An algebraic approach to physical scales

Josef Janyška[1], Marco Modugno[2], Raffaele Vitolo[3]

[1] Department of Mathematics and Statistics, Masaryk University
Janáčkovo nám 2a, 602 00 Brno, Czech Republic
email: janyska@math.muni.cz

[2] Department of Applied Mathematics, Florence University
Via S. Marta 3, 50139 Florence, Italy
email: marco.modugno@unifi.it

[3] Department of Mathematics "E. De Giorgi"
Via per Arnesano, 73100 Lecce, Italy
email: raffaele.vitolo@unile.it

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Abstract

This paper is aimed at introducing an algebraic model for physical scales and units of measurement. This goal is achieved by means of the concept of "positive space" and its rational powers. Positive spaces are "semi–vector spaces" on which the group of positive real numbers acts freely and transitively through the scalar multiplication. Their tensor multiplication with vector spaces yields "scaled spaces" that are suitable to describe spaces with physical dimensions mathematically. We also deal with scales regarded as fields over a given background (*e.g.*, spacetime).

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1 A mathematical approach to physical scales

1.1 Informal approach to scales in physics

Units of measurement, coupling constants, scales and scale dimensions are very standard and basic objects in all fields and formulations of physics. Usually, these objects appear in a very informal way from a mathematical viewpoint: so, we find a gap.

On one hand, nowadays many areas of physics have been formulated in a rigorous and modern mathematical language, in particular in a geometric and algebraic language. Thus, most objects of physics are described by specific and well defined mathematical objects, such as manifolds, bundles, connections, functions, tensor fields and so on.

On the other hand, the concept of physical scales are usually treated intuitively in standard literature. Actually, a rigorous mathematical analysis of relations between such objects can be found in the literature concerning the dimensional analysis. But, still, it is usually omitted to specify the notion itself of physical scale in terms of algebraic objects in a way mathematically homogeneous to other geometric objects such as bundles representing physical fields.

The reason of this gap is that every physicist knows how to deal with scales in a practical way, hence he feels that an intuitive approach is sufficient for his purposes.

1.2 Examples of scales in physical literature

Just as an example of the standard way of dealing with units of measurement and related questions in physics, we quote a few well known textbooks among possible thousands, which are well established references in the area of physics they deal with.

The book "Classical electricity and magnetism" by W.K.H. Panowsky and M. Phillips [27], which is a good classical reference in the field of electromagnetism, provides a valuable comparison between different systems of measurement. The discussion on the experimental definitions of units of measurement is fine. But, as usual in the literature, an introduction of these objects as well defined mathematical concepts is not present, in spite of the thorough mathematical description of the electromagnetic field in terms of tensors.

The book "Gravitation" by C.W. Misner, Kip S. Thorne and J.A. Wheeler [25], which is a well established reference in the field of General Relativity, takes a systematic care of units of measurement and dimensions from the point of view of physics. Here, these aspects of the theory are analysed in detail by means of instructive intuitive operative reasonings. On the other hand, this analysis appears to be not completely satisfactory from a mathematical viewpoint because the treatment of units of measurement is not homogeneous to the geometric setting of the full theory of General Relativity, which makes an essential use of tensor algebra. For instance, due to an insufficient distinction between tensors and their components, the book assigns a dimension to the curvature tensor; clearly, this cannot be true from a mathematical viewpoint, because the curvature tensor is obtained from the connection by a differential operation which does not involve any unit of measurement. So, the reader has to perform additional appropriate interpretations in order to achieve the final correct statements.

Also in quantum mechanics physical scales play a role and further aspects arise; for instance, should the wave functions carry a dimension and how should it be included? Just as an example concerning the usage of units of measurement in quantum mechanics, see the book “Mécanique quantique” by C. Cohen–Tannoudji, B. Diu and F. Laloë [8].

We recall that in theoretical physics it is customary to set some fundamental constants equal to 1, so skipping the problems related to units of measurement. This usage is convenient in some respects, but it can mask some problems. As a possible reference, see, for instance, the book “Quantum Fields” by N.N. Bogoliubov and D.V. Shirkov [3].

1.3 Examples of scales in specialised physical literature

In some specialised fields of physics, units of measurement play a more explicit and protagonist role. In particular, we quote the following topics.

The “conformal field theory” is an important theoretical area of physics which deals with physical fields with an undetermined scale factor; in particular, Weil’s theory and its developments belong to this area (see, for instance, [21]).

A stimulating and deep debate deals with the problem of understanding which is the number of fundamental constants in physics from a theoretical and experimental viewpoint (for instance, see [10, 23]).

The theory of “scale models” and “dimensional analysis” is an applicative field which requires a theoretical approach (see, for instance, [2]).

There is a research topic in physics which investigates the possibility that the fundamental “coupling constants” be not really constant and might be variable with respect to time and space (see, for instance, [32]).

There is another branch of physics devoted just to the best experimental definitions of the units of measurements; a continuous research in this field provides a periodic updating in agreement with the progress of the measurement technique (see, for instance, [9, 20]).

Actually, even in these specialised fields dealing specifically with physical scales, the literature does not pay sufficient attention to the mathematical foundations of these objects. However, the theoretical aspects of these topics could be advantageously reformulated in terms of a well defined mathematical notion of scales and scale bundles.

1.4 Scales as positive spaces used by some authors

Actually, in recent years, a series of papers (see, for instance, [7, 16, 17, 18, 24, 31, 33, 34]) has proposed a geometric formulation of covariant classical and quantum mechanics including also a formal mathematical setting of physical scales and units of measurement in an original algebraic way.

In another series of papers (see, for instance, [4, 5, 6]) a formal mathematical setting of physical scales has been used in the context of spinors. In particular, a nice pure algebraic procedure derives a distinguished space of scales from a complex space of dimension 2 and recovers the full basic stuff of spinors in this context.

The above approaches are based on the notion of "positive space" and its rational powers, as a mathematical model for the spaces of scales and units of measurement. Then, tensor products between positive spaces and vector bundles arising from spacetime yield "scaled objects", i.e. objects with physical dimension.

1.5 Physical motivation and mathematical nature of positive spaces

The starting intuitive idea suggested by physics for this algebraic idea of "positive space" is quite simple.

Let us consider the set of possible scales of measurement of a certain kind: just to fix the idea, let us refer to lengths. Clearly, all possible lengths can be naturally ordered in such a way to constitute a semi-straight line. Then, we wish to emphasise the algebraic properties of such a set and the algebraic operations which can be performed on it. In fact, our purpose is to obtain a structured mathematical object which can be "joined" in a suitable way with vector spaces describing other objects of physics equipped with "scale dimension". It seems quite natural to focus the attention on the sum of two lengths and on the product of a positive real number with a length. Moreover, we emphasise the fact that every length can be represented by a real number in a unique way, once we have chosen a unit of measurement of lengths (i.e. just one length); indeed all such units of measurement are, in this respect, good choices on the same footing. Such a representation of the set of lengths by means of the choice of a unit of measurement fully preserves the algebraic properties and operations selected above. On the other hand, we do not want to confuse the set of lengths with the set of positive real numbers $\mathbb{R}^+$, because we do wish that our theory be independent on the choice of any unit of measurement.

These simple observations lead us to the definition of a positive space as a semi-vector space on which the group of positive real numbers acts freely and transitively.

Clearly, considerations of this kind can be performed on any other set of scales of measurement, such as time intervals, masses, charges, and so on. Actually, in some of the above sets also negative scales might have a meaning, hence might lead to the more usual concept of oriented 1–dimensional vector space. But, such a space can be always recovered from a positive space by means of a tensor product with the space of real numbers $\mathbb{R}$ and the concept of positive space turns out to have a more fundamental character for our purposes (for instance, with respect to the concept of rational powers).

Thus, the concept of positive space is achieved in a very intuitive way and turns out to be a rather simple mathematical notion. However, what is not so trivial is the rigorous mathematical development of this notion with respect to tensor products and rational powers.

1.6 Advantages of this approach

This language for scales and units of measurement works pretty well in all fields of physics, as can be shown by the above papers. Even more, in some cases, a precise mathematical language on the scales allows us to achieve new results. For instance, in the paper [16] the uniqueness of the Schrödinger operator has been proved under the hypothesis of covariance. Here, the covariance is understood in a broad sense: namely, as invariance not only with respect to observers and coordinates, but also with respect to units of measurement. Indeed, in this context, these two aspects of the covariance appear on the same footing in virtue of our language.

1.7 Concrete use of positive spaces in our papers

In order to give an idea of how the language of scales has been implemented in the above series of papers, we sketch a few hints.

One starts by assuming the basic positive spaces of scales of time intervals $\mathbb{T}$, of lengths $\mathbb{L}$ and of masses $\mathbb{M}$.

In non relativistic classical mechanics, the "absolute time" is a 1 dimensional affine space associated with the tensor product $\mathbb{T}\otimes\mathbb{R}$. In the standard language, the "metric" is usually assumed to be a non degenerate symmetric section $g : \boldsymbol{P} \to T^*\boldsymbol{P} \otimes T^*\boldsymbol{P}$, where $\boldsymbol{P}$ is the the configuration space of the theory. Actually, this section g is able to measure the real valued scalar product of vectors only if one fixes the unit of measurement; indeed, this is usually done by an additional choice expressed informally "on a side". Unfortunately, in the usual language, later one can use only informal intuitive reasonings, in order to process this side information. Whereas, according to our language, one assumes the "metric" to be a non degenerate symmetric section $g : \boldsymbol{P} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{P} \otimes T^*\boldsymbol{P})$. Thus, such a metric measures the scalar product of vectors valued into the "scaled space" $\mathbb{L}^2 \otimes \mathbb{R}$. Then, any choice of a unit of measurement of lengths allows us to transform the "scaled metric" g into the "unscaled metric" g. This can be done not only by means of an intuitive reasoning, but through an algebraic contraction. Actually, such a choice of unit of measurements is needed only eventually in some very concrete descriptions of experiments; for most purposes one can carry on such an "abstract" formulation of the metric without any difficulties or complications.

In relativistic mechanics, one can proceed in a similar way. In this case, $\mathbb{T} \otimes \mathbb{R}$ is no longer the vector space associated with the absolute time, but it just describes the proper time intervals of any absolute motion. Moreover, the speed of light is assumed to be an element $c \in \mathbb{L} \otimes \mathbb{T}^*$. Indeed, it becomes a number only after having chosen units of measurement of time intervals and of lengths. Then, the Lorentzian metric can be defined as a section $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$, or, equivalently, as the section $g/c^2 : \boldsymbol{E} \to \mathbb{T}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$, where $\boldsymbol{E}$ is spacetime. Again, such an "abstract" formulations of c and g can be carried on in the theory without any difficulties or complications.

An analogous approach can be used for the electromagnetic field and all other fields involved in the theory, by introducing them as scaled fields.

Moreover, in order to write laws of physics by comparing scaled tensors with different scale factors, one needs to introduce some coupling scales belonging to the tensor product of one scale times the dual of the other scale.

Once the basic scaled objects of the theory are introduced, then the standard algebraic and differential operations usually performed on tensors can be immediately extended to "scaled tensors". So, by using essentially the usual theoretical constructions, we obtain automatically the information on the scale dimension of any new object derived from the starting scaled objects postulated in the theory. In particular, we stress that the Levi–Civita connection, its curvature tensor and the Ricci tensor turn out to be automatically unscaled objects.

Thus, in a sense our procedure starts by taking into due account the essential "conformal" nature of physical theories; actually, in our approach each conformal object is described not by a family of objects defined up to a (positive) real factor, but as a unique object valued in a unique scaled space. The difference between the standard language and our viewpoints is more than a trivial formality. Indeed, several advantages arise when we use our approach in a concrete development of a specific theory; we just quote the result on the uniqueness of the Schrödinger operator proved in [16].

1.8 Need of a paper devoted to the mathematical foundations

In the papers [4, 5, 6, 7, 16, 17, 18, 24, 31, 33, 34] the theory of positive spaces is sketched very briefly. Actually, only the practical basic rules concerning these spaces are declared without any true mathematical justification. So, a comprehensive mathematical foundation of this subject is required in order to fill in this gap.

Indeed, this subject deserves a mathematical paper, as the present one, because a clear and rigorous mathematical treatment of the tensor products and the rational powers concerning positive spaces are more subtle than it could appear at a first insight.

Thus, the present paper is devoted to the mathematical foundations of the concepts of scales, scale dimension and units of measurement.

1.9 Positive spaces and tensor products

The starting concept of positive space is quite simple and intuitive. However, subtle mathematical problems arise when we look for a rigorous construction of the tensor product between positive spaces, of the tensor product between a positive space and a vector space and of the rational powers of a positive space.

We would like to follow as far as possible the modern algebraic definition and explicit construction of tensor product between vector spaces via a universal property in a certain category (see, for instance, [13]). Indeed, such a definition of tensor product works for positive spaces. But, a direct rephrasing of the effective construction of this object is not possible because of the fact that difference operation is lacking. So, we cannot achieve the tensor product between positive spaces directly. One has first to introduce the tensor product between a positive space and a vector space, exploiting the linear structure of the second one. In particular, the tensor product of a positive space with the vector space of reals generates a "universal vectorialising" space of the positive space. Eventually, we can define the tensor product between two positive spaces by considering the tensor product of one with the universal vectorialising space of the other one and by taking into account the natural inclusion of this positive space into its universal vectorialising space.

We observe that in a more general setting of semi–modules the tensor product can be defined without difference operation (but with zero element), [12, 35]. But this approach is far too general for the concrete purpose of the algebraic model of physical scales.

For the definition of rational power of a positive space we try to rephrase the usual properties holding for positive real numbers. But, we need a construction which does not depend on the choice of a unit of measurement. Then, we are led to introduce a rather formal and abstract algebraic approach, which does the job.

The tensor product techniques developed here for positive spaces can be generalised to a larger class of semi–vector spaces and semi–modules with higher dimension. We have studied this subject in [19]. However, we think that the theory developed here is suitable and well sized for the purposes of the present paper devoted to physical scales.

1.10 Possible use of positive spaces in physics

By proceeding in an analogous way, the language of scales based on positive spaces can be conveniently used in all theories of physics. In order to formulate a physical theory according to the language proposed in this paper one should start by selecting the basic scales involved in that theory.

The formal mathematical language should not discourage at all physicists to use this scheme. In fact, on one hand, a rigorous mathematical foundation of positive spaces involves subtle difficulties, which require a theoretical care. On the other hand, once the physicist has checked that the formal definitions and statements concerning these objects are mathematically solid, he can forget for ever these formalities and work with a technique of language which is simple and automatic in practice.

Even more, the physicist who is inclined to trust the mathematical foundations of the theory needs not to study it thoroughly. It is sufficient to use the standard rules of tensor products, of rational powers and the relation between duality and inversion for positive spaces, as it is explained in the present paper. This reproduces the standard treatment of units in mathematical models, as can be seen by close comparison with dimensional analysis (see below). Thus, for practical purposes, the positive spaces can be treated as if they were positive numbers, in many respects.

So, the language proposed here, on one hand is mathematically well established and on the other hand reflects very closely the standard usage.

1.11 Example: the interplay with dimensional analysis

The *dimensional analysis* (here, we use [2] as a reference) is the branch of mathematical physics which studies the properties of physical models which depend on units of measurement.

Many of the foundational ideas of dimensional analysis become very natural facts in our algebraic theory of the units of measurement. Moreover, some basic results of dimensional analysis provide further motivation for the choice of rational tensor powers as scale spaces. Below, we list the main correspondences between the two theories.

A *class of systems of units* [2, p. 14] is, in our language, the choice of basic spaces of scales (like time, length, mass, etc.). The basic spaces of scales can then be raised to a rational power and multiplied tensorially between themselves in order to obtain scale spaces.

The *dimension function, or dimension, of a physical quantity* [2, p. 16] is the expression of the physical quantity with respect to a given scale basis (see Section 3). Such an expression is always a rational function (provided that the quantity depends only on the chosen basic spaces of scales).

The *independence of dimensions* for some quantities [2, p. 20] is just the property of those quantities of being a scale basis (see Section 3), *i.e.*, the tensor product of their rational powers generates scales.

It can be proved [2, p. 17] that the dimension function is always a power-law monomial. This justifies our algebraic setting: we obtain a rigorous formulation of these powers via tensor products and semi–linear duality. On the other hand, polynomials or power series would require additional constructions which seem not to be justified in view of the above property of the dimension function. Indeed, when in physics formulas containing power series occur, they always involve real numbers, i.e. unscaled quantities (usually called "pure numbers"), obtained as ratio of two scales belonging to the same positive space. Often one of the two scales plays the role of a variable and the other one is regarded as a fixed distinguished scale.

Any function that defines some relationship between quantities is homogeneous; this is a proposition from [2, p. 24]. It is a natural consequence of our setting that functions between scale spaces are rational, hence homogeneous. This property leads to the Π-theorem of dimensional analysis [2, 26] (concerning the independence of dimensions), which follows in our approach as an algebraic consequence, rather than an analytic one.

Summarising, after realising that physical quantities transform with a power-law monomial, it is natural to implement scales in physical models as rational tensor powers, and maps between them as rational maps.

1.12 Perspectives and further developments

Besides the formulation of any standard theory in physics, the language of scales proposed in the present paper could be convenient for further specific purposes.

In the final part of our paper we discuss the bundles of positive spaces based on spacetime and their semi–linear connections. These bundles can be fruitfully used as a mathematical contribution to the current debate in the community of physicists concerning the question whether the "physical constants" are really "constant" [32]. Conformal field theories, briefly described in the last sections in our context, are just an example of theories of this type.

Indeed, we believe that the clear mathematical setting provided by the language of scales would help researchers to focus on physical problems.

1.13 Summary of the present paper

The sections of the present paper are organised as follows.

In section 2, we start by introducing "positive spaces". Roughly speaking, they are 1–dimensional "semi–vector" spaces. This concept is not new, and has been used in contexts which differ a lot from the present one: for instance, see [22] for the analysis of some properties of $\mathbb{Z}_2$–valued matrices, [11] for problems of fuzzy analysis, [28] for problems of measure theory, [29, 30] for topological fixed point problems. Then, we introduce the sesqui–tensor product of a semi–vector space with a vector space and the semi–tensor product between semi–vector spaces. These concepts are treated analogously to tensor products of vector spaces, but an additional care in details is strictly necessary. Moreover, we introduce the rational powers of a positive space.

In section 3, we show how positive spaces and their rational powers can be used as scale spaces in a broad class of physical theories. We start by assuming three positive spaces, $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{M}$ as representatives of the spaces of time, length and mass scales. Then, we describe all possible derived scales in terms of semi–tensor products of rational powers of $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{M}$. Next, we introduce the "scaled objects", by considering the sesqui–tensor product of a positive space of scales with a vector bundle arising from spacetime. We also sketch differential calculus with scaled objects. Then, we introduce bundles of positive spaces based on spacetime. Sections of these bundles represent variable scales; problems involving the variability of physical constants find here a natural setting. These bundles can be equipped with geometric structures, like connections. We show that this setting is adequate to reformulate conformal field theories. Instead of defining a conformal field as a family of fields defined up to a numerical factor, we consider the tensor product of a scale bundle with the bundle of the basic field and define a conformal field just as a section of this tensor bundle.

2 Positive spaces

Positive spaces are defined through the concept of semi-vector space. A semi-vector space is defined through axioms which are similar to those of vector spaces but with the field of scalars replaced by the semi-field $\mathbb{R}^+$ [14].

We will use these spaces for achieving an algebraic model of scales and units of measurement.

2.1 Semi-vector spaces

Let $\mathbb{R}^+ \subset \mathbb{R}$ be the subset of positive real numbers and let us set $\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}$.

The set $\mathbb{R}^+$ is a *semi–field* [14] with respect to the operations of addition and multiplication. This means that $(\mathbb{R}^+, +)$ is a commutative semi–group, $(\mathbb{R}^+, \cdot)$ is a commutative group and the distributive law holds. Note that, for each $x, y, z \in \mathbb{R}^+$, the "cancellation law" holds: if $x + z = y + z$, then $x = y$.

2.1 Definition. A *semi–vector space* (over $\mathbb{R}^+$) is defined to be a set $\boldsymbol{U}$ equipped with the operations $+ : \boldsymbol{U} \times \boldsymbol{U} \to \boldsymbol{U}$ and $\cdot : \mathbb{R}^+ \times \boldsymbol{U} \to \boldsymbol{U}$, which fulfill the following properties, for each $r, s \in \mathbb{R}^+$, $u, v, w \in \boldsymbol{U}$,

$$u + (v + w) = (u + v) + w\,, \qquad u + v = v + u\,,$$
$$(rs)\,u = r\,(su)\,, \qquad 1\,u = u\,, \qquad r\,(u + v) = ru + rv\,, \qquad (r + s)\,u = ru + su\,.\square$$

For instance, any vector space is a semi–vector space and the set of linear combinations over $\mathbb{R}^+$ of n independent vectors in a vector space is a semi–vector space.

Despite the fact that some authors (see [11]) require that a semi–vector space has a zero vector, here we do not make such a general assumption. On the other hand, interesting properties arise for semi-vector spaces with a zero vector.

Let $\boldsymbol{U}$ be a semi–vector space.

An element $0 \in \boldsymbol{U}$ is said to be *neutral* if, for each $u \in \boldsymbol{U}$, we have $0 + u = u$.

If two elements $0, 0' \in \boldsymbol{U}$ are neutral, then they coincide; in fact, we have $0' = 0 + 0' = 0' + 0 = 0$.

If $0 \in \boldsymbol{U}$ is a neutral element, then, for each $r \in \mathbb{R}^+$, we have $r0 = 0$; in fact, for each $u \in \boldsymbol{U}$, we have $r0 + u = r(0 + \frac{1}{r}\,u) = r(\frac{1}{r}\,u) = u$.

A semi–vector space $\boldsymbol{U}$ equipped with neutral element 0 turns out to be also a semi-vector space over $\mathbb{R}^+_0$, by setting $0\,u = 0$, for each $u \in \boldsymbol{U}$.

2.2 Definition. A map $f : \boldsymbol{U} \to \boldsymbol{V}$ between semi–vector spaces is said to be *semi–linear* if, for each $u, v \in \boldsymbol{U}$, $r \in \mathbb{R}^+$, we have $f(u + v) = f(u) + f(v)$ and $f(ru) = rf(u)$. $\square$

If $\boldsymbol{U}$ and $\boldsymbol{V}$ are semi–vector spaces, then we obtain the semi–vector space

$$\text{s-Lin}(\boldsymbol{U}, \boldsymbol{V}) := \{f : \boldsymbol{U} \to \boldsymbol{V} \mid f \text{ is semi–linear}\}\,.$$

In particular, if $\boldsymbol{U}$ is a semi–vector space, then its *semi–dual* is defined to be the semi–vector space

$$\boldsymbol{U}^\star := \text{s-Lin}(\boldsymbol{U}, \mathbb{R}^+)\,.$$

2.2 Positive spaces

The positive spaces are the basic objects of our approach.

2.3 Definition. A *positive space* is defined to be a semi-vector space $\mathbb{U}$, such that the product $\cdot : \mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ is a left free and transitive action of the group $(\mathbb{R}^+, \cdot)$ on $\mathbb{U}$. $\square$

Thus, a positive space can be regarded as a “generalized affine space” associated with the group $(\mathbb{R}^+, \cdot)$.

Let us consider a positive space $\mathbb{U}$ and show some immediate consequences of the definition.

The positive space $\mathbb{U}$ has no neutral element 0. In fact, the action $\cdot : \mathrm{I\!R}^+ \times \{0\} \to \mathbb{U}$ would be non transitive.

Let $b \in \mathbb{U}$ be any element. Then, each $u \in \mathbb{U}$ can be written in a unique way as

$$u = (u/b)\, b\,, \qquad \text{with} \qquad u/b \in \mathrm{I\!R}^+ .$$

Moreover, the map $\mathbb{U} \to \mathrm{I\!R}^+ : u \mapsto u/b$ turns out to be a semi–linear isomorphism.

The above fact and the cancellation law for $\mathrm{I\!R}^+$ imply the cancellation law for $\mathbb{U}$: if $u,\, v,\, w \in \mathbb{U}$, then $u + w = v + w$ implies $u = v$.

If $\mathbb{U}$ and $\mathbb{V}$ are positive spaces, then the semi–vector space s-Lin$(\mathbb{U}, \mathbb{V})$ turns out to be a positive space.

In particular, the semi–vector spaces $\mathbb{U}^\star :=$ s-Lin$(\mathbb{U}, \mathrm{I\!R}^+)$ and s-Lin$(\mathbb{U}, \mathbb{U})$ turn out to be positive spaces.

Even more, each semi–linear map $f : \mathbb{U} \to \mathbb{U}$ turns out to be of the type $f : u \mapsto r\, u$, with $r \in \mathrm{I\!R}^+$; hence it is a semi–linear isomorphism. Indeed, this fact yields a natural semi–linear isomorphism s-Lin$(\mathbb{U}, \mathbb{U}) \simeq \mathrm{I\!R}^+$.

2.3 Sesqui-tensor products

The tensor product between a positive space and a vector space can be achieved by rephrasing the procedure for the tensor product of vector spaces. However, it requires an additional care in order to distinguish the role of semi–linear and linear maps. Because of this delicate aspect, we provide a detailed statement and proof, for the convenience of the reader.

Let $\mathbb{U}$ be a positive space and $\boldsymbol{V}$ a vector space.

2.4 Definition. A *(left) sesqui–tensor product* between $\mathbb{U}$ and $\boldsymbol{V}$ is defined to be a vector space $\mathbb{U} \acute{\otimes} \boldsymbol{V}$ along with a map $\acute{\otimes} : \mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \acute{\otimes} \boldsymbol{V}$, which is semi–linear with respect to the 1st factor and linear with respect to the 2nd factor and which fulfills the following universal property: if $\boldsymbol{W}$ is a vector space and $f : \mathbb{U} \times \boldsymbol{V} \to \boldsymbol{W}$ a map which is semi–linear with respect to the 1st factor and linear with respect to the 2nd factor, then there exists a unique linear map $\tilde{f} : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \boldsymbol{W}$, such that $f = \tilde{f} \circ \acute{\otimes}$.□

2.5 Theorem. *The sesqui–tensor product exists, is unique up to a distinguished linear isomorphism and is linearly generated by the image of the map* $\acute{\otimes} : \mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \acute{\otimes} \boldsymbol{V}$.

Proof. *Existence.* We consider the vector space $\boldsymbol{F}$ consisting of all maps $\phi : \mathbb{U} \times \boldsymbol{V} \to \mathrm{I\!R}$, which vanish everywhere except on a finite subset of $\mathbb{U} \times \boldsymbol{V}$. Clearly, the set $\boldsymbol{F}$ becomes a vector space in a natural way. Accordingly, each $\phi \in \boldsymbol{F}$ can be written as a formal sum of the type

$$\phi = \phi^{11}(u_1, v_1) + \cdots + \phi^{nm}(u_n, v_m)\,,$$

where $\phi^{ij} \equiv \phi(u_i, v_j) \in \mathrm{I\!R}$ are the (possibly) non vanishing values of ϕ.

Next, we consider the subset $\boldsymbol{S} \subset \boldsymbol{F}$ consisting of elements of the type

$$(u + u',\, v) - (u, v) - (u', v)\,, \qquad (u,\, v + v') - (u, v) - (u, v')\,,$$
$$(ru, v) - r(u, v)\,, \qquad (u, sv) - s(u, v)\,,$$

with $u, u' \in \mathbb{U}$, $v, v' \in \boldsymbol{V}$, $r \in \mathbb{R}^+$, $s \in \mathbb{R}$. Then, we consider the vector subspace $\langle \boldsymbol{S} \rangle_{\mathbb{R}} \subset \boldsymbol{F}$ linearly generated by $\boldsymbol{S}$ on $\mathbb{R}$. Eventually, we obtain the quotient vector space and the map

$$\mathbb{U} \acute{\otimes} \boldsymbol{V} := \boldsymbol{F}/\langle \boldsymbol{S} \rangle_{\mathbb{R}} \qquad \text{and} \qquad \acute{\otimes} := q \circ \jmath : \mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \acute{\otimes} \boldsymbol{V},$$

where $\jmath : \mathbb{U} \times \boldsymbol{V} \hookrightarrow \boldsymbol{F}$ and $q : \boldsymbol{F} \to \boldsymbol{F}/\langle \boldsymbol{S} \rangle_{\mathbb{R}}$ are the natural inclusion and the quotient projection. Indeed, the map $\acute{\otimes} : \mathbb{U} \times \boldsymbol{V} \to \mathbb{U} \acute{\otimes} \boldsymbol{V}$ is semi–linear with respect to the 1st factor and linear with respect to the 2nd factor.

Clearly, $\mathbb{U} \acute{\otimes} \boldsymbol{V}$ is linearly generated by the image of the map $\acute{\otimes}$.

Now, let us refer to the universal property (Definition 2.4). If the map $\tilde{f} : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \boldsymbol{W}$ exists, then it is unique because $\mathbb{U} \acute{\otimes} \boldsymbol{V}$ is linearly generated by the image of the map $\acute{\otimes}$. Indeed, such a map exists. In fact, we can easily prove that the map $f : \mathbb{U} \times \boldsymbol{V} \to \boldsymbol{W}$ yields naturally a linear map $f' : \boldsymbol{F} \to \boldsymbol{W}$, which passes to the quotient yielding the required linear map $\tilde{f} : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \boldsymbol{W}$.

Uniqueness. The sesqui–tensor product is "unique" in the following sense. If $\mathbb{U} \acute{\otimes} \boldsymbol{V}$ and $\mathbb{U} \underline{\acute{\otimes}} \boldsymbol{V}$ are sesqui–tensor products, then the universal properties of the two sesqui–tensor products yield the following commutative diagram

$$\begin{tikzcd}
\mathbb{U} \acute{\otimes} \boldsymbol{V} \arrow[rr, shift left, "\widetilde{\underline{\acute{\otimes}}}"] & & \mathbb{U} \underline{\acute{\otimes}} \boldsymbol{V} \arrow[ll, shift left, "\widetilde{\acute{\otimes}}"] \\
& \mathbb{U} \times \boldsymbol{V} \arrow[ul, "\acute{\otimes}"] \arrow[ur, "\underline{\acute{\otimes}}"'] &
\end{tikzcd}$$

where $\widetilde{\underline{\acute{\otimes}}} : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \mathbb{U} \underline{\acute{\otimes}} \boldsymbol{V}$ and $\widetilde{\acute{\otimes}} : \mathbb{U} \underline{\acute{\otimes}} \boldsymbol{V} \to \mathbb{U} \acute{\otimes} \boldsymbol{V}$ are mutually inverse linear isomorphisms. QED

2.6 Note. Clearly, for each $u, u' \in \mathbb{U}$, $v, v' \in \boldsymbol{V}$, $r \in \mathbb{R}^+$, $s \in \mathbb{R}$, we have

$$(u + u') \acute{\otimes} v = u \acute{\otimes} v + u' \acute{\otimes} v, \qquad u \acute{\otimes} (v + v') = u \acute{\otimes} v + u \acute{\otimes} v',$$
$$(r\, u) \acute{\otimes} v = r\, (u \acute{\otimes} v), \qquad u \acute{\otimes} (s\, v) = s\, (u \acute{\otimes} v). \square$$

The following annihilation rules follow from the universal property:

2.7 Proposition. If $u \in \mathbb{U}$, $0_{\boldsymbol{V}} \in \boldsymbol{V}$ then we have $u \acute{\otimes} 0_{\boldsymbol{V}} = 0 \in \mathbb{U} \acute{\otimes} \boldsymbol{V}$.

Proof. If $f : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \boldsymbol{W}$ is any linear map, then $\phi := f \circ \acute{\otimes} : \mathbb{U} \times \boldsymbol{V} \to \boldsymbol{W}$ is a linear map with respect to the 2nd factor. Hence, we have $\phi(u, 0_{\boldsymbol{V}}) = 0$, which, in virtue of the universal property, implies $f(u \acute{\otimes} 0_{\boldsymbol{V}}) = 0_{\boldsymbol{W}}$. Therefore, in virtue of the arbitrariness of f, we obtain $u \acute{\otimes} 0_{\boldsymbol{V}} = 0$. QED

2.8 Corollary. Let us consider an element $b \in \mathbb{U}$ and a basis $\mathcal{B} \subset \boldsymbol{V}$. Then,

$$b \acute{\otimes} \mathcal{B} := \{ b \acute{\otimes} b_i \mid b_i \in \mathcal{B} \} \subset \mathbb{U} \acute{\otimes} \boldsymbol{V}$$

is a basis of $\mathbb{U} \acute{\otimes} \boldsymbol{V}$. Hence, we have

$$\dim(\mathbb{U} \acute{\otimes} \boldsymbol{V}) = \dim \boldsymbol{V}.$$

Proof. Clearly, $b \acute{\otimes} \mathcal{B}$ linearly generates $\mathbb{U} \acute{\otimes} \boldsymbol{V}$.

Next, let us prove that the elements of $b \acute{\otimes} \mathcal{B}$ are linearly independent. For this purpose, let us observe that the universal property of the sesqui–linear tensor product yields a bijection $f \mapsto \tilde{f}$ between the maps $f : \mathbb{U} \times \boldsymbol{V} \to \mathbb{R}$, which are semi–linear with respect to the 1st factor and linear with respect to the 2nd factor, and the linear maps $\tilde{f} : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \mathbb{R}$, according to the rule $\tilde{f}(u \acute{\otimes} v) = f(u, v)$, for each $u \in \mathbb{U}$, $v \in \boldsymbol{V}$. Now, let us consider an element $t := \sum_i b \acute{\otimes} (t^i\, b_i) \in \mathbb{U} \acute{\otimes} \boldsymbol{V}$. Indeed, for any $\tilde{f}$ as above, we have $\tilde{f}(\sum_i b \acute{\otimes} (t^i\, b_i)) = \sum_i t^i\, f(b, b_i)$. Then, we can be prove that f is uniquely determined by its values on $\{(b, b_i) \mid b_i \in \mathcal{B}\}$. This implies that $\sum_i t^i f(b, b_i) = 0$, for all f as above, if and only if $t^i = 0$. Hence, $\tilde{f}(\sum_i b \acute{\otimes} (t^i b_i)) = 0$, for all $\tilde{f}$ as above, if and only if $t^i = 0$. QED

We can introduce the right sesqui–tensor product $\boldsymbol{V} \grave{\otimes} \mathbb{U}$ analogously to the left sesqui–tensor product $\mathbb{U} \acute{\otimes} \boldsymbol{V}$. Clearly, we have a natural linear isomorphism $\boldsymbol{V} \grave{\otimes} \mathbb{U} \simeq \mathbb{U} \acute{\otimes} \boldsymbol{V}$, which is characterised by the map $v \grave{\otimes} u \mapsto u \acute{\otimes} v$.

2.4 Universal vectorialising space

The sesqui–tensor product of the vector space of reals with a positive space yields a natural extension of the last one into a vector space. Indeed, this vector space fulfills a universal property.

Let us consider a positive space $\mathbb{U}$.

2.9 Definition. We define the the *universal vector extension* of $\mathbb{U}$ to be the sesqui–tensor product $\bar{\mathbb{U}} := \mathrm{I\!R} \grave{\otimes} \mathbb{U}$. □

If $b \in \mathbb{U}$, then $1 \grave{\otimes} b$ is a basis of $\mathrm{I\!R} \grave{\otimes} \mathbb{U}$.

2.10 Lemma. For each $u\,, u' \in \mathbb{U}$, the equality $1 \grave{\otimes} u = 1 \grave{\otimes} u'$ implies $u = u'$.

Proof. Let us consider any semi–linear map $\phi : \mathbb{U} \to \mathrm{I\!R}$ and the induced map $f : \mathrm{I\!R} \times \mathbb{U} \to \mathrm{I\!R} : (r, u) \mapsto r\,\phi(u)$, which is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor. Then, we obtain the induced linear map $\tilde{f} : \mathrm{I\!R} \grave{\otimes} \mathbb{U} \to \mathrm{I\!R} : r \grave{\otimes} u \mapsto r\,\phi(u)$.

The equality $1 \grave{\otimes} u = 1 \grave{\otimes} \bar{u}$ implies $\tilde{f}(1 \grave{\otimes} u) = \tilde{f}(1 \grave{\otimes} \bar{u})$, hence $\phi(u) = \phi(\bar{u})$. Therefore, the arbitrariness of ϕ implies $u = \bar{u}$. QED

2.11 Lemma. For each $u,\, u' \in \mathbb{U}$, we have $1 \grave{\otimes} u \neq (-1) \grave{\otimes} u'$.

Proof. Let us consider a semi–linear map $\phi : \boldsymbol{U} \to \mathrm{I\!R}$, with positive values. This map yields the map $f : \mathrm{I\!R} \times \boldsymbol{U} \to \mathrm{I\!R} : (r, v) \mapsto r\,\phi(v)$; indeed, the map f is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor. Then, in virtue of the universal property, we obtain the linear map $\tilde{f} : \mathrm{I\!R} \grave{\otimes} \boldsymbol{U} \to \mathrm{I\!R}$. Indeed, we obtain $\tilde{f}(1 \grave{\otimes} u) = \phi(u) \in \mathrm{I\!R}^+$ and $\tilde{f}((-1) \grave{\otimes} u') = -\phi(u') \in \mathrm{I\!R}^-$. Hence, $1 \grave{\otimes} u \neq (-1) \grave{\otimes} u'$, because there is a linear map which takes different values on these elements. QED

2.12 Proposition. The two distinguished subspaces

$$\mathbb{U}_+ := \{1 \grave{\otimes} u \mid u \in \mathbb{U}\} \subset \mathrm{I\!R} \grave{\otimes} \mathbb{U} \qquad \text{and} \qquad \mathbb{U}_- := \{(-1) \grave{\otimes} u \mid u \in \mathbb{U}\} \subset \mathrm{I\!R} \grave{\otimes} \mathbb{U}$$

turn out to be positive spaces. Moreover, we have the disjoint union

$$\bar{\mathbb{U}} = \mathbb{U}_+ \cup \mathbb{U}_- \cup \{0\}$$

and the natural semi–linear inclusion

$$\imath : \mathbb{U} \to \mathrm{I\!R} \grave{\otimes} \mathbb{U} : u \mapsto 1 \grave{\otimes} u\,. \square$$

2.13 Proposition. If $\boldsymbol{W}$ is a vector space and $\phi : \mathbb{U} \to \boldsymbol{W}$ a semi–linear map, then there is a unique linear map $\bar{\phi} : \bar{\mathbb{U}} \to \boldsymbol{W}$, such that $\bar{f} \circ \imath = f$. Indeed, this map is given by $\bar{\phi}(1 \grave{\otimes} u) = \phi(u)$.

Proof. *Uniqueness.* If the map $\bar{\phi}$ exists, then it is unique. To prove this it suffices to show that the value of $\bar{\phi}$ on $\bar{\mathbb{U}}_-$ is determined by the value of $\bar{\phi}$ on $\bar{\mathbb{U}}_+$. In fact, by taking into account that $\bar{\phi}$ is linear, we obtain $\bar{\phi}((-1) \grave{\otimes} u) = \bar{\phi}(-(1 \grave{\otimes} u)) = -\bar{\phi}(1 \grave{\otimes} u) = -\phi(u)$.

Existence. Let $\boldsymbol{W}$ be a vector space and $\phi : \mathbb{U} \to \boldsymbol{W}$ a semi–linear map. Then, we obtain the map $f : \mathrm{I\!R} \times \mathbb{U} \to \boldsymbol{W} : (r, u) \mapsto r\,\phi(u)$, which is linear with respect to the 1st factor and semi–linear with respect to the 2nd factor. Hence, in virtue of the universal property of the sesqui–tensor product, we obtain the linear map $\bar{\phi} := \tilde{f} : \mathrm{I\!R} \grave{\otimes} \mathbb{U} \to \boldsymbol{W}$, whose expression is given by $\bar{\phi}(1 \grave{\otimes} u) = \phi(u)$. QED

2.14 Note. We have the natural linear isomorphism

$$(\bar{\mathbb{U}})^* := (\mathrm{I\!R} \otimes \mathbb{U})^* \simeq \overline{\mathbb{U}^*} := \mathrm{I\!R} \otimes (\mathbb{U}^*) : r\,(1 \otimes (1/b)) \mapsto r \otimes (1/b)\,. \square$$

2.15 Note. If $\boldsymbol{V}$ is a vector space, then, in virtue of the universal property of the sesqui–tensor product, there is a unique linear map

$$j : \mathbb{U} \acute{\otimes} \boldsymbol{V} \to \bar{\mathbb{U}} \otimes \boldsymbol{V}\,,$$

which makes the following diagram commutative

$$\begin{tikzcd}
\mathbb{U} \times \boldsymbol{V} \arrow[r, "\imath \times \mathrm{id}"] \arrow[d, "\acute{\otimes}"'] & \bar{\mathbb{U}} \times \boldsymbol{V} \arrow[d, "\otimes"] \\
\mathbb{U} \acute{\otimes} \boldsymbol{V} \arrow[r, "j"'] & \bar{\mathbb{U}} \otimes \boldsymbol{V}
\end{tikzcd}$$

Indeed, the map j turns out to be a linear isomorphism. $\square$

2.5 Semi–tensor products

The tensor product of two positive spaces cannot be achieved by the same procedure used for the tensor product of vector spaces. In fact, this procedure would fail in some essential aspects. Actually, we overcome this difficulty by passing through the sesqui–tensor product of a positive space with the universal vector extension of the other positive space.

Let us consider two positive spaces $\mathbb{U}$ and $\mathbb{V}$.

2.16 Definition. A *semi–tensor product* between $\mathbb{U}$ and $\mathbb{V}$ is defined to be a positive space $\mathbb{U} \hat{\otimes} \mathbb{V}$ along with a semi–bilinear map $\hat{\otimes} : \mathbb{U} \times \mathbb{V} \to \mathbb{U} \hat{\otimes} \mathbb{V}$, which fulfills the following universal property: if $\mathbb{W}$ is a positive space and $f : \mathbb{U} \times \mathbb{V} \to \mathbb{W}$ a semi–bilinear map, then there exists a unique semi–linear map $\tilde{f} : \mathbb{U} \hat{\otimes} \mathbb{V} \to \mathbb{W}$, such that $f = \tilde{f} \circ \hat{\otimes}$. $\square$

2.17 Theorem. *The semi–tensor product exists and is unique up to a distinguished semi–linear isomorphism.*

Proof. The uniqueness can be proved by a standard procedure as in Theorem 2.5. Then, we have to prove the existence of the semi–tensor product.

For this purpose, we consider the subset $\mathbb{U}\,\hat{\otimes}\,\mathbb{V} \subset \mathbb{U}\,\acute{\otimes}\,\bar{\mathbb{V}}$ consisting of the semi–linear combinations of elements of the type $u\,\acute{\otimes}\,(1\,\grave{\otimes}\,v)$, with $u \in \mathbb{U}$ and $v \in \mathbb{V}$, and the map $\hat{\otimes} : \mathbb{U} \times \mathbb{V} \to \mathbb{U}\,\hat{\otimes}\,\mathbb{V} : (u, v) \mapsto u\,\acute{\otimes}\,(1\,\grave{\otimes}\,v)$.

We can easily see that $\mathbb{U}\,\hat{\otimes}\,\mathbb{V}$ is a positive space and $\hat{\otimes} : \mathbb{U} \times \mathbb{V} \to \mathbb{U}\,\hat{\otimes}\,\mathbb{V}$ a semi–bilinear map.

Next, we prove that the above objects fulfill the required universal property.

Clearly, if the map $\tilde{f} : \mathbb{U}\,\hat{\otimes}\,\mathbb{V} \to \mathbb{W}$ of the universal property exists, then it is unique because $\mathbb{U}\,\hat{\otimes}\,\mathbb{V}$ is semi–linearly generated by the image of the map $\hat{\otimes}$.

Moreover, this map is well defined by the equality $\tilde{f}(u\,\acute{\otimes}\,(1\,\grave{\otimes}\,v)) = f(u, v)$, according to the following commutative diagram

$$
\begin{tikzcd}
\mathbb{U} \times \mathbb{V} \arrow[rrr, "i"] \arrow[dr, "f"] \arrow[dd, "\hat{\otimes}"'] & & & \bar{\mathbb{U}} \times \bar{\mathbb{V}} \arrow[dl, "\overline{\imath \circ f}"'] \arrow[dd, "\otimes"] \\
& \mathbb{W} \arrow[r, "\imath"] & \bar{\mathbb{W}} & \\
\mathbb{U}\,\hat{\otimes}\,\mathbb{V} \arrow[rrr, "j"'] \arrow[ur, "\tilde{f}"'] & & & \bar{\mathbb{U}} \otimes \bar{\mathbb{V}} \arrow[ul, "\widetilde{\overline{\imath \circ f}}"]
\end{tikzcd}
$$

where the maps $i, \imath, j$ are natural inclusions and where the map $(\widetilde{\overline{\imath \circ f}}) \circ j$ uniquely factorises through a map $\tilde{f}$. We can easily see that the map $\tilde{f}$ is semi–linear and that $\tilde{f} \circ \hat{\otimes} = f$. QED

Thus, if $b \in \mathbb{U}$ and $c \in \mathbb{V}$, then $b\,\hat{\otimes}\,c \in \mathbb{U}\,\hat{\otimes}\,\mathbb{V}$ generates the semi–tensor product.

In an analogous way, we can construct the semi–tensor product via the right sesquitensor product (instead of via the left sesqui–tensor product). We can also easily prove that the two constructions are naturally isomorphic.

We have the natural semi–linear isomorphisms

$$
\begin{aligned}
\mathrm{I\!R}^+\,\hat{\otimes}\,\mathbb{U} &\to \mathbb{U} : r\,\hat{\otimes}\,u \mapsto r u\,, \\
\mathbb{U}\,\hat{\otimes}\,\mathrm{I\!R}^+ &\to \mathbb{U} : u\,\hat{\otimes}\,r \mapsto r u\,, \\
\overline{\mathbb{U}\,\hat{\otimes}\,\mathbb{V}} \;&\simeq\; \bar{\mathbb{U}} \otimes \bar{\mathbb{V}}\,, \\
\mathbb{U}^*\,\hat{\otimes}\,\mathbb{V} &\hookrightarrow \text{s-Lin}(\mathbb{U}, \mathbb{V})\,,
\end{aligned}
$$

where the last isomorphism is characterised by $\alpha\,\hat{\otimes}\,v : \mathbb{U} \to \mathbb{V} : u \mapsto \alpha(u)\, v$, for all $\alpha \in \mathbb{U}^*$, $v \in \mathbb{V}$.

We obtain the contravariant “semi–tensor algebra” of a positive space in a way analogous to that of vector spaces.

Let m be a positive integer. If $\mathbb{U}_1, \dots, \mathbb{U}_m$ are positive spaces, then we can easily define the semi–tensor product $\mathbb{U}_1\,\hat{\otimes}\,\dots\,\hat{\otimes}\,\mathbb{U}_m$ and prove its basic properties along the same lines as for vector spaces. In particular, if $\mathbb{U} = \mathbb{U}_1 = \dots = \mathbb{U}_m$, then we set $\mathbb{U}^m \equiv \hat{\otimes}^m\mathbb{U} := \mathbb{U}_1\,\hat{\otimes}\,\dots\,\hat{\otimes}\,\mathbb{U}_m$ and $\hat{\otimes}^0\mathbb{U} := \mathrm{I\!R}^+$. Moreover, the semi–direct sum $\bigoplus_{m\in\mathbb{N}} \mathbb{U}^m$ turns out to be a “semi–algebra” over $\mathrm{I\!R}^+$.

Note that each element $t \in \hat{\otimes}^n\mathbb{U}$ is decomposable; even more, it can be uniquely written as $t = u\,\hat{\otimes}\,\dots\,\hat{\otimes}\,u$, with $u \in \mathbb{U}$.

For positive spaces we shall often adopt a notation similar to the standard notation used for numbers. Namely, if $\mathbb{U}$ and $\mathbb{U}'$ are positive spaces, we shall often write $u\,u' \equiv u\,\hat{\otimes}\,u' \in \mathbb{U}\,\hat{\otimes}\,\mathbb{U}'$, for each $u \in \mathbb{U}$ and $u' \in \mathbb{U}'$.

Moreover, if $\mathbb{U}$ is a positive space and $u \in \mathbb{U}$, then the unique element $1/u \in \mathbb{U}^\star$, such that $\langle 1/u,\, u\rangle = 1$, is called the *inverse* of u (not to be confused with the additive inverse). Clearly, for each $u \in \mathbb{U}$ and $r \in \mathrm{I\!R}^+$, we have $\frac{1}{ru} = \frac{1}{r}\frac{1}{u}$. Moreover, $(1/u)$ is just the dual element of u.

2.6 Rational maps between positive spaces

Next, we discuss the notion of q–rational maps between positive spaces.

Let us consider two positive space $\mathbb{U}$ and $\mathbb{V}$ and a rational number $q \in \mathbb{Q}$.

2.18 Definition. A map $f : \mathbb{U} \to \mathbb{V}$ is said to be q*–rational* (or, *rational of degree* q) if, for each $u \in \mathbb{U}$ and $r \in \mathrm{I\!R}^+$, we have $f(r\,u) = r^q\, f(u)$. $\square$

We denote by $\mathrm{Rat}^q(\mathbb{U}, \mathbb{V}) \subset \mathrm{Map}(\mathbb{U}, \mathbb{V})$ the subspace of q–rational maps between the positive spaces $\mathbb{U}$ and $\mathbb{V}$.

2.19 Proposition. If $u \in \mathbb{U}$ and $v \in \mathbb{V}$, then there exists a unique q–rational map $f : \mathbb{U} \to \mathbb{V}$, such that $f(u) = v$. $\square$

The composition of two rational maps is a rational map, whose degree is the product of the degrees. Hence, positive spaces and rational maps constitute a category.

2.20 Note. Let q' be another rational number. If $f : \mathbb{U} \to \mathrm{I\!R}^+$ is a q–rational map, then the map $f^{q'} : \mathbb{U} \to \mathrm{I\!R}^+ : u \mapsto (f(u))^{q'}$ is (qq')–rational. $\square$

2.21 Proposition. The subspace $\mathrm{Rat}^q(\mathbb{U}, \mathbb{V}) \subset \mathrm{Map}(\mathbb{U}, \mathbb{V})$ turns out to be a semi-vector subspace and a positive space.

Proof. The 1st statement is trivial. Moreover, $\mathrm{Rat}^q(\mathbb{U}, \mathbb{V})$ is a positive space because, for any given $u \in \mathbb{U}$, the map $\mathrm{Rat}^q(\mathbb{U}, \mathbb{V}) \to \mathbb{V} : f \mapsto f(u)$ is a semi–linear isomorphism. QED

2.22 Corollary. A q–rational map $f : \mathbb{U} \to \mathbb{V}$ is a bijection if and only if $q \neq 0$; in this case the inverse map is $(1/q)$–rational. $\square$

2.23 Example. We have the following distinguished cases.

a) The 0–rational maps $f : \mathbb{U} \to \mathbb{V}$ are just the constant maps. Hence, we have the natural semi–linear isomorphism

$$\mathrm{Rat}^0(\mathbb{U}, \mathbb{V}) \simeq \mathbb{V} : f \mapsto f(u),$$

which turns out to be independent of the choice of $u \in \mathbb{U}$. In particular, we have $\mathrm{Rat}^0(\mathbb{U}, \mathrm{I\!R}^+) \simeq \mathrm{I\!R}^+$.

b) The 1–rational maps $f : \mathbb{U} \to \mathbb{V}$ are just the semi–linear maps. Hence, we can write

$$\mathrm{Rat}^1(\mathbb{U}, \mathbb{V}) = \text{s-Lin}(\mathbb{U}, \mathbb{V}).$$

In particular, we have $\mathrm{Rat}^1(\mathbb{U}, \mathrm{I\!R}^+) = \text{s-Lin}(\mathbb{U}, \mathrm{I\!R}^+) = \mathbb{U}^\star$.

c) The (-1)–rational maps $f : \mathbb{U} \to \mathbb{V}$ can be identified with the semi–linear maps $\underline{f} : \mathbb{U}^\star \to \mathbb{V}$, through the natural semi–linear isomorphism

$$\mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{V}) \to \text{s-Lin}(\mathbb{U}^\star, \mathbb{V}) : f \mapsto \underline{f}\,,$$

where $\underline{f} : \mathbb{U}^\star \to \mathbb{V}$ is the unique semi–linear map such that $\underline{f}(1/u) = f(u)$, with reference to a chosen element $u \in \mathbb{U}$. Indeed, this isomorphism turns out to be independent of the choice of $u \in \mathbb{U}$. $\square$

In particular, the map

$$\mathrm{inv} : \mathbb{U} \to \mathbb{U}^\star : u \mapsto 1/u\,,$$

which associates with each element $u \in \mathbb{U}$ its dual form $1/u \in \mathbb{U}^\star$, is a (-1)–rational map.

Indeed, $\mathrm{inv} \in \mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{U}^\star)$ is the distinguished element which corresponds to the element $\mathrm{id}_{\mathbb{U}^\star} \in \text{s-Lin}(\mathbb{U}^\star, \mathbb{U}^\star)$, through the isomorphism $\mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{U}) \simeq \text{s-Lin}(\mathbb{U}^\star, \mathbb{U})$.

We have also the map

$$\mathrm{inv} : \mathbb{U}^\star \to \mathbb{U}^{\star\star} \simeq \mathbb{U}\,. \square$$

2.7 Rational powers of a positive space

Eventually, we introduce the rational powers of a positive space.

The basic idea is quite simple and could be achieved in an elementary way, by referring to a "semi–basis" and showing that the result is independent of this choice.

However, a full understanding of this concept is more subtle than it might appear at first insight and suggests a more sophisticated formal approach.

Let us consider a positive space $\mathbb{U}$ and a rational number $q \in \mathbb{Q}$.

2.24 Lemma. The map

$$\pi^q : \mathbb{U} \to \mathrm{Rat}^q(\mathbb{U}^\star, \mathbb{R}^+) : u \mapsto u^q\,,$$

where $u^q \in \mathrm{Rat}^q(\mathbb{U}^\star, \mathbb{R}^+)$ is the unique element such that $u^q(1/u) = 1$, turns out to be q–rational.

Proof. In fact, we have $1 = u^q(1/u)$ and $1 = (ru)^q(1/(ru)) = (ru)^q(1/r\,1/u) = (1/r)^q\,(ru)^q(1/u)$. Hence, we obtain $u^q(1/u) = (1/r)^q\,(ru)^q(1/u)$, which yields $(ru)^q = r^q u^q$. QED

2.25 Definition. The *q–power* of $\mathbb{U}$ is defined to be the pair $(\mathbb{U}^q,\, \pi^q)$ defined by

$$\mathbb{U}^q := \mathrm{Rat}^q(\mathbb{U}^\star, \mathbb{R}^+) \qquad \text{and} \qquad \pi^q : \mathbb{U} \to \mathbb{U}^q : u \mapsto u^q\,,$$

where $u^q : \mathbb{U}^\star \to \mathbb{R}^+$ is the unique q–rational map such that $u^q(1/u) = 1$. $\square$

We can re–interpret the above notion in a natural way in terms of semi–tensor powers as follows.

2.26 Note. Clearly, for each $r \in \mathrm{I\!R}^+$, the following diagram commutes

$$\begin{tikzcd} \mathbb{U} \arrow[r] \arrow[d, "s_r"'] & \mathbb{U}^q \arrow[d, "s_{r^q}"] \\ \mathbb{U} \arrow[r] & \mathbb{U}^q \end{tikzcd},$$

where s_r and s_{r^q} denote the scalar multiplications by r and r^q.

Thus, the rational power of positive spaces emulates the rational power of positive numbers, according to the above commutative diagram. □

2.27 Note. We have the following distinguished cases.

1) If $q = 0$, then we have a natural semi–linear isomorphism

$$\mathbb{U}^0 := \mathrm{Rat}^0(\mathbb{U}^\star, \mathrm{I\!R}^+) \simeq \mathrm{I\!R}^+,$$

and π^0 turns out to be the constant map with value 1.

2) Let $q \equiv n$ be a positive integer.

Then, we have the natural mutually inverse semi–linear isomorphisms

$$\hat{\otimes}^n \mathbb{U} \to \mathrm{Rat}^n(\mathbb{U}^\star, \mathrm{I\!R}^+) : u \,\hat{\otimes} \ldots \hat{\otimes}\, u \to f_u,$$
$$\mathrm{Rat}^n(\mathbb{U}^\star, \mathrm{I\!R}^+) \to \hat{\otimes}^n \mathbb{U} : f \mapsto u_f \,\hat{\otimes} \ldots \hat{\otimes}\, u_f,$$

where $f_u : \mathbb{U}^\star \to \mathrm{I\!R}^+ : \omega \mapsto \omega(u) \ldots \omega(u)$ and where $u_f := 1/\omega_f \in \mathbb{U}$, being $\omega_f \in \mathbb{U}^\star$ the unique element such that $f(\omega_f) = 1$.

Moreover, according to the above isomorphisms, the map π^n is given by

$$\pi^n : \mathbb{U} \to \mathrm{Rat}^n(\mathbb{U}^\star, \mathrm{I\!R}^+) : u \mapsto f_u.$$

In particular, in the case $q \equiv n = 1$ we have the natural semi–linear isomorphism

$$\mathbb{U}^1 := \mathrm{Rat}^1(\mathbb{U}^\star, \mathrm{I\!R}^+) = \text{s-Lin}(\mathbb{U}^\star, \mathrm{I\!R}^+) := \mathbb{U}^{\star\star} \simeq \mathbb{U}.$$

3) Let $q \equiv 1/n$ be the inverse of a positive integer n.

Then, we have the natural mutually inverse semi–linear isomorphisms

$$\hat{\otimes}^n \mathrm{Rat}^{1/n}(\mathbb{U}^\star, \mathrm{I\!R}^+) \to \text{s-Lin}(\mathbb{U}^\star, \mathrm{I\!R}^+) := \mathbb{U}^{\star\star} \simeq \mathbb{U} : f \,\hat{\otimes} \ldots \hat{\otimes}\, f \mapsto f^n,$$
$$\mathbb{U} \simeq \mathbb{U}^{\star\star} := \text{s-Lin}(\mathbb{U}^\star, \mathrm{I\!R}^+) \to \hat{\otimes}^n \mathrm{Rat}^{1/n}(\mathbb{U}^\star, \mathrm{I\!R}^+) : f \mapsto f^{1/n} \,\hat{\otimes} \ldots \hat{\otimes}\, f^{1/n},$$

where $f^n : \mathbb{U}^\star \to \mathrm{I\!R}^+ : \omega \mapsto f(\omega) \ldots f(\omega)$ and $f^{1/n} : \mathbb{U}^\star \to \mathrm{I\!R}^+ : \omega \mapsto (f(\omega))^{1/n}$.

Moreover, according to the above isomorphisms, the map $\pi^{1/n}$ is given by

$$\pi^{1/n} : \mathbb{U} \simeq \mathbb{U}^{\star\star} \to \mathbb{U}^{1/n} := \mathrm{Rat}^{1/n}(\mathbb{U}^\star, \mathrm{I\!R}^+) : f \mapsto f^{1/n}.$$

4) Let $q \equiv -n$ be a negative integer.

Then, we have the natural mutually inverse semi–linear isomorphisms

$$\hat{\otimes}^n \mathbb{U}^\star \to \mathrm{Rat}^{-n}(\mathbb{U}^\star, \mathbb{R}^+) : \omega \,\hat{\otimes}\, \dots \,\hat{\otimes}\, \omega \to f_\omega \,,$$
$$\mathrm{Rat}^{-n}(\mathbb{U}^\star, \mathbb{R}^+) \to \hat{\otimes}^n \mathbb{U}^\star : f \mapsto \omega_f \,\hat{\otimes}\, \dots \omega_f \,,$$

where $f_\omega : \mathbb{U}^\star \to \mathbb{R}^+ : \alpha \mapsto \omega(1/\alpha) \dots \omega(1/\alpha)$ and $\omega_f \in \mathbb{U}^\star$ is the unique element such that $f(\omega_f) = 1$.

Moreover, according to the above isomorphisms, the map π^{-n} is given by

$$\pi^{-n} : \mathbb{U} \to \mathrm{Rat}^{-n}(\mathbb{U}^\star, \mathbb{R}^+) : u \mapsto f_{1/u} \,.$$

In particular, in the case $q = -1$, we have the natural semi–linear isomorphism

$$\mathbb{U}^{-1} := \mathrm{Rat}^{-1}(\mathbb{U}^\star, \mathbb{R}^+) \simeq \text{s-Lin}(\mathbb{U}^{\star\star}, \mathbb{R}^+) \simeq \text{s-Lin}(\mathbb{U}, \mathbb{R}^+) := \mathbb{U}^\star \,. \square$$

Next, we analyse the natural behaviour of the exponents of rational powers. Indeed, this behaviour is just what we expect and is analogous to that of powers of positive real numbers. We leave to the reader the easy proofs of the following Propositions.

2.28 Proposition. Let p and q be rational numbers. Then, we obtain the natural semi–bilinear map

$$b : \mathrm{Rat}^p(\mathbb{U}^*, \mathbb{R}^+) \times \mathrm{Rat}^q(\mathbb{U}^*, \mathbb{R}^+) \to \mathrm{Rat}^{p+q}(\mathbb{U}^*, \mathbb{R}^+) : (f, g) \mapsto fg \,,$$

which yields the unique semi–linear isomorphism $\tilde{b} : \mathbb{U}^p \,\hat{\otimes}\, \mathbb{U}^q \to \mathbb{U}^{p+q}$, such that $\pi^{p+q} = \tilde{b} \circ (\pi^p \,\hat{\otimes}\, \pi^q)$, in virtue of the universal property of the semi–tensor product. $\square$

2.29 Proposition. If p and q are rational numbers, then we have the natural semi–linear isomorphism

$$c : (\mathbb{U}^p)^q := \mathrm{Rat}^q\big(\mathrm{Rat}^p(\mathbb{U}^*, \mathbb{R}^+), \mathbb{R}^+\big) \to \mathbb{U}^{p+q} := \mathrm{Rat}^{p+q}(\mathbb{U}^*, \mathbb{R}^+) : f \mapsto g_f \,,$$

where $g_f : \mathbb{U}^\star \to \mathbb{R}^+ : 1/u \mapsto f(1/u^p)$. Moreover, we have $\pi^{pq} = c \circ (\pi^q \circ \pi^p)$. $\square$

2.30 Corollary. If q is a rational number, then

$$(\mathbb{U}^q)^* \simeq (\mathbb{U}^*)^q \,.$$

If $p < q$ are two positive integers, then

$$\hat{\otimes}^q \mathbb{U} \,\hat{\otimes}\, (\hat{\otimes}^p \mathbb{U}^*) \simeq \mathbb{U}^q \,\hat{\otimes}\, \mathbb{U}^{-p} = \mathbb{U}^{q-p} \simeq \hat{\otimes}^{q-p} \mathbb{U}$$
$$\hat{\otimes}^p \mathbb{U} \,\hat{\otimes}\, (\hat{\otimes}^q \mathbb{U}^*) \simeq \mathbb{U}^p \,\hat{\otimes}\, \mathbb{U}^{-q} = \mathbb{U}^{p-q} \simeq \hat{\otimes}^{|p-q|} \mathbb{U}^* \,. \square$$

3 Algebraic model of physical scales

Next, we discuss the physical model of scales and units of measurement through positive spaces. Thus, we introduce the fundamental scale spaces and related notions, including scale dimension, scale basis and coupling scales. Finally, we show how unit spaces can be employed in physical theories by the language of differential geometry.

The formalism discussed in this section has been widely used in several papers dealing with physical theories (see, for instance, [7, 16, 17, 24, 31, 33, 34]). In the present paper we analyse the mathematical foundations of this formalism for the first time. We also discuss the interplay of our theory with dimensional analysis.

3.1 Units and scales

We introduce the fundamental scale spaces and related notions.

In this paper, we shall be concerned just with scales derived from time, length and mass scales via rational powers. Of course, the treatment could be extended to other types of systems in an analogous way.

In several theories of physics it is convenient to assume the following positive spaces as *basic spaces of scales*:

(1) the space $\mathbb{T}$ of *time scales*,

(2) the space $\mathbb{L}$ of *length scales*,

(3) the space $\mathbb{M}$ of *mass scales*.

The elements of the above spaces are called *basic scales*. More precisely,

(1) each element $u_0 \in \mathbb{T}$ is said to be a *time scale*,

(2) each element $\mathfrak{l} \in \mathbb{L}$ is said to be a *length scale*,

(3) each element $\mathfrak{m} \in \mathbb{M}$ is said to be a *mass scale*.

For each time scale $u_0 \in \mathbb{T}$, we denote its dual by $u^0 := 1/u_0 \in \mathbb{T}^*$.

3.1 Definition. A *scale space* is defined to be a positive space of the type

$$\mathbb{S} \equiv \mathbb{S}[d_1, d_2, d_3] := \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_2}, \qquad \text{where} \qquad d_i \in \mathbb{Q}.$$

A *scale* is defined to be an element $k \in \mathbb{S}$.

A scale $k \in \mathbb{S}$, regarded as a generator of the scale space $\mathbb{S}$, is called a *unit of measurement.*

For each scale space $\mathbb{S} = \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_2}$ and for each scale $k \in \mathbb{S}$, we set

$$|\mathbb{S}| := (d_1, d_2, d_3), \qquad |k| \equiv (|k|_1, |k|_2, |k|_3) := (d_1, d_2, d_3),$$
$$\lceil k \rceil := \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_2}.$$

The above 3-plet $|k| := (d_1, d_2, d_3)$ of rational numbers is called the *scale dimension* of $\mathbb{S}$ and of k. □

3.2 Note. The scale dimension $|k|$ of a scale k determines the corresponding scale space $\mathbb{S}$. In other words, for two scales k and k', we have $|k| = |k'|$ if and only if the two scales belong to the same scale space. If this is the case, then $k = rk'$, where $r = k/k'$. Hence, the scale dimension $|k|$ of a scale k determines k up to a positive real factor.

The map $k \mapsto |k|$ fulfills the following properties, for each $k \in \mathbb{S}$, $k' \in \mathbb{S}'$, $r \in \mathrm{I\!R}^+$ and $q \in \mathbb{Q}$,

$$|rk| = |k|\,, \qquad |1/k| = -|k|\,, \qquad |k \hat{\otimes} k'| = |k| + |k|'\,, \qquad |k^q| = q\,|k|\,.\square$$

3.3 Definition. A 3–plet of scales (e_1, e_2, e_3) is said to be a *scale basis* if each scale k can be written in a unique way as

$$k = r\,(e_1)^{c_1} \hat{\otimes} (e_2)^{c_2} \hat{\otimes} (e_3)^{c_3}\,, \qquad \text{with} \qquad r \in \mathrm{I\!R}^+\,,\; c_i \in \mathbb{Q}\,.\square$$

3.4 Proposition. A 3–plet of scales (e_1, e_2, e_3) is a scale basis if and only if

$$\det(|e_j|_i) \neq 0\,.$$

Moreover, let (e_1, e_2, e_3) be a scale basis and k a scale. Then, the 3–plet of rational exponents (c_1, c_2, c_3) is the unique solution of the linear rational system

$$|k|_i = \sum_j |e_j|_i\, c_j\,.$$

Proof. Let us consider a 3–plet of scales (e_1, e_2, e_3) and a scale k. Then,

$$k = r\,(e_1)^{c_1} \hat{\otimes} (e_2)^{c_2} \hat{\otimes} (e_3)^{c_3} \qquad \Leftrightarrow \qquad |k|_i = \sum_j |e_j|_i\, c_j\,.$$

Hence, the above left hand side expression holds and is unique if and only if $\det(|e_j|_i) \neq 0$. QED

3.5 Example. Clearly, each 3–plet of the type

$$(u_0, \mathfrak{l}, \mathfrak{m}) \in \mathbb{T} \times \mathbb{L} \times \mathbb{M}$$

is a scale basis. More generally, each 3–plet of the type

$$(u_0^{d_1}, \mathfrak{l}^{d_2}, \mathfrak{m}^{d_3}) \in \mathbb{T}^{d_1} \times \mathbb{L}^{d_2} \times \mathbb{M}^{d_3}\,, \qquad \text{with} \qquad d_i \in \mathbb{Q} - \{0\}\,,$$

is a scale basis. $\square$

Of course, we can also consider variable scales. Indeed, given a manifold $\boldsymbol{M}$, we define a *scale* of $\boldsymbol{M}$ to be a map of the type $k : \boldsymbol{M} \to \mathbb{S}$.

3.2 Scaled objects

In geometric models of physical theories one is often concerned with vector bundle valued maps which have physical dimensions. Our theory of positive spaces allows us to keep into account this fact in a formal algebraic way. In fact, we consider maps with values in vector bundles tensorialised with positive spaces. The positive factors can be treated as numerical constants, with respect to differential operators.

3.6 Note. Let $\mathbb{U}$ be a positive space. We observe that $\mathbb{U}$ has a natural structure of 1-dimensional manifold. Moreover, it is easy to prove that the tangent space $T\mathbb{U}$ is naturally isomorphic to the cartesian product $T\mathbb{U} \simeq \mathbb{U} \times \bar{\mathbb{U}}$. $\square$

Now, let us consider a scale space $\mathbb{S} := \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_3}$, and two vector bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ and a manifold $\boldsymbol{M}$.

We can easily define the sesqui–tensor product bundle $(\mathbb{U} \acute{\otimes} \boldsymbol{F}) \to \boldsymbol{B}$. We can regard this vector bundle as the sesqui–tensor product over $\boldsymbol{B}$ of the trivial semi–vector bundle $\widetilde{\mathbb{U}} := (\boldsymbol{B} \times \mathbb{U}) \to \boldsymbol{B}$ and the vector bundle $\boldsymbol{F} \to \boldsymbol{B}$.

3.7 Definition. The bundle $(\mathbb{S} \acute{\otimes} \boldsymbol{F}) \to \boldsymbol{B}$, the sections $s : \boldsymbol{B} \to \mathbb{S} \acute{\otimes} \boldsymbol{F}$ and the linear differential operators $\phi : \sec(\boldsymbol{B}, \boldsymbol{G}) \to \sec(\boldsymbol{B}, \mathbb{S} \acute{\otimes} \boldsymbol{F})$ are said to be *scaled.*

Moreover, they are said to have *scale dimension*

$$|\mathbb{S} \acute{\otimes} \boldsymbol{F}| = |s| = |\phi| = |\mathbb{S}| = (d_1, d_2, d_3)$$

and we set

$$\lceil \mathbb{S} \acute{\otimes} \boldsymbol{F} \rceil = \lceil s \rceil = \lceil \phi \rceil = \mathbb{T}^{d_1} \hat{\otimes} \mathbb{L}^{d_2} \hat{\otimes} \mathbb{M}^{d_3} \, . \square$$

3.8 Note. Let $\mathcal{D} : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, \boldsymbol{G})$ be a linear differential operator. Then, we obtain the linear differential operator (defined by the same symbol)

$$\mathcal{D} : \sec(\boldsymbol{B}, \mathbb{S} \acute{\otimes} \boldsymbol{F}) \to \sec(\boldsymbol{B}, \mathbb{S} \acute{\otimes} \boldsymbol{G}) : s \mapsto \mathcal{D}s := u \acute{\otimes} \mathcal{D}\langle \alpha, 1/u \rangle \, ,$$

where $u \in \mathbb{S}$ and $\langle \alpha, 1/u \rangle \in \sec(\boldsymbol{B}, \boldsymbol{F})$. Of course, this definition does not depend on the choice of u . $\square$

The above construction applies, for instance, to the cases when $\mathcal{D}$ is the exterior differential, a Lie derivative, a covariant derivative, and so on.

3.9 Example. If $\alpha \in \sec(\boldsymbol{M}, \mathbb{S} \acute{\otimes} \Lambda^r T^*\boldsymbol{M})$ is a scaled form, then we obtain the "*scaled exterior differential*"

$$d\alpha := u \acute{\otimes} d\alpha' \in \sec(\boldsymbol{M}, \mathbb{S} \acute{\otimes} \Lambda^{r+1} T^*\boldsymbol{M}) \, ,$$

where $u \in \mathbb{S}$ and α' is the form $\alpha' := \langle \alpha, 1/u \rangle \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M})$. $\square$

3.10 Example. If $t \in \sec(\boldsymbol{M}, \otimes^r T\boldsymbol{M})$ is a form and $X \in \sec(\boldsymbol{M}, \mathbb{S} \acute{\otimes} T\boldsymbol{M})$ a scaled vector field. Then, we obtain the "*scaled Lie derivative*"

$$L_X \, t := u \acute{\otimes} L_{X'} \, t \in \sec(\boldsymbol{M}, \mathbb{S} \acute{\otimes} (\otimes^r T\boldsymbol{M})) \, ,$$

where $u \in \mathbb{S}$ and X' is the vector field $X' := \langle X, 1/u \rangle \in \sec(\boldsymbol{M}, T\boldsymbol{M})$. $\square$

3.11 Example. If c is a linear connection of the vector bundle $\boldsymbol{F} \to \boldsymbol{B}$, $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ a vector field and $s \in \sec(\boldsymbol{B}, \mathbb{S} \acute{\otimes} \boldsymbol{F})$ a section. Then, we obtain the "*scaled covariant derivative*"

$$\nabla_X s := u \acute{\otimes} \nabla_X s' \in \sec(\boldsymbol{B}, \mathbb{S} \acute{\otimes} V\boldsymbol{F}) \, ,$$

where $u \in \mathbb{U}$ and s' is the section $s' := \langle s, 1/u \rangle \in \sec(\boldsymbol{B}, \boldsymbol{F})$.

We can re–interpret the above result in the following way.

The trivial linear connection of $(\boldsymbol{B} \times \bar{\mathbb{U}}) \to \boldsymbol{B}$ and the linear connection c of $\boldsymbol{F} \to \boldsymbol{B}$ yield a linear connection c' of $(\mathbb{U} \acute{\otimes} \boldsymbol{F}) \to \boldsymbol{B}$, which has the same symbols of c. The scaled covariant derivative can be regarded as the covariant derivative with respect to the above product connection.

By abuse of language, we shall denote by c also the product connection c'. □

3.3 Distinguished scales

In this section, we discuss the algebraic model of distinguished scales occurring in physics.

Let us consider a vector bundle $\boldsymbol{F} \to \boldsymbol{B}$. Suppose that in a physical theory we meet two scaled sections with different scale factors

$$s : \boldsymbol{M} \to \mathbb{S} \acute{\otimes} \boldsymbol{F} \qquad \text{and} \qquad s' : \boldsymbol{M} \to \mathbb{S}' \acute{\otimes} \boldsymbol{F}.$$

Then, we can compare the two scales and write $s = k \acute{\otimes} s'$, provided we avail of a scale factor $k : \boldsymbol{B} \to \mathbb{S} \hat{\otimes} \mathbb{S}'^*$, whose scale dimension is $|k| = |s| - |s'|$. We call such a factor a *coupling scale* (or, according to the traditional use, a *coupling constant*).

Some coupling scales, such as, for instance, the speed of the light, the Planck constant, the gravitational constant and the positron charge have a fixed value, without reference to specific systems. For this reason, we shall call these coupling scales *universal.*

Other types of coupling scales, such as, for instance, masses and charges, arise, case by case, and are associated with different particles.

For instance, we have the following *universal coupling scales*:

1) the *speed of the light* $c \in \mathbb{T}^{-1} \hat{\otimes} \mathbb{L}$,

2) the *Planck constant* $\hbar \in \mathbb{T}^{-1} \hat{\otimes} \mathbb{L}^2 \hat{\otimes} \mathbb{M}$,

3) the *gravitational constant* $\mathsf{g} \in \mathbb{T}^{-2} \hat{\otimes} \mathbb{L}^3 \hat{\otimes} \mathbb{M}^{-1}$,

4) the *positron charge* $\mathsf{e} \in \mathbb{T}^{-1} \hat{\otimes} \mathbb{L}^{3/2} \hat{\otimes} \mathbb{M}^{1/2}$.

Besides the above universal coupling scales, there are the following scales which depend on the choice of a particle:

1) a *mass* $m \in \mathbb{M}$,

2) a *charge* $q \in \bar{\mathbb{T}}^{-1} \hat{\otimes} \mathbb{L}^{3/2} \hat{\otimes} \mathbb{M}^{1/2}$. We stress that a charge is a scale tensorialised with real numbers; hence, a charge might be positive, vanishing, or negative.

3.12 Note. The following 3–plets are scale bases (for $q \neq 0$):

$$\begin{aligned} &1) \quad (e_1, e_2, e_3) := (m, q, \hbar), \\ &2) \quad (e_1, e_2, e_3) := (m, \hbar, \mathsf{g}), \\ &3) \quad (e_1, e_2, e_3) := (q, \hbar, \mathsf{g}). \end{aligned}$$

Conversely, the following 3–plet is not a scale basis (for $q \neq 0$):

$$4) \quad (e_1, e_2, e_3) := (m, q, \mathfrak{g})\,.$$

In fact, we have the following values of determinants in the above cases, respectively:

$$1)\ \det(|e_j|_i) = -1/2\,, \quad 2)\ \det(|e_j|_i) = 1\,, \quad 3)\ \det(|e_j|_i) = 1\,, \quad 4)\ \det(|e_j|_i) = 0\,.$$

Note that $|\mathfrak{g}| = |q^2/m^2|$. □

It may be algebraically correct, but not physically reasonable to express certain scales by means of some of the above scale bases. For instance, it may not be physically reasonable to express the gravitational coupling scale $\mathfrak{g}$ through the scale basis $(m, q, \hbar)$, because $\mathfrak{g}$ is a universal coupling scale, while m and q depend on the choice of a specific particle.

4 Fields of scales

We can regard scales as fields described by sections of bundles ruled by a dynamical equation formulated in terms of a covariant derivative. In view of such developments, we introduce bundles of scales equipped with semi-linear connections.

4.1 Bundles of scales

We naturally regard positive spaces as manifolds and define bundles structured in the category of positive spaces.

In the following, we denote all kinds of tensor products by the standard symbol $\otimes$, for the sake of simplicity.

Let us consider a positive space $\mathbb{U}$.

Each element $b \in \mathbb{U}$ induces a bijection $\mathbb{U} \to \mathrm{I\!R}^+ \subset \mathrm{I\!R} : s \to s/b$. Indeed, these maps yield a smooth manifold structure on $\mathbb{S}$.

By taking into account the natural smooth inclusion $\mathbb{U} \subset \bar{\mathbb{U}} := \mathrm{I\!R} \otimes \mathbb{U}$ and the natural linear fibred isomorphism $\mathbb{U} \times \bar{\mathbb{U}} \to \mathbb{U} \times \mathrm{I\!R} : (b, v) \mapsto (b, v/b)$ over $\mathbb{U}$, we obtain the natural linear fibred isomorphisms over $\mathbb{U}$

$$T\mathbb{U} \simeq \mathbb{U} \times \bar{\mathbb{U}} \simeq \mathbb{U} \times \mathrm{I\!R}\,.$$

Now, let us consider n positive spaces $\mathbb{S}_1, \dots, \mathbb{S}_n$ and the tensor product $\mathbb{P} := \mathbb{S}_1 \otimes \dots \otimes \mathbb{S}_n$. Moreover, let us consider n smooth curves $c_1 : \mathrm{I\!R} \to \mathbb{S}_1, \dots, c_n : \mathrm{I\!R} \to \mathbb{S}_n$ and the tensor product $c := c_1 \otimes \dots \otimes c_n : \mathrm{I\!R} \to \mathbb{P}$. By considering the natural inclusions $\mathbb{S}_i \subset \bar{\mathbb{S}}_i$ and $\mathbb{P} \subset \bar{\mathbb{P}}$, we have also the smooth curves $\bar{c}_i : \mathrm{I\!R} \to \bar{\mathbb{S}}_i$ and $\bar{c} : \mathrm{I\!R} \to \bar{\mathbb{P}}$.

Then, we obtain the differentials $d\bar{c}_i : \mathrm{I\!R} \to \bar{\mathbb{S}}_i$ and $d\bar{c} : \mathrm{I\!R} \to \bar{\mathbb{P}}$. Indeed, by taking into account the natural isomorphism $\bar{\mathbb{S}}_1 \otimes \mathbb{S}_2 \otimes \ldots \otimes \mathbb{S}_n \simeq \ldots \simeq \mathbb{S}_1 \otimes \ldots \otimes \mathbb{S}_{n-1} \otimes \bar{\mathbb{S}}_n \simeq \bar{\mathbb{P}}$, we can prove the equality

$$d\bar{c} = d\bar{c}_1 \otimes (c_2 \otimes \ldots \otimes c_n) + \cdots + (c_1 \otimes \ldots \otimes c_{n-1}) \otimes d\bar{c}_n \,.$$

Next, let us consider a positive space $\mathbb{S}$, a smooth curve $c : \mathrm{I\!R} \to \mathbb{S}$, a rational number m/n and the smooth curve $c^{m/n} : \mathrm{I\!R} \to \mathbb{S}^{m/n}$. By considering the natural inclusions $\mathbb{S} \subset \bar{\mathbb{S}}$ and $\mathbb{S}^{m/n} \subset \overline{\mathbb{S}^{m/n}}$, we have also the smooth curves $\bar{c} : \mathrm{I\!R} \to \bar{\mathbb{S}}$ and $\overline{c^{m/n}} : \mathrm{I\!R} \to \overline{\mathbb{S}^{m/n}}$.

Then, we obtain the differentials $d\bar{c} : \mathrm{I\!R} \to \bar{\mathbb{S}}$ and $d(\overline{c^{m/n}}) : \mathrm{I\!R} \to \overline{\mathbb{S}^{m/n}}$. Indeed, we can prove the equality

$$d(\overline{c^{m/n}}) = (m/n)\, c^{(m/n)-1} \otimes d\bar{c} \,.$$

Moreover, for each $b_a \in \mathbb{S}$, we have the following coordinate expressions

$$c = c^a\, b_a \,, \qquad \bar{c} = c^a\, 1 \otimes b_a \,,$$
$$d\bar{c} = Dc^a\, 1 \otimes b_a \,, \qquad d(\overline{c^{m/n}}) = (m/n)\, (c^a)^{(m/n)-1}\, Dc^a\, 1 \otimes (b_a)^{m/n} \,.$$

In the above formula and later, in order to follow the standard coordinate notation, we quote the index a explicitly, even if its range is just $\{1\}$.

Indeed, the above rules of differentiation are consistent with the algebraic rules concerning exponentials and tensor products.

Now, let us consider a manifold $\boldsymbol{E}$.

We define *a positive bundle* over $\boldsymbol{E}$ to be a bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$, whose fibres are smoothly endowed with a structure in the category of positive spaces. Thus, each fibre $\boldsymbol{S}_e \subset \boldsymbol{S}$, with $e \in \boldsymbol{E}$, is a positive space and $\pi : \boldsymbol{S} \to \boldsymbol{E}$ turns out to be an Abelian principal bundle with structure group $\mathrm{I\!R}^+$.

Thus, a (local) section $s : \boldsymbol{E} \to \boldsymbol{S}$ assigns the choice of the scale $s(e) \in \boldsymbol{S}_e$, for each $e \in \boldsymbol{E}$.

Due to a well-known theorem [15, p. 21], bundles whose fibre is topologically trivial (like $\mathrm{I\!R}^+$) admit a global section. In our case, this implies that positive bundles are trivial bundles. However, we do not assume any distinguished trivialization here.

The tangent prolongation of the operations $+$ and $\cdot$ on the fibres of the positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ makes the bundle $T\pi : T\boldsymbol{S} \to T\boldsymbol{E}$ a semi–vector bundle (but not a positive bundle).

Moreover, let us consider the Abelian group $\mathrm{I\!R}^+ \times \mathrm{I\!R}$, with the multiplication $(r,\lambda)\cdot(s,\mu) = (rs, r\mu + s\lambda)$; we have the natural subgroup $\mathrm{I\!R}^+ \subset \mathrm{I\!R}^+ \times \mathrm{I\!R} : r \mapsto (r, 0)$.

The tangent prolongation of the fibred action $\boldsymbol{S} \times \mathrm{I\!R}^+ \to \boldsymbol{S}$ makes the bundle $T\pi : T\boldsymbol{S} \to T\boldsymbol{E}$ a principal bundle with structure group $\mathrm{I\!R}^+ \times \mathrm{I\!R}$. On the other hand, the subgroup $\mathrm{I\!R}^+$ acts freely (but not transitively) on the fibres of this bundle.

The positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ yields also the 1–dimensional vector bundle $\bar{\pi} : \bar{\boldsymbol{S}} := \mathrm{I\!R} \otimes \boldsymbol{S} \to \boldsymbol{E}$; we have a natural semi–linear fibred inclusion $\boldsymbol{S} \subset \bar{\boldsymbol{S}}$ over $\boldsymbol{E}$.

The vertical bundle $V\boldsymbol{S} \to \boldsymbol{E}$ turns out to be naturally isomorphic to the 1-dimensional vector bundle $\boldsymbol{S} \underset{\boldsymbol{E}}{\times} \bar{\boldsymbol{S}}$ and to the trivial 1–dimensional vector bundle $\boldsymbol{S} \times \mathrm{I\!R}$. Indeed, $\mathrm{I\!R}$ is the trivial Lie algebra of the structure group $\mathrm{I\!R}^+$.

From now on, we shall refer to fibred charts (x^λ, y^a) of $\pi : \boldsymbol{S} \to \boldsymbol{E}$, which are adapted to the $\mathrm{I\!R}^+$–affine structure of the fibres. Namely, the fibre coordinate of a chart is defined by choosing a (local) section b_a and by setting $y^a \circ s := s/b_a : \boldsymbol{E} \to \mathrm{I\!R}^+$, for each section $s : \boldsymbol{E} \to \boldsymbol{S}$.

The transition rule between the fibred charts (x^λ, y^a) and (x^λ, y'^a), associated with the sections b_a and b'_a, is $y'^a = f^a_a \, y^a$, where $f^a_a := b_a/b'_a : \boldsymbol{E} \to \mathrm{I\!R}^+$.

We shall also refer to linear fibred charts $(x^\lambda, \bar{y}^a)$ of the vector bundle $\bar{\pi} : \bar{\boldsymbol{S}} \to \boldsymbol{E}$.

4.2 Semi–linear connections on positive bundles

We analyse the semi–linear connections on positive bundles and discuss the tensor product and the rational powers of semi–linear connections.

Let us consider the positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$.

A semi–linear connection of the bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ is defined to be a connection $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$, which is a semi–linear map over $T\boldsymbol{E}$. Clearly, a semi–linear connection is equivariant with respect to the action of the group $\mathrm{I\!R}^+$, i.e. it makes the following diagram commutative, for each $r \in \mathrm{I\!R}^+$,

$$\begin{array}{ccc}
\boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\quad c \quad} & T\boldsymbol{S} \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\
\boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\quad c \quad} & T\boldsymbol{S}
\end{array} \qquad \circ$$

Indeed, a connection $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$ is semi–linear if and only if it is principal.

Now, let us suppose that the scale bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ be equipped with a semi–linear connection $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$.

The coordinate expression of c is of the type

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^a{}_b \, y^b \, \partial_a) , \qquad \text{with} \qquad c_\lambda{}^a{}_b : \boldsymbol{E} \to \mathrm{I\!R} .$$

Then, the coordinate expression of the covariant differential of a section $s : \boldsymbol{E} \to \boldsymbol{S}$

$$\nabla s : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{S} \simeq T^*\boldsymbol{E} \otimes \bar{\boldsymbol{S}} \simeq \boldsymbol{S} \underset{\boldsymbol{E}}{\times} (T^*\boldsymbol{E} \otimes \mathrm{I\!R}^+) \simeq \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}$$

is

$$\nabla s = (\partial_\lambda s^a - c_\lambda{}^a{}_b \, s^b) \, d^\lambda \otimes b_a , \qquad \text{where} \qquad c_\lambda{}^a{}_b := -y^a(\nabla_\lambda b_b) .$$

Thus, the connection c is characterised by the 1–form $\omega := -c_\lambda{}^a_a \, d^\lambda : \boldsymbol{E} \to T^*\boldsymbol{E}$, which depends on the choice of the chart. The transition rule for the above 1–form, with respect to two fibred charts (x^λ, y^a) and (x^λ, y'^a), is $c'_\lambda{}^a_a = c_\lambda{}^a_a + \partial_\lambda f^a_a / f^a_a$.

A section $s : \boldsymbol{E} \to \boldsymbol{S}$ is said to be *c–constant* if $\nabla s = 0$, i.e. if, in coordinates, $\partial_\lambda s^a = c_{\lambda}{}^a_a \, s^a$.

The curvature r of the connection c is the 2–form

$$r := -[c, c] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} \otimes V \boldsymbol{S} \otimes V^* \boldsymbol{S} \simeq \Lambda^2 T^* \boldsymbol{E} \,,$$

where $[,]$ denotes the Frölicher–Nijenhuis bracket. We have the coordinate expression

$$r = 2 \, d\omega = -2 \, \partial_\lambda c_{\mu}{}^a_a \, d^\lambda \wedge d^\mu \,.$$

We stress that, the above 2–form does not depend on the choice of the fibred chart, even if the 1–form ω depends on the the choice of the fibred chart.

Semi–linear connections yield further distinguished connections on the bundles associated with a positive bundle in the following way.

4.1 Lemma. If $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$ is a semi–linear connection of the positive bundle, then there is a unique linear connection $\bar{c} : \bar{\boldsymbol{S}} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{S}}$ of the vector bundle $\bar{\pi} : \bar{\mathbb{S}} \to \boldsymbol{E}$, which makes the following diagram commutative

$$\begin{tikzcd}
\boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \arrow[r, "c"] \arrow[d, "\cap"'] & T\boldsymbol{S} \arrow[d, "\cap"] \\
\bar{\boldsymbol{S}} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \arrow[r, "\bar{c}"] & T\bar{\boldsymbol{S}}
\end{tikzcd}$$

$\circ$

Indeed, both connections c and $\bar{c}$ are represented in coordinates by the same symbols $c_\lambda{}^a{}_b$.

Conversely, if $\bar{c} : \bar{\boldsymbol{S}} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{S}}$ is a linear connection of the vector bundle $\bar{\boldsymbol{S}} \to \boldsymbol{E}$, then the restricted map $\boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \subset \bar{\boldsymbol{S}} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{S}}$ factorises through a semi–linear connection $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$ of the positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$, according to the above commutative diagram.

In this way, we obtain a bijection between semi–linear connections of the positive bundle $\boldsymbol{S} \to \boldsymbol{E}$ and linear connections of the vector bundle $\bar{\boldsymbol{S}} \to \boldsymbol{E}$. $\square$

4.2 Proposition. We have a natural bijection between semi–linear connections c of the positive bundle $\boldsymbol{S} \to \boldsymbol{E}$ and semi–linear connections c^* of the dual positive bundle $\boldsymbol{S}^* \to \boldsymbol{E}$. Indeed, in coordinates we have the equality $c^*_\lambda{}^a{}_b = -c_\lambda{}^a{}_b$.

Proof. Let us consider a semi–linear connection c of the positive bundle $\boldsymbol{S} \to \boldsymbol{E}$ and the associated linear connection $\bar{c}$ of the vector bundle $\bar{\boldsymbol{S}} \to \boldsymbol{E}$. Then, $\bar{c}$ yields the dual linear connection $\bar{c}^* : \bar{\boldsymbol{S}}^* \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{S}}^*$ of the dual vector bundle $\bar{\boldsymbol{S}}^* \to \boldsymbol{E}$. Moreover, by considering the natural linear fibred isomorphism $\bar{\boldsymbol{S}}^* \simeq \overline{\boldsymbol{S}^*}$ over $\boldsymbol{E}$, we can regard $\bar{c}^*$ as a linear connection $\bar{c}^* : \overline{\boldsymbol{S}^*} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\overline{\boldsymbol{S}^*}$ of the vector bundle $\overline{\boldsymbol{S}^*} \to \boldsymbol{E}$. Hence, in virtue of Lemma 4.1, we obtain a semi–linear connection $c^* : \boldsymbol{S}^* \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}^*$ of the positive bundle $\boldsymbol{S}^* \to \boldsymbol{E}$. QED

4.3 Proposition. Let us consider n positive bundles $\pi_i : \boldsymbol{S}_i \to \boldsymbol{E}$, equipped with the semi–linear connections $c_i : \boldsymbol{S}_i \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}_i$. Then, we obtain naturally a semi–linear connection $c := c_1 \otimes \ldots \otimes c_n : \boldsymbol{P} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \to T\boldsymbol{P}$ of the positive bundle $\boldsymbol{P} := \boldsymbol{S}_1 \otimes \ldots \otimes \boldsymbol{S}_n \to \boldsymbol{E}$.

Proof. In virtue of Lemma 4.1, the semi–linear connections $c_i : \boldsymbol{S}_i \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}_i$ yield the linear connections $\bar{c}_i : \bar{\boldsymbol{S}}_i \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{S}}_i$, hence the linear connection $\bar{c} := \bar{c}_1 \otimes \ldots \otimes \bar{c}_n : (\bar{\boldsymbol{S}}_1 \otimes \ldots \otimes \bar{\boldsymbol{S}}_n) \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T(\bar{\boldsymbol{S}}_1 \otimes \ldots \otimes \bar{\boldsymbol{S}}_n)$, by means of the Leibniz' rule. Moreover, by considering the natural linear fibred isomorphism $\bar{\boldsymbol{P}} \simeq \bar{\boldsymbol{S}}_1 \otimes \ldots \otimes \bar{\boldsymbol{S}}_n$ over $\boldsymbol{E}$, we can regard $\bar{c}$ as a connection $\bar{c} : \bar{\boldsymbol{P}} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\bar{\boldsymbol{P}}$ of the vector bundle $\bar{\boldsymbol{S}} \to \boldsymbol{E}$.

Eventually, again in virtue of Lemma 4.1, the above linear connection yields naturally a semi–linear connection $c : \boldsymbol{P} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \to T\boldsymbol{P}$ of the positive bundle $\boldsymbol{P} \to \boldsymbol{E}$. QED

4.4 Proposition. Let us consider a positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ equipped with a semi–linear connection $c : \boldsymbol{S} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{S}$ and a rational number m/n. Then, the covariant differentiation rule $\nabla(s^{m/n}) := (m/n)\, s^{(m/n)-1} \otimes \nabla s$ yields a semi–linear connection on the positive bundle $\boldsymbol{S}^{m/n} \to \boldsymbol{E}$. $\square$

The above rules of covariant differentiation are consistent with the algebraic rules concerning exponentials and tensor products.

We can interpret physically the above setting as follows.

The manifold $\boldsymbol{E}$ represents spacetime and a positive bundle $\pi : \boldsymbol{S} \to \boldsymbol{E}$ represents all possible values of a certain physical scale $\mathbb{S}$ (for instance, units of lengths, or of time, etc.), for each event $e \in \boldsymbol{E}$. Then, a section $s : \boldsymbol{E} \to \boldsymbol{S}$ represents a choice of the above type of scale for each event.

Moreover, the parallel transport associated with the connection c represents a physical way to transport a unit of measurement of the chosen type from an event to another. Clearly, in every physical model we consider, such a transport needs to be operatively specified by suitable experimental rules.

If the physical model under consideration is based on a certain scale basis, then the semi–linear connections chosen on the corresponding bundles yield a semi–linear connection on each scale bundle obtained via tensor product or rational powers.

This mathematical setting can yield some possible consequences on the physical problems of defining "global units of measurement" and of understanding whether fundamental scaled and unscaled coupling factors are really constant or not. We just quote a couple of remarks, just as examples. If the holonomy of the connection c is non trivial, then there is no way to choose a global c–constant unit of measurement of the considered type. Another remark arises with respect to an unscaled coupling factor obtained through a tensor product of scale bundles. In principle, it might be that the connections of these scale bundles have non vanishing curvature, even if their tensor product be a trivial connection. In such a case, we might have a "constant" unscaled coupling factor generated by scales which do not admit global constant sections.

4.3 Conformal fields

Eventually, we analyse a pair of examples of scaled fields, in order to show how typical differential equations for the unscaled fields appear if we do not choose a covariantly constant scale as unit of measurement.

Let us consider a manifold $\boldsymbol{E}$ equipped with a linear connection Γ.

Let $\boldsymbol{L} \to \boldsymbol{E}$ be a positive bundle equipped with a semi–linear connection c.

Then, we consider the scaled vector bundle $\boldsymbol{G} := \boldsymbol{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \to \boldsymbol{E}$ and the linear connection $K := c^2 \otimes \Gamma^* \otimes \Gamma^*$ induced on it by Γ and c.

Let $g : \boldsymbol{E} \to \boldsymbol{G}$ be a scaled symmetric non degenerate section. Chosen a section $l : \boldsymbol{E} \to \boldsymbol{L}$, we can write the section g as the tensor product $g = l^2 \otimes \mathsf{g}$, where g is the symmetric non degenerate unscaled section $\mathsf{g} := l^{-2} \otimes g : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$.

4.5 Proposition. The Levi–Civita condition $\nabla g = 0$ for the scaled metric g can be written, in terms of the unscaled metric g, as

$$\nabla \mathsf{g} = -2\, l^{-1} \otimes \nabla l \otimes \mathsf{g} \,. \square$$

In the particular case when the chosen scale section l is covariantly constant, the above condition reduces to the classical equality $\nabla \mathsf{g} = 0$.

Let $\boldsymbol{S} \to \boldsymbol{E}$ be a positive bundle equipped with a semi–linear connection c.

Then, we consider the vector bundle $\boldsymbol{F} := \boldsymbol{S} \otimes \Lambda^2 T^*\boldsymbol{E} \to \boldsymbol{E}$ of scaled 2–forms.

Let $F : \boldsymbol{E} \to \boldsymbol{F}$ be a scaled 2–form. Chosen a scale section $s : \boldsymbol{E} \to \boldsymbol{S}$, we can write the section F as the tensor product $F = s \otimes \mathsf{F}$, where F is the unscaled 2–form $\mathsf{F} := s^{-1} \otimes F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

4.6 Proposition. The condition of (exterior) covariant closure $\nabla F = 0$ for the scaled 2–form F can be written, in terms of the unscaled 2–form F, as

$$d\mathsf{F} = -s^{-1} \otimes \nabla s \otimes \mathsf{F} \,. \square$$

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