_i{}^h{}_j - u^h_x(\Gamma_0{}^0{}_0 + 2u^j_x\Gamma_0{}^0{}_j + u^i_x u^j_x \Gamma_i{}^0{}_j) = 0. \tag{14}$$

This equation plays a key role in the discussion of projectively equivalent connections [25]. In fact, point symmetries of Eq. (14) are the projective vector fields of the manifold E, and conversely. We shall analyze the case $m = 1$; we show that its invariant solutions behave as predicted by theorem 12.

16 Theorem. *Let E be a complete simply connected Riemannian 2-dimensional manifold. Then Eq. (14) is strongly Lie remarkable if and only if E has constant Gaussian curvature. Furthermore, it is locally isomorphic to the group of isometries of E.*

Proof. Let us suppose that Eq. (14) is strongly Lie remarkable. Then, taking into consideration that $\dim \mathcal{E} = 3$, in view of theorem 5, we conclude that E must have constant Gaussian curvature. In fact, a classical result of projective geometry stays that the dimension of the algebra of projective vector fields on a 2-dimensional Riemannian manifold can be $1, 2, 3$, or 8, where the maximal dimension is attained just in the case of constant Gaussian curvature (see for instance [21, 22]).

Now let us suppose that E has constant Gaussian curvature. Then, due to well-known results of Riemannian geometry, E is isometric to either the Euclidean plane, or the sphere, or the hyperbolic plane, depending on the sign of the Gaussian curvature. In the table below we show as, in these cases, the 8-dimensional algebra of point symmetries is realized as vector fields on E:

	$dx^2 + du^2$	$dx^2 + \sin^2(x)du^2$	$\frac{1}{u^2}(dx^2 + du^2)$
Ξ_1	∂_x	∂_u	∂_x
Ξ_2	∂_u	$-\cos(u)\partial_x + \cot(x)\sin(u)\partial_u$	$x\partial_x + u\partial_u$
Ξ_3	$x\partial_u - u\partial_x$	$\sin(u)\partial_x + \cot(x)\cos(u)\partial_u$	$(x^2 - u^2)\partial_x + 2xu\partial_u$
Ξ_4	$x\partial_u$	$\sin(2x)\partial_x$	$2x(x^2 + u^2)\partial_x + \frac{u^4 - x^4}{u}\partial_u$
Ξ_5	$u\partial_u$	$\sin^2(x)\cos(u)\partial_x$	$-(3x^2 + u^2)\partial_x + \frac{2x^3}{u}\partial_u$
Ξ_6	$x\partial_x$	$\sin^2(x)\sin(u)\partial_x$	$\frac{x}{u}\partial_u$
Ξ_7	$xu\partial_x + u^2\partial_u$	$-\frac{\sin(2x)\cos(2u)}{2}\partial_x + \sin(2u)\partial_u$	$\frac{1}{u}\partial_u$
Ξ_8	$x^2\partial_x + xu\partial_u$	$\frac{\sin(2x)\sin(2u)}{2}\partial_x + \cos(2u)\partial_u$	$\frac{x^2+u^2}{u}\partial_u$

We see that in these three cases we have that the algebra of point symmetries spans a distribution of rank 4 on the whole jet spaces except for the equation under consideration. Then the theorem follows in view of theorem 7.

The second part of the theorem follows taking into account that Ξ_1, Ξ_2 and Ξ_3 form a basis of the Lie algebra of local isometries of the corresponding metrics. Since the prolongations span a distribution of rank 3 on the equation, by using the same arguments after Eq. (9), the assertion is proved in view of theorem 9. □

17 Remark. Actually, in the case in which the Lie algebra of point symmetries of equation (14), for $m = 1$, is 8-dimensional, then the equation is point equivalent to $y_{xx} = 0$. Then the Lie remarkability in the case of constant curvature could be reduced to study the equation $y_{xx} = 0$.

It could be interesting to see when an equation of type $(II^H)^{-1}(0)$ (see (5)) satisfies the hypotheses of theorems 11 and 12. A first step in this direction is given by the previous theorem. In fact, in the hypotheses of the theorem, Eq. (14) is strongly (and then weakly) Lie remarkable, and invariant solutions always exist. Actually, we can assume weaker hypotheses (for instance, to work in the case when there are only three point symmetries), and to see if we get a weakly Lie remarkable equation. This will be the object of a future work.

18 Proposition.

- *The geodesic equation on $\mathbb{R}^2$ is diffeomorphic to the group of isometries of $\mathbb{R}^2$.*

- *The geodesic equation on S^2 is diffeomorphic to the connected component of the identity of the group of isometries of S^2.*

Proof. We have $J^1(\mathbb{R}^2, 1) \simeq \mathbb{R}^2 \times S^1$ and $J^1(S^2, 1) \simeq SO(3)$, and the geodesic equations are diffeomorphic to the respective first jet spaces (see also the proof of theorem 11). □

5 Lie remarkability of Monge–Ampère equations

In this section we apply our results to Monge–Ampère equations of various order. Since we deal with local point symmetries of these equations, as we are interested at the moment to local aspects, we will interpret them as submanifolds of jets of a trivial bundle $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. See [23] for a geometric treatment of second-order Monge–Ampère equations.

5.1 Second order Monge-Ampère equation

In 1968 Boillat [8] discovered a remarkable characterization of the celebrated Monge–Ampère equation [45]. In fact, he proved that the most general scalar second order partial DE for the unknown $u(x, y)$, assumed to be hyperbolic (so one of the independent variables has the meaning of time), having the property of complete exceptionality [7, 20], is given as a linear combination of all minors extracted from the Hessian matrix with coefficients depending on the independent variables x and y, the dependent variable u and first order derivatives u_x and u_y, that is

$$\kappa_1(u_{xx}u_{yy} - u_{xy}^2) + \kappa_2 u_{xx} + \kappa_3 u_{xy} + \kappa_4 u_{yy} + \kappa_5 = 0, \tag{15}$$

where $\kappa_i = \kappa_i(x, y, u, u_x, u_y)$ $(i = 1, \ldots, 5)$ are arbitrary functions.

The complete exceptionality requirement allowed to derive Monge–Ampère equations for the unknown u depending on three independent variables [41], on four independent variables [14], and on arbitrary number of independent variables [9]: all these equations share the property of being given as linear combinations of all minors extracted from the Hessian matrix with coefficients depending on the independent variables, the dependent variable u and first order derivatives.

Also, Boillat [10], by using once again the criterion of complete exceptionality, derived the partial DEs of higher order and called them higher order Monge–Ampère equations: all these equations are given as a linear combination of all minors extracted from suitable Hankel matrices built with the higher order derivatives.

Now, let us restrict ourselves to consider Eq. (15) in the case in which the coefficients κ_i are constant (in particular we assume $\kappa_1 \neq 0$ in order to have a nonlinear equation).

Through the substitution $u \to u + \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2$, where $\alpha_1 = -\kappa_4/2\kappa_1$, $\alpha_2 = \kappa_3/2\kappa_1$, $\alpha_3 = -\kappa_2/2\kappa_1$, Eq. (15) is mapped to

$$u_{xx}u_{yy} - u_{xy}^2 = \kappa, \tag{16}$$

where $\kappa = (4\kappa_2\kappa_4 - 4\kappa_1\kappa_5 - \kappa_3^2)/4\kappa_1^2$.

If $\kappa = 0$ we have the homogeneous Monge-Ampère equation for the surface $u(x, y)$ with zero Gaussian curvature.

19 Theorem. *Eq.* (16) *is weakly Lie remarkable if* $\kappa \neq 0$*, whereas it is strongly Lie remarkable if* $\kappa = 0$.

Proof. If $\kappa \neq 0$, then Eq. (16) admits a 9–parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\text{(17a)} \qquad \Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \frac{\partial}{\partial y}, \qquad \Xi_3 = \frac{\partial}{\partial u},$$

$$\text{(17b)} \qquad \Xi_4 = x\frac{\partial}{\partial y}, \qquad \Xi_5 = y\frac{\partial}{\partial x}, \qquad \Xi_6 = x\frac{\partial}{\partial u},$$

$$\text{(17c)} \qquad \Xi_7 = y\frac{\partial}{\partial u}, \qquad \Xi_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad \Xi_9 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

The second order prolonged vector fields give rise to a distribution of rank 7 on the whole jet space provided that we exclude the 5–dimensional submanifold locally described by (9) where the rank reduces to 5. We note that (9) is not a submanifold of (16), then the above second order vector fields give rise to a distribution on Eq. (16) of rank 7. Thus, according to theorem 6, Eq. (16) results weakly Lie remarkable.

On the contrary, if $\kappa = 0$, Eq. (16) admits a 15–parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\text{(18)} \qquad \frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}\right),$$

$\forall\, a, b \in \{x, y, u\}$.

In this case, the second order prolonged vector fields give rise to a distribution of rank 8 on the whole jet space provided that we exclude the 7-dimensional submanifold locally described by the equation (13), where the rank reduces to 7, provided we exclude the 5-dimensional submanifold of (13) locally described by (9), where the rank reduces to 5. Thus, according to theorem 7, Eq. (13) results strongly Lie remarkable. □

There are various 7-dimensional Lie subalgebras of the 9-dimensional Lie algebra of the symmetries of Eq. (13) whose second order prolonged vector fields give rise to a distribution of rank 7. For instance, we can consider the subalgebra whose generators are $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_6, \Xi_7, \Xi_8\}$. Then, in view of theorem 9, this equation has the structure of a local Lie group diffeomorphic to a local subgroup of the group of its point symmetries.

Moreover, it is natural to ask if among the 15-dimensional algebra of point symmetries of (13) it is possible to extract some subalgebra with respect to Eq. (13) is still strongly Lie remarkable. In fact, in view of theorems 5 and 7, it could be sufficient an 8-dimensional Lie subalgebra. We see that there are many 8-dimensional Lie subalgebras whose second order prolonged vector fields generate a distribution of rank 8 (provided we exclude, of course, Eq. (13)), even if we confine ourselves to consider the

8-dimensional subalgebras spanned by subsets containing 8 elements chosen in the set of basic vector fields given by (18). Here we quote the Lie subalgebra spanned by the operators

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y}, \quad u\frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial u}.$$

20 Remark. In [38] the author showed that the dimension of the subalgebra of the algebra of point symmetries of Eq. (13) which determines this equation has to be at least 8, which we can easily obtain from theorem 5. Furthermore, in [38, 39, 40] it is claimed that the 6–dimensional subalgebra

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial u}.$$

characterizes completely Eq. (13) provided we restrict our attention to fully symmetric systems. This last requirement is essential as the previous symmetries can not determine Eq. (13) in view of theorem 5. For instance, the equation $\Delta(u_y, u_{yy}) = 0$ where Δ is an arbitrary smooth function, admits the same 6-dimensional Lie subalgebra.

21 Remark. We observe that $I = u_{xx}u_{yy} - u_{xy}^2$ is a second order differential invariant of (17), and then, in view of Lie remarkability, it is unique up to functional dependence. Therefore it follows that the most general second order scalar partial DE invariant by (17) must be given as $\Delta(I) = 0$, where Δ is an arbitrary function of the invariant. For instance, $I = \kappa$, with $\kappa \in \mathbb{R}$, results again a weakly Lie remarkable in view of proposition 10.

5.2 Third order Monge-Ampère equation

The third order Monge-Ampère equation [10, 15] in two independent variables is built by starting with the Hankel matrix

$$H_3 = \begin{bmatrix} u_{xxx} & u_{xxy} & u_{xyy} \\ u_{xxy} & u_{xyy} & u_{yyy} \end{bmatrix}$$

and taking a linear combination (with coefficients depending on independent variables, dependent variable and derivatives up to the order 2) of all minors extracted from the matrix H_3:

$$(19)\quad \kappa_1(u_{xxy}u_{yyy} - u_{xyy}^2) + \kappa_2(u_{xxx}u_{yyy} - u_{xxy}u_{xyy}) + \kappa_3(u_{xxx}u_{xyy} - u_{xxy}^2) + \kappa_4 u_{xxx} + \kappa_5 u_{xxy} + \kappa_6 u_{xyy} + \kappa_7 u_{yyy} + \kappa_8 = 0.$$

By limiting ourselves to consider the coefficients κ_i $(i = 1, \dots, 8)$ to be constant and using the substitution $u \to u + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3$, with suitable coefficients a_i $(i = 1, \dots, 4)$, we are led to the equation

$$(20)\quad \kappa_1(u_{xxy}u_{yyy} - u_{xyy}^2) + \kappa_2(u_{xxx}u_{yyy} - u_{xxy}u_{xyy}) + \kappa_3(u_{xxx}u_{xyy} - u_{xxy}^2) + \kappa = 0,$$

where κ is constant.

In general, this equation admits a 10-dimensional Lie algebra of point symmetries; since the dimension of the jet space is 12 it follows (see theorem 5) that Eq. (20) is neither strongly nor weakly Lie remarkable.

An interesting case of Eq. (20) is obtained by choosing $\kappa_1 = \kappa_2^2/\kappa_3$, whereupon Eq. (20) becomes

$$(u_{xxy}u_{yyy} - u_{xyy}^2) + \lambda(u_{xxx}u_{yyy} - u_{xxy}u_{xyy}) + \lambda^2(u_{xxx}u_{xyy} - u_{xxy}^2) + \mu = 0 \tag{21}$$

with $\lambda = \kappa_3/\kappa_2$, $\mu = \kappa\kappa_3/\kappa_2^2$. In such a case the following theorem holds true.

22 Theorem. *Eq.* (21) *is weakly Lie remarkable (provided we remove a singular subset).*

Proof. The Lie algebra of the admitted point symmetries of Eq. (21) is infinite–dimensional and is spanned by the vector fields

(22a)
$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \frac{\partial}{\partial y}, \qquad \Xi_3 = \frac{\partial}{\partial u},$$
(22b)
$$\Xi_4 = (x-\lambda y)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \Xi_5 = (x-\lambda y)\frac{\partial}{\partial y} - \lambda u\frac{\partial}{\partial u}, \quad \Xi_6 = \lambda y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u},$$
(22c)
$$\Xi_7 = x\frac{\partial}{\partial u}, \qquad \Xi_8 = y\frac{\partial}{\partial u}, \qquad \Xi_9 = x^2\frac{\partial}{\partial u},$$
(22d)
$$\Xi_{10} = xy\frac{\partial}{\partial u}, \qquad \Xi_{11} = y^2\frac{\partial}{\partial u}, \qquad \Xi_{12} = F(x-\lambda y)\frac{\partial}{\partial u},$$

where F is an arbitrary function of the indicated combination of x and y.

If $F''' \neq 0$ (where the $'$ denotes the differentiation with respect to the argument), the corresponding third order prolonged vector fields give rise to a distribution of rank 11 on the whole jet space, provided that we exclude:

- The 11-dimensional submanifold locally described by the equation

$$\lambda^3 u_{xxx} + 3\lambda^2 u_{xxy} + 3\lambda u_{xyy} + u_{yyy} = 0,$$

- The 10-dimensional submanifolds locally described by the following systems:

$$\left\{ \begin{array}{l} 2\lambda^3 u_{xxx} - 3\lambda u_{xyy} - u_{yyy} = 0, \\ \lambda u_{xxx} + u_{xxy} = 0; \end{array} \right. \tag{23a}$$

$$\left\{ \begin{array}{l} 5\lambda^3 u_{xxx} + 3\lambda^2 u_{xxy} + 2u_{yyy} = 0, \\ \lambda^2 u_{xxx} - \lambda u_{xxy} - 2u_{xyy} = 0. \end{array} \right. \tag{23b}$$

It remains to remove from equation (21) the points lying in the above submanifolds.

By excluding the operator Ξ_5, and taking $F(x-\lambda y) = (x-\lambda y)^3$ in (22), we have a 11-dimensional Lie subalgebra whose vector fields prolonged up to the third order generate a distribution of rank 11. $\square$

23 Remark. By following the reasoning of remarks 15 and 21, we see that the unique third order differential invariant of (21), up to functional dependence, is

$$I = (u_{ttx}u_{xxx} - u_{txx}^2) + \lambda(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \lambda^2(u_{ttt}u_{txx} - u_{ttx}^2). \tag{24}$$

5.3 Fourth order Monge-Ampère equation

Finally, let us consider the fourth order Monge-Ampère equation [10] in two independent variables; it is built by starting with the Hankel matrix

$$H_4 = \begin{bmatrix} u_{xxxx} & u_{xxxy} & u_{xxyy} \\ u_{xxxy} & u_{xxyy} & u_{xyyy} \\ u_{xxyy} & u_{xyyy} & u_{yyyy} \end{bmatrix}$$

and taking a linear combination (with coefficients depending on independent variables, dependent variable and derivatives up to the order 3) of all minors (including the determinant of H_4) extracted from the matrix H_4:

$$\begin{aligned} \kappa_1 \det(H_4) + \kappa_2 \det(H_4^{(1,1)}) + \kappa_3 \det(H_4^{(1,2)}) + \kappa_4 \det(H_4^{(1,3)}) \\ + \kappa_5 \det(H_4^{(2,2)}) + \kappa_6 \det(H_4^{(2,3)}) + \kappa_7 \det(H_4^{(3,3)}) \\ + \kappa_8 u_{xxxx} + \kappa_9 u_{xxxy} + \kappa_{10} u_{xxyy} + \kappa_{11} u_{xyyy} + \kappa_{12} u_{yyyy} + \kappa_{13} = 0, \end{aligned} \tag{25}$$

where $H_4^{(i,j)}$ denotes the 2×2 matrix obtained by removing from H_4 the i-th row and the j-th column. By limiting ourselves to consider the coefficients κ_i ($i = 1, \dots, 13$) to be constant and using the substitution $u \to u + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4$, with suitable coefficients a_i ($i = 1, \dots, 5$), we are led to the equation

$$\begin{aligned} \kappa_1 \det(H_4) + \kappa_2 \det(H_4^{(1,1)}) + \kappa_3 \det(H_4^{(1,2)}) + \kappa_4 \det(H_4^{(1,3)}) \\ + \kappa_5 \det(H_4^{(2,2)}) + \kappa_6 \det(H_4^{(2,3)}) + \kappa_7 \det(H_4^{(3,3)}) + \kappa = 0, \end{aligned} \tag{26}$$

where κ is constant.

The Lie algebra of point symmetries of Eq. (26) is 13-dimensional; since Eq. (26) defines in the jet space a submanifold of dimension 17, in general Eq. (26) is neither strongly nor weakly Lie remarkable (see theorem 5). Nevertheless, if we consider the special case $\det(H_4) = 0$, then the following theorem holds true.

24 Theorem. *The equation*

$$\det(H_4) = 0, \tag{27}$$

is weakly Lie remarkable (provided we remove a singular subset).

Proof. The Lie algebra of the admitted point symmetries of Eq. (27) is 19–dimensional and is spanned by the vector fields

$$\begin{aligned}
&\text{(28a)}\quad \Xi_1 = \frac{\partial}{\partial x}, && \Xi_2 = \frac{\partial}{\partial y}, && \Xi_3 = \frac{\partial}{\partial u},\\
&\text{(28b)}\quad \Xi_4 = x\frac{\partial}{\partial u}, && \Xi_5 = y\frac{\partial}{\partial u}, && \Xi_6 = x^2\frac{\partial}{\partial u},\\
&\text{(28c)}\quad \Xi_7 = xy\frac{\partial}{\partial u}, && \Xi_8 = y^2\frac{\partial}{\partial u}, && \Xi_9 = x^3\frac{\partial}{\partial u},\\
&\text{(28d)}\quad \Xi_{10} = x^2y\frac{\partial}{\partial u}, && \Xi_{11} = xy^2\frac{\partial}{\partial u}, && \Xi_{12} = y^3\frac{\partial}{\partial u},\\
&\text{(28e)}\quad \Xi_{13} = x\frac{\partial}{\partial x}, && \Xi_{14} = y\frac{\partial}{\partial x}, && \Xi_{15} = x\frac{\partial}{\partial y},\\
&\text{(28f)}\quad \Xi_{16} = y\frac{\partial}{\partial y}, && \Xi_{17} = u\frac{\partial}{\partial u}, &&
\end{aligned}$$

$$\text{(28g)}\quad \Xi_{18} = x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u}\right),\ \Xi_{19} = y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u}\right).$$

The corresponding fourth order prolonged vector fields give rise to a distribution of rank 16 on the whole jet space, provided that we exclude the points of the singular subset, *i.e.* the solutions of the following systems:

$$\text{(29a)}\quad \begin{cases} u_{xxxx}u_{xyyy}^{1/2} + 2u_{xxxy}^{3/2} = 0,\\ u_{yyyy}u_{xxxy}^{1/2} + 2u_{xyyy}^{3/2} = 0,\\ u_{xxyy} + (u_{xxxy}u_{xyyy})^{1/2} = 0; \end{cases}$$

$$\text{(29b)}\quad \begin{cases} u_{xxxx}u_{xyyy}^{1/2} + u_{xxxy}^{3/2} = 0,\\ u_{yyyy}u_{xxxy}^{1/2} + u_{xyyy}^{3/2} = 0,\\ u_{xxyy} + (u_{xxxy}u_{xyyy})^{1/2} = 0; \end{cases}$$

$$\text{(29c)}\quad \begin{cases} u_{xxxx}u_{xyyy}^{1/2} - 2u_{xxxy}^{3/2} = 0,\\ u_{yyyy}u_{xxxy}^{1/2} - 2u_{xyyy}^{3/2} = 0,\\ u_{xxyy} - (u_{xxxy}u_{xyyy})^{1/2} = 0; \end{cases}$$

$$\text{(29d)}\quad \begin{cases} 2u_{xxxx}u_{xyyy}^2 - 3u_{xxxy}u_{xxyy}u_{xyyy} + u_{xxxy}u_{xyyy}(9u_{xxyy}^2 - 8u_{xxxy}u_{xyyy})^{1/2} = 0,\\ 2u_{xxxy}u_{yyyy} + 3u_{xxyy}u_{xyyy} - u_{xyyy}(9u_{xxyy}^2 - 8u_{xxxy}u_{xyyy})^{1/2} = 0; \end{cases}$$

$$\text{(29e)}\quad \begin{cases} 2u_{xxxx}u_{xyyy}^2 - 3u_{xxxy}u_{xxyy}u_{xyyy} - u_{xxxy}u_{xyyy}(9u_{xxyy}^2 - 8u_{xxxy}u_{xyyy})^{1/2} = 0,\\ 2u_{xxxy}u_{yyyy} - 3u_{xxyy}u_{xyyy} - u_{xyyy}(9u_{xxyy}^2 - 8u_{xxxy}u_{xyyy})^{1/2} = 0; \end{cases}$$

$$\text{(29f)}\quad \begin{cases} u_{xxxx}u_{xyyy}^2 + 2u_{xxyy}^3 - 3u_{xxxy}u_{xxyy}u_{xyyy} - 2(u_{xxyy}^2 - u_{xxxy}u_{xyyy})^{3/2} = 0,\\ u_{yyyy}u_{xxxy}^2 + 2u_{xxyy}^3 - 3u_{xxxy}u_{xxyy}u_{xyyy} + 2(u_{xxyy}^2 - u_{xxxy}u_{xyyy})^{3/2} = 0; \end{cases}$$

$$\text{(29g)}\quad \begin{cases} u_{xxxx}u_{xyyy}^2 + 2u_{xxyy}^3 - 3u_{xxxy}u_{xxyy}u_{xyyy} + 2(u_{xxyy}^2 - u_{xxxy}u_{xyyy})^{3/2} = 0,\\ u_{yyyy}u_{xxxy}^2 + 2u_{xxyy}^3 - 3u_{xxxy}u_{xxyy}u_{xyyy} - 2(u_{xxyy}^2 - u_{xxxy}u_{xyyy})^{3/2} = 0; \end{cases}$$

It remains to remove from the equation (27) the points fulfilling one of the above systems of equations.

Finally, it is worth of noticing that the vector fields $\{\Xi_1, \dots, \Xi_{16}\}$ span a 16-dimensional Lie subalgebra whose fourth order prolongations provide a distribution of rank 16. $\square$

25 Remark. Again, by following the reasoning of remarks 15 and 21, we see that the unique fourth order differential invariant of (27), up to functional dependence, is

$$I = u_{tttt}(u_{ttxx}u_{xxxx} - u_{txxx}^2) + 2u_{tttx}u_{ttxx}u_{txxx} - u_{ttxx}^3 - u_{tttx}^2 u_{xxxx}. \tag{30}$$

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