Integrability of Kupershmidt deformations[*]

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Abstract

We prove that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. Moreover, Magri hierarchies of the initial system give rise to Magri hierarchies of Kupershmidt deformations as well. Since Kupershmidt deformations are not written in evolution form, we start with an outline a geometric framework to study Hamiltonian properties of general non-evolution differential equations, developed in [2] (see also [4]).

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1 Introduction

Recently, in the paper [3] the authors derived the new equation

$$(1) \qquad (\partial_x^2 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0,$$

called the KdV6 equation, which turned out to pass the Painlevé test. The authors introduced the new variables $v = u_x$ and $w = u_t + u_{xxx} + 6u_x^2$, transforming (1) into the system

$$(2) \qquad v_t + v_{xxx} + 12vv_x - w_x = 0, \quad w_{xxx} + 8vw_x + 4wv_x = 0,$$

and posed the problem to study conservation laws and Hamiltonian structures of the above system.

In [6], Kupershmidt proved the existence of an infinite series of conservation laws for (2) in the following way. He noted that (2) can be written in the form

$$(3) \qquad F + A_1^*(w) = 0, \quad A_2^*(w) = 0,$$

where $F = 0$ is the KdV equation in the unknown v, and A_1, A_2 are the two standard Hamiltonian operators for $F = 0$. So, (3) can be considered to be a deformation of $F = 0$; we call it the *Kupershmidt deformation*. The following result yields an infinite series of conservation laws for (3).

Theorem (Kupershmidt). *Let $F = u_t - f = 0$ be an evolution bi-Hamiltonian system, with A_1, A_2 being the corresponding Hamiltonian operators. If this equation has a Magri hierarchy of conserved densities $\frac{dH_i}{dt} = 0$, $A_1\left(\frac{\delta H_i}{\delta u}\right) = A_2\left(\frac{\delta H_{i+1}}{\delta u}\right)$, then H_1, H_2, ... are conserved densities for* (3).

Proof.

$$\frac{dH_i}{dt} = \left\langle \frac{\delta H_i}{\delta u}, f + A_1(w) \right\rangle = \left\langle -A_1\left(\frac{\delta H_i}{\delta u}\right), w \right\rangle$$
$$= \left\langle -A_2\left(\frac{\delta H_{i+1}}{\delta u}\right), w \right\rangle = \left\langle \frac{\delta H_{i+1}}{\delta u}, A_2(w) \right\rangle = 0.$$

$\square$

Further, Kupershmidt conjectured that these conservation laws commute (in a sense) so that (3) is integrable.

What makes this conjecture especially interesting is that system (3) is not in evolution form. Recently, in [2] (see also [4]) we, together with S. Igonin, have introduced a generalization of Hamiltonian formalism for general, not necessarily evolution, systems.

In the present paper, we apply this formalism to the Kupershmidt conjecture. Namely we prove that the Kupershmidt deformation of every bi-Hamiltonian equation is again a bi-Hamiltonian system and that every hierarchy of conservation laws of the original bi-Hamiltonian system gives rise to a hierarchy of conservation laws of the Kupershmidt deformation.

2 Preliminaries

An adequate setting for dealing with symmetries and conservation laws of differential equations is provided by jet bundles. Our main sources are [1, 5]; we will shortly describe our notation below.

From now on all manifolds and maps are C^∞.

Jets and differential equations. Let $\pi\colon E \to M$ be a vector bundle. We denote by $J^k(\pi)$ the corresponding jet manifold and by $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$ for $k > l$ and $\pi_k\colon J^k(\pi) \to E$ the standard projections. The inverse limit of the chain of projections $\cdots \to \pi_{k+1,k} \to \pi_{k,k-1} \to \cdots$ is said to be the *infinite order jet space* and is denoted by $J^\infty(\pi)$.

Let $x^1, \ldots, x^n$ be local coordinates in M, $u^1, \ldots, u^m$ and be local fiber coordinates in E. Then u_σ^j, where σ is a multiindex of arbitrary length, denote local derivative coordinates on the fibers of $J^\infty(\pi)$. If $s\colon M \to E$ is a section then its prolongation $j_\infty s\colon M \to J^\infty(\pi)$ fulfills $u_\sigma^j \circ j_\infty s = \partial^{|\sigma|}(u^j \circ s)/\partial x^\sigma$.

We denote by $\mathscr{F}(\pi)$ the algebra of smooth functions on $J^\infty(\pi)$. This is defined as the direct limit of the chain of inclusions of smooth functions on $J^k(\pi)$ into smooth functions on $J^{k+1}(\pi)$ via pull-back. The $\mathscr{F}(\pi)$-module $\Lambda^*(\pi)$ of differential forms on $J^\infty(\pi)$ is defined in the similar way.

A *horizontal module* is the $\mathscr{F}(\pi)$-module of sections of $\pi_\infty^*(\alpha)$, where α is a vector bundle over M. Denote by $\varkappa(\pi)$ the horizontal module corresponding to the bundle π itself.

Let $P_1(\pi)$ and $P_2(\pi)$ be horizontal modules. A linear differential operator $\Delta\colon P_1(\pi) \to P_2(\pi)$ is called $\mathscr{C}$-*differential* if it can be restricted to the graphs of all infinitely prolonged sections of the bundle. The set of all $\mathscr{C}$-differential operators from $P_1(\pi)$ to $P_2(\pi)$ is denoted by $\mathscr{C}\mathrm{Diff}(P_1(\pi), P_2(\pi))$. In coordinates, $\mathscr{C}$-differential operators have the form of a matrix $(a_{ij}^\sigma D_\sigma)$, where $a_{ij}^\sigma \in \mathscr{F}(\pi)$, $D_\sigma = D_{i_1} \circ \cdots \circ D_{i_r}$ for $\sigma = i_1 \ldots i_r$ and $D_i = \partial/\partial x^i + u_{\sigma i}^j \partial/\partial u_\sigma^j$ is the total derivative operator with respect to x^{i1}.

A π_∞-vertical vector field on $J^\infty(\pi)$ is called an *evolutionary field* if it commutes with all D_i (this property does not depend on the choice of coordinates). In coordinates, each evolutionary field is of the form $\eth_\varphi = D_\sigma(\varphi^j)\partial/\partial u_\sigma^j$, where $\varphi^j \in \mathscr{F}(\pi)$. It can be proved that each evolutionary vector field is uniquely determined by its *generating function* $\varphi \in \varkappa(\pi)$ and vice verse to any $\varphi \in \varkappa(\pi)$ there corresponds an evolutionary field $\eth_\varphi$.

Let $P(\pi)$ be a horizontal module. For each element $p \in P(\pi)$ there is a $\mathscr{C}$-differential operator $\ell_p\colon \varkappa(\pi) \to P(\pi)$ called the *universal linearization* of p and defined by $\ell_p(\varphi) = \eth_\varphi(p)$, with $\varphi \in \varkappa(\pi)$. In coordinates, ℓ_p is the matrix of the form $(\partial p^i/\partial u_\sigma^j \cdot D_\sigma)$.

A differential form $\omega \in \Lambda^q(\pi)$ on $J^\infty(\pi)$ is called a *Cartan form* if its pull-back through any prolonged section vanishes. In coordinates, Cartan forms contain factors of the type $\omega_\sigma^j = du_\sigma^j - u_{\sigma i}^j\, dx^i$. Denote the module of all Cartan q-forms by $\mathscr{C}\Lambda^q(\pi)$. It is not difficult to show that $d(\mathscr{C}\Lambda^q(\pi)) \subset \mathscr{C}\Lambda^{q+1}(\pi)$. Therefore the quotient $\bar{d}$ of d, acting

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[1]The Einstein summation convention will be used throughout the paper.

on $\bar\Lambda^*(\pi) = \Lambda^*(\pi)/\mathscr{C}\Lambda^*(\pi)$, is well defined. The module $\bar\Lambda^*(\pi)$ is identified with the submodule in $\Lambda^*(\pi)$ generated by $\pi_\infty^*\Lambda^*(M)$. Elements of $\bar\Lambda^q(\pi)$ are called *horizontal forms*. In coordinates, $\bar\Lambda^q(\pi)$ is generated by $f\,dx^{i_1}\wedge\cdots\wedge dx^{i_q}$, where $f\in\mathscr{F}(\pi)$, and $\bar d(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_q}) = D_i(f)\,dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_q}$. The cohomology $\bar H^i(\pi)$ of the complex $(\bar\Lambda^*(\pi),\bar d)$ is said to be the *horizontal cohomology*, and coincides with the de Rham cohomology of M for all degrees i up to $n-1$.

If $P(\pi)$ is an $\mathscr{F}(\pi)$-module, we write $\hat P(\pi) = \mathrm{Hom}_{\mathscr{F}(\pi)}(P(\pi),\bar\Lambda^n(\pi))$ and consider the natural pairing $\langle\cdot,\cdot\rangle\colon \hat P(\pi)\times P(\pi)\to\bar\Lambda^n(\pi)$. We recall that for each operator $\Delta\in\mathscr{C}\mathrm{Diff}(P_1(\pi),P_2(\pi))$ there exists a unique operator $\Delta^*\in\mathscr{C}\mathrm{Diff}(\hat P_2(\pi),\hat P_1(\pi))$ such that

$$[\langle\hat p_2,\Delta(p_1)\rangle] = [\langle\Delta^*(\hat p_2),p_1\rangle],\qquad \hat p_2\in\hat P_2(\pi),\quad p_1\in P_1(\pi),$$

where $[\omega]$ denotes the horizontal cohomology class of $\omega\in\bar\Lambda^n$. The operator Δ^* is called *adjoint* to Δ. In coordinates, $(a_{ij}^\sigma D_\sigma)^* = ((-1)^{|\sigma|}D_\sigma\circ a_{ji}^\sigma)$, where $a_{ij}^\sigma\in\mathscr{F}(\pi)$.

A *differential equation* is a submanifold $\mathscr{E}\subset J^k(\pi)$. We assume that $\mathscr{E} = \{F=0\}$, where $F\in\pi_k^*(\alpha)$ is a section of the pull-back bundle of a vector bundle α on M.

Let us set $P(\pi)=\pi_\infty^*(\alpha)$. The equation $\mathscr{E}$ can be prolonged to a submanifold of $J^\infty(\pi)$, that we still denote by $\mathscr{E}$. In local coordinates, we have $\mathscr{E} = \{D_\sigma F^k = 0\}$ for such a prolongation. Throughout the paper we will assume equations to be *regular*: this means that any function $f\in\mathscr{F}(\pi)$ vanishing on $\mathscr{E}$ can be expressed as $f=\Delta(F)$, where $\Delta\in\mathscr{C}\mathrm{Diff}(P(\pi),\mathscr{F}(\pi))$.

Symmetries. For any horizontal module $Q(\pi)$ we will denote by Q the restriction of $Q(\pi)$ to $\mathscr{E}$. We denote by $\ell_\mathscr{E}\colon\varkappa\to P$ the restriction of the universal linearization ℓ_F to $\mathscr{E}$. The space $\ker\ell_\mathscr{E}$ coincides with the space $\mathrm{Sym}\,\mathscr{E}$ of *symmetries* of the differential equation $\mathscr{E}$. The space $\ker\ell_\mathscr{E}^*$ is the space of *cosymmetries*, and is denoted by $\mathrm{CoSym}\,\mathscr{E}$.

It can be proved that there exists a sequence

$$(4)\qquad\qquad 0\to\mathscr{C}\mathrm{Diff}(P,\mathscr{F})\to\mathscr{C}\mathrm{Diff}(\varkappa,\mathscr{F})\to\mathscr{C}\Lambda^1\to 0.$$

Here the first nontrivial map is defined by $\Delta\mapsto\Delta\circ\ell_\mathscr{E}$ and the second one is a natural projection (see [1, 5] for details). The differential equation $\mathscr{E}$ is said to be *normal* if the above sequence is exact. In other words, $\mathscr{E}$ is normal, if the equality $\Delta\circ\ell_\mathscr{E}$ implies $\Delta=0$. Most differential equations of mathematical physics fulfill this property; however, the gauge equations do not.

Conservation laws. Consider the horizontal de Rham complex $(\bar\Lambda^q,\bar d)$ on $\mathscr{E}$. A *conservation law* is an element $[\omega]\in\bar H^{n-1}/H^{n-1}(M)$ (we quotient out the topological conservation laws). The Vinogradov $\mathscr{C}$-spectral sequence, see [1, 5, 7] for a detailed treatment) yields the complex $(E_1^{*,n-1},d_1)$ which plays the same role as the de Rham complex on jets. In particular the first two terms of the complex are $E_1^{0,n-1} = \bar H^{n-1}$ and $E_1^{1,n-1} = \ker\ell_\mathscr{E}^* = \mathrm{CoSym}\,\mathscr{E}$, and the first differential $d_1\colon E_1^{0,n-1}\to E_1^{1,n-1}$ is defined by $d_1([\omega]) = \Delta^*(1)$, where $\Delta\in\mathscr{C}\mathrm{Diff}(P,\bar\Lambda^n)$ fulfills $\bar d\omega = \Delta(F)$. The element $\Delta^*(1)\in\hat P$ is said to be a *generating function* of the conservation law $[\omega]$.

A $\mathscr{C}$-spectral sequence argument shows that there exists an exact sequence $0 \rightarrow H^{n-1} \rightarrow \bar{H}^{n-1} \rightarrow \ker \ell_{\mathscr{E}}^*$, where the last map is just d_1. So, the space of conservation laws $\mathrm{cl}(\mathscr{E}) = \bar{H}^{n-1}/H^{n-1}(M)$ is a subset of the kernel of $\ell_{\mathscr{E}}^*$, $\mathrm{cl}(\mathscr{E}) \subset \ker \ell_{\mathscr{E}}^* = \mathrm{CoSym}\,\mathscr{E}$.

3   Hamiltonian bivectors on general equations

In this section we collect formulas related to the Hamiltonian formalism on general equations. For details and geometric definitions we refer the reader to [2] (see also [4]).

Let $\mathscr{E} \subset J^\infty(\pi)$ be a normal equation given by $F = 0$.

A *variational bivector* on $\mathscr{E}$ is the equivalence class of $\mathscr{C}$-differential operators $A \colon \hat{P} \rightarrow \varkappa$ on $\mathscr{E}$ that satisfy the condition

$$(5) \qquad \ell_{\mathscr{E}} \circ A = A^* \circ \ell_{\mathscr{E}}^*,$$

where two operators are equivalent if they differ by an operator of the form $\square \circ \ell_{\mathscr{E}}^*$, $\square = \square^* \colon \hat{\varkappa} \rightarrow \varkappa$.

It is straightforward to see that an action of variational bivectors on cosymmetries (and, in particular, on conservation laws) is well-defined and the result is a symmetry.

If A is a bivector then on $J^\infty(\pi)$ we have

$$(6) \qquad \ell_F \circ A - A^* \circ \ell_F^* = B(F, \cdot),$$

where A is extended onto $J^\infty(\pi)$, $B \colon P(\pi) \times \hat{P}(\pi) \rightarrow P(\pi)$ is a $\mathscr{C}$-differential operator. We denote by $B^* \colon \hat{P} \times \hat{P} \rightarrow \hat{P}$ the operator adjoint to the operator B in the first argument and restricted to $\mathscr{E}$.

Consider an equivalence relation on the set of operators $\mathscr{C}\mathrm{Diff}(\hat{P} \times \hat{P}, \hat{P})$ on $\mathscr{E}$ such that two operators are equivalent if they differ by an operator of the form

$$(7) \qquad \square_1(\ell_{\mathscr{E}}^*(\,\cdot\,), \,\cdot\,) + \square_2(\,\cdot\,, \ell_{\mathscr{E}}^*(\,\cdot\,)),$$

where $\square_1 \colon \hat{\varkappa} \times \hat{P} \rightarrow \hat{P}$, $\square_2 \colon \hat{P} \times \hat{\varkappa} \rightarrow \hat{P}$

1 Proposition. *For every bivector A the equivalence class of B^* is uniquely defined and contains a skew-symmetric operator $B^* \colon \hat{P} \times \hat{P} \rightarrow \hat{P}$.*

2 Remark. If $\mathscr{E}$ is written in evolution form (in this case $P = \varkappa$) then we can put $B^*(\psi_1, \psi_2) = \ell_{A,\psi_2}^*(\psi_1)$, here we use the notation $\ell_{\Delta,p} = \ell_{\Delta(p)} - \Delta \circ \ell_p$. Skew-symmetricity follows from the formula

$$(8) \qquad \ell_{\Delta,p}^*(\hat{q}) = \ell_{\Delta^*,\hat{q}}^*(p).$$

The Schouten bracket of two bivectors is defined by the formula

$$
\begin{aligned}
(9) \qquad & [\![A_1, A_2]\!](\psi_1, \psi_2) \\
&= \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\quad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\quad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}
$$

where $\psi_1, \psi_2 \in \hat{P}$ and B_1, B_2 are the operators which play the role of B in Eq. (6) for A_1, A_2, respectively.

The Schouten bracket is a bracket between multivectors. This means that it is defined on a wide class of operators. In particular, 0-vectors are conservation laws and the bracket between a variational bivector A and a conservation law ω has the form

$$[\![A, \omega]\!] = A(\psi),$$

where $\psi = d_1[\omega]$ is the generating function of ω; 1-vectors are symmetries, the bracket between a symmetry φ and a conservation law $[\omega]$:

$$[\![\varphi, \omega]\!] = L_\varphi(\omega) = \eth_\varphi(\psi) + \Delta^*(\psi),$$

where L_φ is the Lie derivative and the operator $\Delta\colon P \to P$ is defined by the relation $\ell_F(\varphi) = \Delta(F)$ on J^∞ (for equations in evolution form one can take $\Delta = \ell_\varphi$); the bracket between two symmetries

$$[\![\varphi_1, \varphi_2]\!] = [\varphi_1, \varphi_2] \quad \text{(the usual commutator)}.$$

and the bracket between two conservation laws

$$[\![\omega_1, \omega_2]\!] = 0.$$

A bivector A is called *Hamiltonian* if $[\![A, A]\!] = 0$.

A Hamiltonian bivector A on $\mathscr{E}$ gives rise to a Lie algebra structure on the space of conservation laws of $\mathscr{E}$:

$$\{\omega_1, \omega_2\}_A = [\![[\![A, \omega_1]\!], \omega_2]\!].$$

An equation $\mathscr{E}$ is called *bi-Hamiltonian* if it has two Hamiltonian bivectors A_1 and A_2 such that $[\![A_1, A_2]\!] = 0$.

Magri hierarchy on a bi-Hamiltonian equation $\mathscr{E}$ is an infinite sequence ω_1, ω_2, … of conservation laws of $\mathscr{E}$ such that $A_1(\psi_i) = A_2(\psi_{i+1})$.

3 Proposition. *For Magri hierarchy we get*

$$\{\omega_i, \omega_j\}_{A_1} = 0, \quad \{\omega_i, \omega_j\}_{A_2} = 0$$
$$\{\varphi_i, \varphi_j\} = 0,$$

where $\varphi_i = A_1(\psi_i) = A_2(\psi_{i+1})$ are symmetries and the bracket between them is the commutator: $\eth_{\{\varphi_i, \varphi_j\}} = [\eth_{\varphi_i}, \eth_{\varphi_2}]$.

4 Remark. The reader is invited to check that, if $\mathscr{E}$ has an evolutionary form, then the above Hamiltonian formalism reduces to the usual one.

4   The Kupershmidt deformation

Let $\mathscr{E}$ be a bi-Hamiltonian equation with Hamiltonian operators A_1 and A_2 given by $F = 0$ as above. Let us consider the bundle $\hat{\pi}\colon \hat{E} = E^* \otimes_M \Lambda^n(T^*M) \to M$, where $E^* \to M$ is the dual bundle to $\pi\colon E \to M$. We denote by $w = (w^1, \ldots, w^m)$ fiber coordinates on $\hat{\pi}$.

5 Definition. The *Kupershmidt deformation* $\tilde{\mathscr{E}} \subset J^\infty(\pi) \times J^\infty(\hat{\pi})$ has the form

$$(10) \qquad\qquad F + A_1^*(w) = 0, \qquad A_2^*(w) = 0.$$

We will write $\tilde{F} = (F + A_1^*(w), A_2^*(w)) \in \tilde{P} = P \oplus P$, so that $\tilde{\mathscr{E}}$ is given by $\tilde{F} = 0$. The linearization $\ell_{\tilde{\mathscr{E}}}\colon \varkappa \oplus \hat{\varkappa} \to P \oplus P$ and its adjoint have the form

$$(11) \qquad \ell_{\tilde{\mathscr{E}}} = \begin{pmatrix} \ell_{F+A_1^*(w)} & A_1^* \\ \ell_{A_2^*(w)} & A_2^* \end{pmatrix}, \quad \ell_{\tilde{\mathscr{E}}}^* = \begin{pmatrix} \ell_{F+A_1^*(w)}^* & \ell_{A_2^*(w)}^* \\ A_1 & A_2 \end{pmatrix}$$

The linearizations in the left- and right-hand sides of these formulas have different meaning: the left-hand ones are usual linearizations on $\tilde{\mathscr{E}}$, while the right-hand ones are linearizations with respect to the dependent variables u only, that is, linearizations on $J^\infty(\pi)$.

In what follow we use the following notational rule: linearization of something marked with tilde is the $J^\infty(\pi) \times J^\infty(\hat{\pi})$-linearization (i.e., the linearization with respect to the dependent variables u, w), otherwise it is the $J^\infty(\pi)$-linearization.

Below we often use the obvious relation $\ell_{A^*,w} = \ell_{A^*(w)}$, which is true since the linearization here is the $J^\infty(\pi)$-linearization.

6 Lemma. *We have the following equalities:*

(12a) $\ell_{F+A_1^*(w)} \circ A_1 - A_1^* \circ \ell_{F+A_1^*(w)}^* = B_1(F + A_1^*(w), \cdot)$

$\qquad \ell_{F+A_1^*(w)} \circ A_2 - A_1^* \circ \ell_{A_2^*(w)}^* - A_2^* \circ \ell_{F+A_1^*(w)}^* - \ell_{A_2^*(w)} \circ A_1$

$$(12b) \qquad\qquad\qquad\qquad\qquad = B_2(F + A_1^*(w), \cdot) + B_1(A_2^*(w), \cdot)$$

(12c) $\ell_{A_2^*(w)} \circ A_2 - A_2^* \circ \ell_{A_2^*(w)}^* = B_2(A_2^*(w), \cdot).$

Proof. We will only prove the statement for the first operator. For every ψ_1, $\psi_2 \in \hat{P}$ we have the following equalities modulo $\operatorname{im} \bar{d}$:

$$\langle \ell_F(A_1(\psi_1)) + \ell_{A_1^*,w}(A_1(\psi_1)), \psi_2 \rangle - \langle \ell_F(A_1(\psi_2)) + \ell_{A_1^*,w}(A_1(\psi_2)), \psi_1 \rangle$$

$$= \langle (\ell_F \circ A_1 - A_1^* \circ \ell_F^*)(\psi_1), \psi_2 \rangle + \langle A_1(\psi_1), \ell_{A_1^*,w}^*(\psi_2) \rangle - \langle A_1(\psi_2), \ell_{A_1^*,w}^*(\psi_1) \rangle$$

$$= \langle B_1(F, \psi_1), \psi_2 \rangle + \langle A_1(\psi_1), \ell_{A_1,\psi_2}^*(w) \rangle - \langle A_1(\psi_2), \ell_{A_1,\psi_1}^*(w) \rangle$$

$$= \langle B_1^*(\psi_2, \psi_1), F \rangle + \langle (\ell_{A_1,\psi_2}(A_1(\psi_1)) - \ell_{A_1,\psi_1}(A_1(\psi_2))), w \rangle$$

$$= \langle B_1^*(\psi_2, \psi_1), F \rangle + \langle A_1(B_1^*(\psi_2, \psi_1)), w \rangle$$

$$= \langle B_1^*(\psi_2, \psi_1), F + A_1^*(w) \rangle$$

$$= \langle B_1(F + A_1^*(w), \psi_1), \psi_2 \rangle.$$

The equalities in the statement turn out to be true in view of the fact that both sides of the above equalities are linear (zero order) operators in ψ_2, and the only linear operator which is the composition of $\bar{d}$ and an operator in ψ_2 is the zero operator.

The other two equalities are proved similarly. $\square$

Of course the right-hand sides of the equations in Lemma 6 vanish on $\tilde{\mathscr{E}}$, hence the following corollary.

7 Corollary. *The following operators are selfadjoint on $\tilde{\mathscr{E}}$:*

$$\ell_F \circ A_1 + \ell_{A_1^*(w)} \circ A_1$$

$$\ell_F \circ A_2 + \ell_{A_1^*(w)} \circ A_2 - A_1^* \circ \ell_{A_2^*,w}^*$$

$$\ell_{A_2^*(w)} \circ A_2.$$

In our computations we will also need the linearization of Eqs. (12).

8 Lemma. *The linearization of Eqs. (12) computed in $\psi_1 \in \hat{P}$ yields the equations*

(13a)

$$\ell_{\ell_{F+A_1^*(w)},A_1}(\psi_1) + \ell_{F+A_1^*(w)} \circ \ell_{A_1,\psi_1} - \ell_{A_1^*,\ell_{F+A_1^*(w)}^*}(\psi_1) - A_1^* \circ \ell_{\ell_{F+A_1^*(w)}^*,\psi_1}$$
$$= B_1(\cdot,\psi_1) \circ \ell_{F+A_1^*(w)}$$

(13b)

$$\ell_{\ell_{F+A_1^*(w)},A_2}(\psi_1) + \ell_{F+A_1^*(w)} \circ \ell_{A_2,\psi_1} - \ell_{A_1^*,\ell_{A_2^*(w)}^*}(\psi_1) - A_1^* \circ \ell_{\ell_{A_2^*(w)}^*,\psi_1}$$
$$- \ell_{A_2^*,\ell_{F+A_1^*(w)}^*}(\psi_1) - A_2^* \circ \ell_{\ell_{F+A_1^*(w)}^*,\psi_1} + \ell_{\ell_{A_2^*(w)},A_1}(\psi_1) + \ell_{A_2^*(w)} \circ \ell_{A_1,\psi_1}$$
$$= B_2(\cdot,\psi_1) \circ \ell_{F+A_1^*(w)} + B_1(\cdot,\psi_1) \circ \ell_{A_2^*(w)}$$

(13c)

$$\ell_{\ell_{A_2^*(w)},A_2}(\psi_1) + \ell_{A_2^*(w)} \circ \ell_{A_2,\psi_1} - \ell_{A_2^*,\ell_{A_2^*(w)}^*}(\psi_1) - A_2^* \circ \ell_{\ell_{A_2^*(w)}^*,\psi_1} = B_2(\cdot,\psi_1) \circ \ell_{A_2^*(w)}$$

up to terms which vanish on $\tilde{\mathscr{E}}$.

Proof. We make use of the formula

$$\ell_{\Delta \circ \square(\alpha)} = \ell_{\Delta,\square(\alpha)} + \Delta \circ \ell_{\square,\alpha} + \Delta \circ \square \circ \ell_\alpha.$$

Let us apply the above formula to Eq. (12a) computed at ψ_1:

$$\ell_{\ell_{F+A_1^*(w)},A_1}(\psi_1) + \ell_{F+A_1^*(w)} \circ \ell_{A_1,\psi_1} + \ell_{F+A_1^*(w)} \circ A_1 \circ \ell_{\psi_1}$$
$$- \ell_{A_1^*,\ell_{F+A_1^*(w)}^*}(\psi_1) - A_1^* \circ \ell_{\ell_{F+A_1^*(w)}^*,\psi_1} - A_1^* \circ \ell_{F+A_1^*(w)}^* \circ \ell_{\psi_1}$$
$$= \ell_{B_1(\cdot,\cdot),(F+A_1^*(w),\psi_1)} + B_1(\cdot,\psi_1) \circ \ell_{F+A_1^*(w)} + B_1(F+A_1^*(w),\cdot) \circ \ell_{\psi_1}.$$

We obtain Eq. (13a) by observing that the term $\ell_{B_1(\cdot,\cdot),(F+A_1^*(w),\psi_1)}$ vanishes on $\tilde{\mathscr{E}}$ and that terms composed with ℓ_{ψ_1} cancel by virtue of Eq. (12a). The two remaining equations can be derived with similar reasoning. $\square$

Now let us consider the operators $\tilde{A}_1$, $\tilde{A}_2 \colon \hat{P} \oplus \hat{P} \to \varkappa \oplus \hat{\varkappa}$ defined by

(14) $\qquad \tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell^*_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix}, \qquad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell^*_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}.$

9 Proposition. *We have the equalities*

(15) $\qquad\qquad\qquad\qquad\qquad \ell_{\tilde{F}} \circ \tilde{A}_1 - \tilde{A}_1^* \circ \ell^*_{\tilde{F}} = \tilde{B}_1(\tilde{F}, \cdot)$

(16) $\qquad\qquad\qquad\qquad\qquad \ell_{\tilde{F}} \circ \tilde{A}_2 - \tilde{A}_2^* \circ \ell^*_{\tilde{F}} = \tilde{B}_2(\tilde{F}, \cdot)$

where

$\qquad \tilde{B}_1((p_1, p_1'), (\psi_2, \psi_2')) =$
(17) $\qquad (B_1(p_1, \psi_2) - B_1(p_1, \psi_2'), -B_1(p_1, \psi_2) - B_2(p_1, \psi_2') - B_1(p_1', \psi_2') - B_2(p_1', \psi_2'))$
$\qquad \tilde{B}_2((p_1, p_1'), (\psi_2, \psi_2')) =$
(18) $\qquad (B_1(p_1, \psi_2) + B_2(p_1, \psi_2) + B_1(p_1', \psi_2) + B_2(p_1', \psi_2'), B_2(p_1', \psi_2) - B_2(p_1', \psi_2'))$

It follows that the operators $\tilde{A}_1$, $\tilde{A}_2$ define two variational bivectors on $\tilde{\mathscr{E}}$.

Proof. We have

$\ell_{\tilde{F}} \circ \tilde{A}_1 - \tilde{A}_1^* \circ \ell^*_{\tilde{F}}$

$= \begin{pmatrix} \ell_{F+A_1^*(w)} & A_1^* \\ \ell_{A_2^*(w)} & A_2^* \end{pmatrix} \cdot \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell^*_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix}$

$\qquad\qquad - \begin{pmatrix} A_1^* & 0 \\ -A_1^* & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \cdot \begin{pmatrix} \ell^*_{F+A_1^*(w)} & \ell^*_{A_2^*(w)} \\ A_1 & A_2 \end{pmatrix}$

$= \begin{pmatrix} B_1(F + A_1^*(w), \cdot) & -B_1(F + A_1^*(w), \cdot) \\ -B_1(F + A_1^*(w), \cdot) & -B_2(F + A_1^*(w), \cdot) - (B_1 + B_2)(A_2^*(w), \cdot) \end{pmatrix},$

where the last equality is obtained by Lemma 6. Eqs. (16) and (18) can be proved in the same way.

The last statement follows directly from the definitions (Eqs. (5) and (6)). $\qquad \square$

Our next task is to prove that the bivectors from the above proposition endow $\tilde{\mathscr{E}}$ with a bi-Hamiltonian structure. To this aim we have to compute the Schouten brackets $[\![\tilde{A}_i, \tilde{A}_j]\!]$ for i, $j = 1, 2$. To do this, we have to compute the linearization $\ell_{\tilde{A}_i, (\psi_1, \psi_1')}$ and the operators $\tilde{B}_i^*$, with $i = 1, 2$ (here the adjoint is taken with respect to the first argument). We have

(19) $\qquad\qquad \ell_{\tilde{A}_1, (\psi_1, \psi_1')} = \begin{pmatrix} \ell_{A_1, \psi_1} - \ell_{A_1, \psi_1'} & 0 \\ \ell_{\ell^*_{F+A_1^*(w)+A_2^*(w)}, \psi_1'} & \ell^*_{A_1 + A_2, \psi_1} \end{pmatrix},$

and

(20) $\qquad\qquad \ell_{\tilde{A}_2, (\psi_1, \psi_1')} = \begin{pmatrix} \ell_{A_2, \psi_1} - \ell_{A_2, \psi_1'} & 0 \\ -\ell_{\ell^*_{F+A_1^*(w)+A_2^*(w)}, \psi_1} & -\ell^*_{A_1 + A_2, \psi_1} \end{pmatrix}$

directly from the definitions.

10 Lemma. *We have*

$$\tilde{B}_1^*((\psi_1, \psi_1'), (\psi_2, \psi_2'))$$

(21)
$$= (B_1^*(\psi_1, \psi_2) - B_1^*(\psi_1, \psi_2') - B_1^*(\psi_1', \psi_2) - B_2^*(\psi_1', \psi_2'), -B_1^*(\psi_1', \psi_2') - B_2^*(\psi_1', \psi_2'))$$

$$\tilde{B}_2^*((\psi_1, \psi_1'), (\psi_2, \psi_2'))$$

(22)
$$= (B_1^*(\psi_1, \psi_2) + B_2^*(\psi_1, \psi_2), B_1^*(\psi_1, \psi_2) + B_2^*(\psi_1, \psi_2') + B_2^*(\psi_1', \psi_2) - B_2^*(\psi_1', \psi_2'))$$

Proof. In fact we can write

$$\tilde{B}_1((p_1, p_1'), (\psi_2, \psi_2')) = \begin{pmatrix} B_1(\cdot, \psi_2) - B_1(\cdot, \psi_2') & 0 \\ -B_1(\cdot, \psi_2) - B_2(\cdot, \psi_2') & -B_1(\cdot, \psi_2') - B_2(\cdot, \psi_2') \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_1' \end{pmatrix}.$$

It follows that

$$\tilde{B}_1^*((\psi_1, \psi_1'), (\psi_2, \psi_2')) = \begin{pmatrix} B_1^*(\cdot, \psi_2) - B_1^*(\cdot, \psi_2') & -B_1^*(\cdot, \psi_2) - B_2^*(\cdot, \psi_2') \\ 0 & -B_1^*(\cdot, \psi_2') - B_2^*(\cdot, \psi_2') \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix}.$$

The other identity follows in an analogous way. $\qquad\square$

Note that $\tilde{B}_1^*$ and $\tilde{B}_2^*$ are skew-symmetric with respect to the interchange of the arguments (ψ_1, ψ_1') and (ψ_2, ψ_2').

11 Theorem. *The Kupershmidt deformation $\tilde{\mathscr{E}}$ is a bi-Hamiltonian equation with respect to the variational bivectors $\tilde{A}_1$, $\tilde{A}_2$.*

Proof. We have to prove the conditions

(23)
$$\begin{aligned}
&[\![\tilde{A}_i, \tilde{A}_j]\!]((\psi_1, \psi_1'), (\psi_2, \psi_2')) \\
&= \ell_{\tilde{A}_i,(\psi_1,\psi_1')}(\tilde{A}_j(\psi_2, \psi_2')) - \ell_{\tilde{A}_i,(\psi_2,\psi_2')}(\tilde{A}_j(\psi_1, \psi_1')) \\
&+ \ell_{\tilde{A}_j,(\psi_1,\psi_1')}(\tilde{A}_i(\psi_2, \psi_2')) - \ell_{\tilde{A}_j,(\psi_2,\psi_2')}(\tilde{A}_i(\psi_1, \psi_1')) \\
&- \tilde{A}_i(\tilde{B}_j^*((\psi_1, \psi_1'), (\psi_2, \psi_2'))) - \tilde{A}_j(\tilde{B}_i^*((\psi_1, \psi_1'), (\psi_2, \psi_2'))) = 0,
\end{aligned}$$

for $i, j = 1, 2$. We will only prove them for $i = 1$, $j = 2$, other computations being very similar. First of all we compute the summands in the expression (23):

$$\begin{aligned}
&\ell_{\tilde{A}_1,(\psi_1,\psi_1')}(\tilde{A}_2(\psi_2, \psi_2')) \\
&= \Big((\ell_{A_1,\psi_1} - \ell_{A_1,\psi_1'})(A_2(\psi_2) - A_2(\psi_2')), \\
&\qquad \mathscr{L}(\psi_1, A_2(\psi_2) - A_2(\psi_2')) + \ell_{A_1+A_2,\psi_1}^*(-\ell_{F+A_1^*(w)+A_2^*(w)}^*(\psi_2')) \Big), \\
&\ell_{\tilde{A}_2,(\psi_1,\psi_1')}(\tilde{A}_1(\psi_2, \psi_2')) \\
&= \Big((\ell_{A_2,\psi_1} - \ell_{A_2,\psi_1'})(A_1(\psi_2) - A_1(\psi_2')),
\end{aligned}$$

$$- \mathscr{L}(\psi_1, A_1(\psi_2) - A_1(\psi_2')) - \ell^*_{A_1+A_2,\psi_1}(\ell^*_{F+A_1^*(w)+A_2^*(w)}(\psi_2'))\Big),$$

$$\tilde{A}_1(\tilde{B}_2^*((\psi_1, \psi_1'), (\psi_2, \psi_2')))$$
$$= (A_1(B_2^*(\psi_1, \psi_2) - B_2^*(\psi_1, \psi_2') - B_2^*(\psi_1', \psi_2) + B_2^*(\psi_1', \psi_2')),$$
$$\ell^*_{F+A_1^*(w)+A_2^*(w)}(B_1^*(\psi_1, \psi_2) + B_2^*(\psi_1, \psi_2') + B_2^*(\psi_1', \psi_2) - B_2^*(\psi_1', \psi_2'))),$$

$$\tilde{A}_2(\tilde{B}_1^*((\psi_1, \psi_1'), (\psi_2, \psi_2')))$$
$$= (A_2(B_1^*(\psi_1, \psi_2) - B_1^*(\psi_1, \psi_2') - B_1^*(\psi_1', \psi_2) + B_1^*(\psi_1', \psi_2')),$$
$$- \ell^*_{F+A_1^*(w)+A_2^*(w)}(B_1^*(\psi_1, \psi_2) - B_1^*(\psi_1, \psi_2') - B_1^*(\psi_1', \psi_2) - B_2^*(\psi_1', \psi_2'))),$$

were we introduced the notation $\mathscr{L}(\psi_1, \varphi_2) = \ell_{\ell^*_{F+A_1^*(w)+A_2^*(w)},\psi_1}(\varphi_2)$. Let us set

$$(\varphi_3, \psi_3') = [\![\tilde{A}_1, \tilde{A}_2]\!]((\psi_1, \psi_1'), (\psi_2, \psi_2')).$$

We have

$$\varphi_3 = [\![A_1, A_2]\!](\psi_1, \psi_2) + [\![A_1, A_2]\!](\psi_1'\psi_2') - [\![A_1, A_2]\!](\psi_1', \psi_2) - [\![A_1, A_2]\!](\psi_1, \psi_2') = 0$$

because $[\![A_1, A_2]\!] = 0$.

As for the second component ψ_3', we first observe that the operator

(24) $$\mathscr{L}(\psi_1', \cdot) \colon \varkappa \to \hat{\varkappa}$$

is selfadjoint. Now, we take the adjoint of Eqs. (13) and compute them in $\psi_2 \in \hat{P}$ using Eq. (8). We obtain

(25a)

$$\ell^*_{\ell^*_{F+A_1^*(w)},\psi_2}(A_1(\psi_1)) + \ell^*_{A_1,\psi_1}(\ell^*_{F+A_1^*(w)}(\psi_2))$$
$$- \ell^*_{A_1,\psi_2}(\ell^*_{F+A_1^*(w)}(\psi_1)) - \ell^*_{\ell^*_{F+A_1^*(w)},\psi_1}(A_1(\psi_2)) = \ell^*_{F+A_1^*(w)}(-B_1^*(\psi_1, \psi_2))$$

(25b)

$$\ell^*_{\ell^*_{F+A_1^*(w)},\psi_2}(A_2(\psi_1)) + \ell^*_{A_2,\psi_1}(\ell^*_{F+A_1^*(w)}(\psi_2)) - \ell^*_{A_1,\psi_2}(\ell^*_{A_2^*(w)}(\psi_1)) - \ell^*_{\ell^*_{A_2^*(w)},\psi_1}(A_1(\psi_2))$$
$$- \ell^*_{A_2,\psi_2}(\ell^*_{F+A_1^*(w)}(\psi_1)) - \ell^*_{\ell^*_{F+A_1^*(w)},\psi_1}(A_2(\psi_2)) + \ell^*_{\ell^*_{A_2^*(w)},\psi_2}(A_1(\psi_1)) + \ell^*_{A_1,\psi_1}(\ell^*_{A_2^*(w)}(\psi_2))$$
$$= \ell^*_{F+A_1^*(w)}(-B_2^*(\psi_1, \psi_2)) + \ell^*_{A_2^*(w)}(-B_1(\psi_1, \psi_2)),$$

(25c)

$$\ell^*_{\ell^*_{A_2^*(w)},\psi_2}(A_2(\psi_1)) + \ell^*_{A_2,\psi_1}(\ell^*_{A_2^*(w)}(\psi_2)) - \ell^*_{A_2,\psi_2}(\ell^*_{A_2^*(w)}(\psi_1)) - \ell^*_{\ell^*_{A_2^*(w)},\psi_1}(A_2(\psi_2))$$
$$= \ell^*_{A_2^*(w)}(-B_2^*(\psi_1, \psi_2)).$$

Then, we sum the above three equations to get one single equation, that we compute two times in (ψ_1, ψ_2') and (ψ_1', ψ_2) respectively. By replacing the result into ψ_3' we obtain

$$\psi_3' = \mathscr{L}(\psi_2', A_1(\psi_1) + A_2(\psi_1')) + \mathscr{L}(\psi_2, A_1(\psi_1) + A_2(\psi_1'))$$

$$- \mathscr{L}(\psi_1', A_1(\psi_2) + A_2(\psi_2')) - \mathscr{L}(\psi_1, A_1(\psi_2) + A_2(\psi_2')).$$

We define the operators

$$\Box_1, \Box_1' \colon (\hat{\varkappa} \times \varkappa) \times (\hat{P} \times \hat{P}) \to (\varkappa \times \hat{\varkappa}), \quad \Box_2, \Box_2' \colon (\hat{P} \times \hat{P}) \times (\hat{\varkappa} \times \varkappa) \to (\varkappa \times \hat{\varkappa}),$$

as follows:

$$\Box_1((\varphi_1, \varphi_1'), (\psi_2, \psi_2')) = (0, -\mathscr{L}(\psi_2', \varphi_1')), \quad \Box_2((\psi_1, \psi_1'), (\varphi_2, \varphi_2')) = (0, \mathscr{L}(\psi_1', \varphi_2')),$$
$$\Box_1'((\varphi_1, \varphi_1'), (\psi_2, \psi_2')) = (0, \mathscr{L}(\psi_2, \varphi_1')), \quad \Box_2'((\psi_1, \psi_1'), (\varphi_2, \varphi_2')) = (0, +\mathscr{L}(\psi_1, \varphi_2')).$$

We have

$$(0, \psi_3') = \Box_1(\ell_{\tilde{\mathscr{E}}}^*(\psi_1, \psi_1'), (\psi_2, \psi_2')) + \Box_2((\psi_1, \psi_1'), \ell_{\tilde{\mathscr{E}}}^*(\psi_2, \psi_2'))$$
$$+ \Box_1'(\ell_{\tilde{\mathscr{E}}}^*(\psi_1, \psi_1'), (\psi_2, \psi_2')) + \Box_2'((\psi_1, \psi_1'), \ell_{\tilde{\mathscr{E}}}^*(\psi_2, \psi_2')),$$

hence the three-vector $[\![\tilde{A}_1, \tilde{A}_2]\!]$ is zero up to the equivalence relation (7). $\Box$

12 Theorem. *Let* ω_1, ω_2, $\ldots$ *is a Magri hierarchy for* $\mathscr{E}$. *Suppose that there exist extensions of* ω_i, A_1, *and* A_2 *onto* J^∞ *such that on* J^∞ *we have*

$$\bar{d}\omega_i = \langle \psi_i, F \rangle,$$
$$A_1(\psi_i) = A_2(\psi_{i+1}),$$

here ψ_i *is an extension of the generating function of* ω_i. *Then* $(\psi_i, -\psi_{i+1})$, $i = 1, 2, \ldots$ *is a Magri hierarchy for the Kupershmidt deformation* $\tilde{\mathscr{E}}$.

Proof. On $\tilde{J}^\infty$ we have

$$\begin{aligned}
\bar{d}\omega_i = \langle \psi_i, F \rangle &= \langle \psi_i, F + A_1^*(w) \rangle - \langle \psi_i, A_1^*(w) \rangle \\
&= \langle \psi_i, F + A_1^*(w) \rangle - \langle A_1(\psi_i), w \rangle + \bar{d}\chi_1 \\
&= \langle \psi_i, F + A_1^*(w) \rangle - \langle A_2(\psi_{i+1}), w \rangle + \bar{d}\chi_1 \\
&= \langle \psi_i, F + A_1^*(w) \rangle - \langle \psi_{i+1}, A_2^*(w) \rangle + \bar{d}\chi_2.
\end{aligned}$$

Thus, the form $\omega_i - \chi_2$ is a conservation law with the generating function $(\psi_i, -\psi_{i+1})$. The condition $\tilde{A}_1(\psi_i, -\psi_{i+1}) = \tilde{A}_2(\psi_{i+1}, -\psi_{i+2})$ can be easily checked by direct computation. $\Box$

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