Absolute contact differentiation on submanifolds of Cartan spaces

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Abstract

We introduce the general concept of higher order absolute contact differentiation that is based on the idea of semiholonomic contact elements. We clarify how the moving frame method leads to the coordinate functions of the field of r-th order contact elements on a submanifold of Klein space and of the r-th absolute contact differential of a submanifold of Cartan space. We point out that the standard geometric objects of submanifolds are defined on contact elements, so that they are of universal character. In examples, we use heavily the concept of universal horizontal and vertical bundle over contact elements.

Keywords: semiholonomic contact elements, absolute contact differentiation, submanifolds of Cartan spaces, geometric objects of submanifolds, universal horizontal and vertical bundles.

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Introduction

The present paper was initiated by a conference talk by the second author on the contact element approach to geometric objects of submanifolds of Riemannian manifolds, [24]. Generally speaking, he pointed out that these objects are of universal character. Indeed, they are defined on the bundles of contact (n, r)-elements, so that they can be applied to every n-submanifold and are independent of its parametrization. This fact was also observed by the first author for submanifolds of Klein spaces in connection with the Cartan method of moving frames, [14], as well as for submanifolds of Cartan spaces, [13]. (We replace the term "Cartan geometry" from [22] by "Cartan space", see Section 3 for justification.)

In the course of the present research we realized that our approach to submanifolds of Cartan spaces is essentially based on the ideas of semiholonomic contact element and absolute contact differentiation. So, in the present paper we start with basic properties of nonholonomic and semiholonomic contact elements. In Section 2 we introduce the general concept of r-th order absolute contact differentiation, that leads to semiholonomic contact elements. Section 3 is devoted to two equivalent definitions of Cartan space. The interrelations between both points of view are essential for our research.

In Section 4 we recall, in the case of an arbitrary Klein space $S = G/H$, how the moving frame approach leads to the coordinate functions of the field of r-th order contact elements determined by a submanifold $N \subset S$. Then we present the general concept of r-th order geometric object for n-submanifolds of S that is motivated by the computational procedures related with the Cartan method of moving frames from [17]. We also clarify that the Cartan prolongation procedure leads to the equations of the infinitesimal action of H on the standard fiber of the bundle of contact (n, r)-elements on S, that can be used for evaluating the geometric objects. In Section 5 we modify these ideas to submanifolds of a Cartan space $\mathcal{S}$ of type S. This is based on the concept of semiholonomic (n, r)-object. In Section 6, Proposition 8 reads that if $\mathcal{S}$ is torsion-free, then the values of the second order absolute contact differentiation are holonomic. In particular, this is true for the submanifolds of a Riemannian manifold, that is considered as a Cartan space with respect to the Levi-Civita connection.

In Section 7 we define the universal horizontal and vertical bundles for n-submanifolds. As an example, we discuss the universal version of the fundamental vertical-valued quadratic form for submanifolds of affine spaces. In Section 8 we introduce the concept of reduced torsion and clarify that its universal version coincides with the difference tensor of second order semiholonomic contact elements. This yields another proof of Proposition 8. At this occasion we also illustrate the use of the algorithm from Section 5. Further we point out that in the case of a 2-submanifold of a 3-space with projective connection, the reduced torsion gives rise to an invariant discovered already by É. Cartan in [4]. In the last section, we extend the idea of universality to a wide class of geometric objects for submanifolds.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [16].

1 Semiholonomic contact elements

The bundle of contact (n,r)-elements $K^r_n M$ on a manifold M can be defined as the factor space

$$K^r_n M = \operatorname{reg} T^r_n M / G^r_n, \qquad n < m = \dim M \tag{1}$$

of the space of regular (n,r)-velocities on M with respect to the right action, determined by the jet composition, of the r-th differential group G^r_n in dimension n, [16]. So every n-submanifold $N \subset M$ defines a contact (n,r)-element $k^r_x N$ for every $x \in N$. This gives rise to a map $k^r_N \colon N \to K^r_n M$, that can be viewed as a section of the restriction $(K^r_n M)_N$ of $K^r_n M$ over N. We write $k\colon \operatorname{reg} T^r_n M \to K^r_n M$ for the factor projection and denote by the same symbol the induced map $k\colon \operatorname{reg} J^r(N,M) \to K^r_n M$. We remark that these classical contact elements are also discussed by P.J. Olver, [20], and are called jets of submanifolds in [1].

We extend the idea of nonholonomic and semiholonomic jets by C. Ehresmann [5, p. 361] to contact elements. To this aim, we recall the definition of jet prolongations of fibered manifolds, [15]. Let $p\colon Y \to M$ be a fibered manifold and $J^r Y$ denote the bundle of standard (holonomic) r-jets of local sections of Y. The r-th nonholonomic jet prolongation $\tilde J^r Y$ is defined by the induction $\tilde J^1 Y = J^1 Y$, $\tilde J^r Y = J^1(\tilde J^{r-1} Y \to M)$. The r-th semiholonomic jet prolongation $\bar J^r Y$ is defined by induction as the space of first-order jets $j^1_x s$ of local sections s of $\bar{J}^{r-1}Y \to M$ such that $s(x) = j^1_x(\beta_{r-1} \circ s)$, where $\beta_{r-1}\colon \bar{J}^{r-1}Y \to \bar{J}^{r-2}Y$ is defined in the induction procedure with $\bar{J}^1 Y = J^1 Y \to Y$. We have the inclusions $J^r Y \subset \bar{J}^r Y \subset \tilde{J}^r Y$. The first one is given by the iteration $j^r_x s \mapsto j^1_x(u \mapsto j^{r-1}_u s)$, the second one is straightforward. If $p\colon Y = M \times N \to M$ is the first product projection, then we set $\tilde{J}^r(M,N) = \tilde{J}^r(M \times N)$ and $\bar{J}^r(M,N) = \bar{J}^r(M \times N)$.

Definition 1. The space $\tilde{K}^r_n M$ of nonholonomic contact (n,r)-elements on M is defined by the iteration $\tilde{K}^r_n M = K^1_n(\tilde{K}^{r-1}_n M)$, $\tilde{K}^1_n M = K^1_n M$.

Hence $\tilde{K}^r_n M \to \tilde{K}^{r-1}_n M$ is a fibered manifold. The injection $K^r_n M \hookrightarrow \tilde{K}^r_n M$ is determined by the rule

$$k^r_x N \mapsto k^1_X(k^{r-1}_N), \qquad X = k^{r-1}_x N,$$

where k^{r-1}_N is interpreted as a submanifold of $K^{r-1}_n M \subset \tilde{K}^{r-1}_n M$.

We are going to clarify up to what extent the manifold of nonholonomic contact elements can be regarded as a quotient manifold analogously to (1). To this aim, we recall the definition of composition of nonholonomic jets [5], [15]. Let M, N, Q be three manifolds. For $r = 1$ we have the standard composition of 1-jets in $J^1(N,Q)$ with 1-jets in $J^1(M,N)$, yielding 1-jets in $J^1(M,Q)$. If $\beta\colon \tilde{J}^{r-1}(M,N) \to N$ is the target jet projection, $X = j^1_x s(u) \in \tilde{J}^r_x(M,N)_y$, $u \in M$, and $Z = j^1_y \sigma \in \tilde{J}^r_y(N,Q)_z$, $y = \beta(s(x))$, then we define

$$Z \circ X = j^1_x(\sigma(\beta(s(u))) \circ s(u)) \in \tilde{J}^r_x(M,Q)_z$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. We say that $X \in \tilde{J}^r_x(M,N)_y$ is regular, if there exists $Z \in \tilde{J}^r_y(N,M)_x$ such that $Z \circ X = j^r_x \,\mathrm{id}_M$. There are r underlying 1-jets of X and X is regular iff all of them correspond to injective linear maps $T_x M \to T_y N$.

We define $\tilde{T}^r_n M = \tilde{J}^r_0(\mathbb{R}^n, M)$. This is extended into a bundle functor $\tilde{T}^r_n$ on $\mathcal{M}f$ in the standard way, [5]. A natural equivalence of functors

$$\mu^r_M\colon \tilde{T}^r_n M \to T^1_n(\tilde{T}^{r-1}_n M)$$

is defined as follows. Every $X \in \tilde{T}^r_n M$ is of the form $X = j^1_0 \varphi$, where $\varphi\colon \mathbb{R}^n \to \tilde{J}^{r-1}(\mathbb{R}^n, M)$ is a section of the source projection $\tilde{J}^{r-1}(\mathbb{R}^n, M) \to \mathbb{R}^n$. On the other hand, $Z \in T^1_n(\tilde{T}^{r-1}_n M)$ means $Z = j^1_0 \psi$ with $\psi\colon \mathbb{R}^n \to \tilde{J}^{r-1}_0(\mathbb{R}^n, M)$.

Write $t_u\colon \mathbb{R}^n \to \mathbb{R}^n$ for the translation $x \mapsto x+u$. Then $u \mapsto \big(\varphi(u) \circ j_0^{r-1} t_u\big)$ is a map $\mathbb{R}^n \to \tilde{J}_0^{r-1}(\mathbb{R}^n, M)$ and we set

$$\mu_M^r(X) = j_0^1\big(\varphi(u) \circ j_0^{r-1} t_u\big).$$

To obtain the inverse map, we consider $u \mapsto \psi(u) \circ j_u^{r-1} t_u^{-1}$. Then we have

$$(\mu_M^r)^{-1}(Z) = j_0^1\big(\psi(u) \circ j_u^{r-1} t_u^{-1}\big).$$

One verifies easily that μ_M^r maps $\operatorname{reg} \tilde{T}_n^r M$ into $\operatorname{reg} T_n^1(\operatorname{reg} \tilde{T}_n^{r-1} M)$.

On every fibered manifold $p\colon Y \to M$, a contact element $X \in K_n^1 Y$ is said to be transversal, if the underlying linear n-space of X has zero intersection with the vertical tangent space of Y. We write $\operatorname{tr} K_n^1 Y \subset K_n^1 Y$ for the subset of all transversal contact $(n,1)$-elements on Y. This is an open subset of $K_n^1 Y$ and $\operatorname{tr} K_n^1 Y \to Y$ is a fibered manifold. For $r \geq 2$, the bundle $\operatorname{tr} \tilde{K}_n^r M \subset \tilde{K}_n^r M$ of nonholonomic transversal contact (n,r)-elements on M is defined by the iteration

$$\operatorname{tr} \tilde{K}_n^r M = \operatorname{tr} K_n^1(\operatorname{tr} \tilde{K}_n^{r-1} M \to M).$$

We recall that $\tilde{G}_n^r = \operatorname{reg} \tilde{J}_0^r(\mathbb{R}^n, \mathbb{R}^n)_0$ is a group with respect to the composition of nonholonomic jets.

Proposition 1. *We have*

$$\operatorname{tr} \tilde{K}_n^r M = \operatorname{reg} \tilde{T}_n^r M / \tilde{G}_n^r. \tag{2}$$

We write $k\colon \operatorname{reg} \tilde{T}_n^r M \to \operatorname{tr} \tilde{K}_n^r M$ for the factor projection.

Proof. Assume by induction $\operatorname{tr} \tilde{K}_n^{r-1} M = \operatorname{reg} \tilde{T}_n^{r-1} M / \tilde{G}_n^{r-1}$. So we have defined $k\colon \operatorname{reg} \tilde{T}_n^{r-1} M \to \operatorname{tr} \tilde{K}_n^{r-1} M$. Consider $X \in \operatorname{reg} T_n^1(\operatorname{reg} \tilde{T}_n^{r-1} M)$, $X = j_0^1 \varphi(u)$, $\varphi\colon \mathbb{R}^n \to \tilde{T}_n^{r-1} M$. Then $u \mapsto k\big(\varphi(u)\big)$ is the parametrization of an n-dimensional submanifold of $\operatorname{tr} \tilde{K}_n^{r-1} M$ that is transversal to $\operatorname{tr} \tilde{K}_n^{r-1} M \to M$. Hence we have $k\big(j_0^1(k(\varphi(u)))\big) \in \operatorname{tr} \tilde{K}_n^r M$. Consider another $\psi(v)\colon \mathbb{R}^n \to \tilde{T}_n^{r-1} M$ such that $k\big(j_0^1 k(\psi(v))\big) = k\big(j_0^1 k(\varphi(u))\big)$. First of all, there is a map $v = f(u)$ such that

$$j_0^1 k\big(\psi(f(u))\big) = j_0^1 k\big(\varphi(u)\big).$$

By the induction hypothesis, there is a map $g\colon \mathbb{R}^n \to \tilde{G}_n^{r-1}$ such that $\psi\big(f(u)\big) \circ g(u) = \varphi(u)$. We have $j_0^1 f \in G_n^1$ and $j_0^1 g \in T_n^1 \tilde{G}_n^{r-1}$ and our construction is in accordance with the well known expression $\tilde{G}_n^r = G_n^1 \rtimes T_n^1 \tilde{G}_n^{r-1}$, [16]. $\square$

Definition 2. The bundle of semiholonomic contact (n,r)-elements $\bar{K}_n^r M \subset \tilde{K}_n^r M$ is the subset of all $k_X^1 Q$ such that $Q \subset \bar{K}_n^{r-1} M$, $X = k^1_{\beta_{r-1}(X)}\big(\beta_{r-1}(Q)\big)$, where $\beta_{r-1}\colon \bar{K}_n^{r-1} M \to \bar{K}_n^{r-2} M$ is defined in the induction procedure starting with the bundle projection $K_n^1 M \to M$.

We easily deduce $\bar{K}_n^r M \subset \operatorname{tr} \tilde{K}_n^r M$ and $K_n^r M \subset \bar{K}_n^r M$. Analogously to Proposition 1, we obtain

$$\bar{K}_n^r M = \operatorname{reg} \bar{T}_n^r M / \bar{G}_n^r, \tag{3}$$

where $\bar{G}_n^r = \operatorname{reg} \bar{J}_0^r(\mathbb{R}^n, \mathbb{R}^n)_0$ is a subgroup of $\tilde{G}_n^r$. The canonical projection to lower order semiholonomic contact elements will be denoted by the same symbol π_s^r, $s < r$, as in the jet case.

The underlying contact $(n,1)$-element X_1 of $X \in (K_n^r \mathbb{R}^m)_x$ is identified with a linear n-space in $T_x \mathbb{R}^m$. Write $\mathbb{R}^{n,m-n}$ for the product bundle $\mathbb{R}^n \times \mathbb{R}^{m-n}$ and $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$ for its bundle projection. Denote by $\tau K_n^r \mathbb{R}^m \subset K_n^r \mathbb{R}^m$ the open subset of all X such that X_1 is transversal to π. It is well known that $\tau K_n^r \mathbb{R}^m$ is identified with the jet prolongation $J^r \mathbb{R}^{n,m-n}$, [16].

In the nonholonomic case $X \in \tilde{K}_n^r \mathbb{R}^m$, we have r underlying contact $(n,1)$-elements $X_1^{(1)}, \dots, X_1^{(r)}$. Write $\tau \tilde{K}_n^r \mathbb{R}^m \subset \tilde{K}_n^r \mathbb{R}^m$ for the open subset of all X such that all $X_1^{(1)}, \dots, X_1^{(r)}$ are transversal to π.

Proposition 2. *$\tau \tilde{K}_n^r \mathbb{R}^m$ is identified with the r-th nonholonomic prolongation $\tilde{J}^r \mathbb{R}^{n,m-n}$.*

Proof. By Proposition 1, $X \in \tau \tilde{K}_n^r \mathbb{R}^m$ can be expressed as $X = Z \circ \tilde{G}_n^r$ with $Z \in (\operatorname{reg} \tilde{T}_n^r \mathbb{R}^m)_x$. Write $Z_1^{(1)}, \dots, Z_1^{(r)}$ for the underlying 1-velocities of Z. Then $j_x^1 \pi \circ Z_1^{(1)}, \dots j_x^1 \pi \circ Z_1^{(r)}$ are invertible 1-jets, so that $\zeta := (j_x^r \pi) \circ Z \in \tilde{J}_0^r(\mathbb{R}^n, \mathbb{R}^n)$ is invertible. Then $Z \circ \zeta^{-1}$ satisfies $(j_0^r \pi) \circ (Z \circ \zeta^{-1}) = j^r_{\pi(x)} \operatorname{id}_{\mathbb{R}^n}$, which implies $Z \circ \zeta^{-1} \in \tilde{J}^r \mathbb{R}^{n,m-n}$. □

In the semiholonomic case $X \in \bar{K}_n^r \mathbb{R}^m$, we have $X_1^{(1)} = \dots = X_1^{(r)} =: X_1$. We write $\tau \bar{K}_n^r \mathbb{R}^m \subset \bar{K}_n^r \mathbb{R}^m$ for the open subset of all X such that X_1 is transversal to $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$. In the same way as in Proposition 2, we construct an identification

$$\tau \bar{K}_n^r \mathbb{R}^m \approx \bar{J}^r \mathbb{R}^{n,m-n}, \tag{4}$$

where $\bar{J}^r \mathbb{R}^{n,m-n}$ denotes the r-th semiholonomic prolongation of $\mathbb{R}^{n,m-n}$. In particular, for $n = 1$ we have $\bar{K}_1^r M = K_1^r M$.

2 The absolute contact differentiation

Consider a principal bundle $P(M, G)$ with a principal connection Γ, a left G-space F and the associated bundle $E = P[F]$. For every section s of E, its absolute differential can be viewed as a section

$$\nabla_\Gamma s\colon M \to \bigcup_{x\in M} J^1_x(M, E_x)_{s(x)},$$

[16]. If $\Gamma(u) = j^1_x\rho$ for a local section ρ of P, then $(\nabla_\Gamma s)(x)$ is transformed by $\tilde{u}^{-1}$ into

$$j^1_x\big(\widetilde{\rho(y)}^{-1}(s(y))\big) \in J^1_x(M, F), \qquad y \in M, \tag{5}$$

where $\tilde{u}\colon F \to E_x$ denotes the frame map corresponding to $u \in P$ (see also the beginning of Section 4).

Having in mind submanifolds of Cartan spaces, we introduce the concept of absolute contact differential. Replace M by N and assume $n = \dim N < \dim F$. Clearly, all spaces $K^1_n(E_x)$, $x \in N$, form an associated bundle

$$\bigcup_{x\in N} K^1_n(E_x) = P[K^1_n F].$$

Assume further that each $\nabla_\Gamma s(x)$ is a regular 1-jet.

Definition 3. The absolute contact differential $k\nabla_\Gamma$ of s is defined by

$$\big((k\nabla_\Gamma)s\big)(x) = k\big((\nabla_\Gamma s)(x)\big), \qquad x \in N.$$

Hence $k\nabla_\Gamma s$ is a section $N \to P[K^1_n F]$. Since we have a section of another bundle associated to P, we can construct

$$\nabla_\Gamma\big((k\nabla_\Gamma)s\big)\colon N \to \bigcup_{x\in N} J^1_x\big(N, K^1_n(E_x)\big).$$

This is formed by regular 1-jets and we define

$$((k\nabla^2_\Gamma)s)(x) = k\big(\nabla_\Gamma((k\nabla_\Gamma)s)(x)\big).$$

One verifies easily that this is an element of $\bar{K}^2_n(E_x)$. Hence $(k\nabla^2_\Gamma)s$ is a section $N \to P[\bar{K}^2_n F]$.

Definition 4. The r-th absolute contact differential of s is defined by the iteration

$$\big((k\nabla^r_\Gamma)s\big)(x) = k\big(\nabla_\Gamma((k\nabla^{r-1}_\Gamma)s)(x)\big). \tag{6}$$

By the very definition of semiholonomic contact (n,r)-elements, we deduce that (6) form a section

$$(k\nabla^r_\Gamma)s\colon N \to P[\bar{K}^r_n F].$$

Remark 1. If Γ is curvature free, then P can be locally viewed as the product $N \times G$ with the canonical flat connection. Then (5) implies that the values of $(k\nabla^r_\Gamma)s$ are holonomic contact elements for every section $s\colon M \to E$.

3 Cartan spaces

We recall that a Klein space is a manifold S with a transitive left action $(g,x) \mapsto gx$ of a Lie group G. Fix a point $c \in S$ and write H for its stability group. Then S coincides with the coset space $S = G/H$, $c = \{H\}$ and G can be viewed as a principal H-bundle over S with bundle projection $g \mapsto gc$. Every $g \in G(S,H)$ is interpreted as a frame $\tilde{g}\colon S \to S$, $\tilde{g}(a) = ga$, $a \in S$.

A "curved" version of S can be defined in two formally different ways. First we present the viewpoint from the book by Sharpe, [22]. Consider a pair (G,H) of a Lie group G and a closed subgroup $H \subset G$.

Definition 5a. Cartan geometry *of type* (G,H) *is a principal bundle* $Q(M,H)$ *with* 1*-form* $\omega\colon TQ \to \mathfrak{g}$ *(which is said to be* Cartan connection*) such that*

(i) $\omega(u)\colon T_uQ \to \mathfrak{g}$ *is a linear isomorphism for every* $u \in Q$*,*

(ii) $(R_h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ *for every* $h \in H$*,*

(iii) $\omega\big(X^*(u)\big) = X$ *for every* $X \in \mathfrak{h}$ *and every* $u \in Q$*, where* X^* *is the fundamental vector field on* Q *induced by* X*.*

We remark that, in addition to [22], further interesting examples of Cartan spaces can be found in [2] and [23].

In what follows we assume G acts effectively on the coset space G/H. So $S = G/H$ is a Klein space. Clearly, $T_cS = \mathfrak{g}/\mathfrak{h}$.

On the other hand, consider $P(M,G)$, F, $E = P[F]$ as in Section 2 and fix a section $s\colon M \to E$. The following definition in [10] or [11] was based directly on some ideas by Ehresmann, [5].

Definition 5b. *Space with Cartan connection of type* (G,H) *over* M *is a quadruple* $\mathcal{S} = \mathcal{S}(M) = \big(P(M,G),\ \Gamma,\ E = P[G/H], s\big)$ *such that* $\dim M = \dim S$ *and the absolute differential* $\nabla_\Gamma s$ *is formed by regular* 1*-jets.*

We deduce that both concepts are naturally equivalent. In the case b), s defines a reduction to subgroup H

$$Q = \big\{u \in P, \tilde{u}(c) = s(p(u))\big\},$$

where $p\colon P \to M$ is the bundle projection. Write ω for the restriction of the connection form $\omega_\Gamma\colon TP \to \mathfrak{g}$ to Q. Clearly, $\omega\colon TQ \to \mathfrak{g}$ satisfies (ii) and (iii) from Definition 5a.

Lemma 1. ω *satisfies* (i), *iff the* 1*-jets* $(\nabla_\Gamma s)(x)$ *are regular for all* $x \in M$.

Proof. The vertical tangent bundle VE is an associated bundle $P[TS]$ and $T\tilde{u}\colon TS \to TE_x$ is the induced frame map on VE. Consider $X = \frac{d\gamma(0)}{dt} \in T_xM$, $\gamma\colon \mathbb{R} \to M$. Then

$$(T\tilde{u})^{-1}\big((\nabla_\Gamma s)(X)\big) = \frac{d}{dt}\Big|_0 \widetilde{\varrho\big(\gamma(t)\big)}^{-1}\big(s(\gamma(t))\big).$$

Since G acts transitively on S, we have

$$\widetilde{\varrho\big(\gamma(t)\big)}^{-1}\big(s(\gamma(t))\big) = \delta(t)c, \qquad \delta\colon \mathbb{R} \to G.$$

Write $Z = \frac{d}{dt}\big|_0 \varrho\big(\gamma(t)\big)\delta(t) \in T_uQ$. By the definition of the connection form, we have

$$(T\tilde{u})^{-1}\big((\nabla_\Gamma s)(X)\big) = \omega(Z) + \mathfrak{h} \in T_cS.$$

These vectors are linearly independent for a basis of T_xM, iff $\omega(u)$ is a linear isomorphism. □

Using Lemma 1 one easily verifies that Definitions 5a and 5b are equivalent. In what follows, $\mathcal{S}(M)$ will be called a Cartan space and ω will be called its connection form.

Consider an n-submanifold $N \subset M$. If we restrict all objects in question over N, we obtain

$$(P_N, \Gamma_N, E_N, s_N) = \mathcal{S}_N = (Q_N, \omega_N).$$

Then we have the situation from Section 2. By induction we deduce that $\big((k\nabla^r_{\Gamma_N})s_N\big)(x)$ depends on k^r_xN only. Write $S^r_n = (K^r_nS)_c$ and $\bar{S}^r_n = (\bar{K}^r_nS)_c$. Clearly, both S^r_n and $\bar{S}^r_n$ are H-spaces.

Definition 6. The map

$$\Gamma^r_n\colon K^r_n M \to Q[\bar{S}^r_n], \quad \Gamma^r_n(k^r_x N) = \big((k\nabla^r_{\Gamma_N})s_N\big)(x)$$

is called the formal absolute contact (n, r)-differentiation on $\mathcal{S}$. The map

$$\Gamma^r_N = \Gamma^r_n \circ k^r_N\colon N \to Q_N[\bar{S}^r_n]$$

is said to be the r-th absolute contact differential of N.

If $\mathcal{S}$ is the Klein space G/H with the canonical flat connection, we have $Q[S^r_n] = K^r_n S$. Then, by Remark 1, Γ^r_n is the identity of $K^r_n S$ composed with the injection $K^r_n S \hookrightarrow \bar{K}^r_n S = Q[\bar{S}^r_n]$.

Remark 2. In [13], our investigation of the r-th absolute contact differential of N was based on Ehresmann's idea of higher order prolongations of connection Γ. But (5) implies directly that both approaches coincide. So we can use our results from [13] in what follows.

4 Submanifolds of Klein spaces

If N is a submanifold of Klein space S, we write G_N for the restriction of principal bundle $G(S, H)$ over N. Then k^r_N is a section $N \to G_N[S^r_n]$, that is sometimes called the fundamental r-th order field on N. It was pointed out by G. F. Laptev, [17] (but in a computational form only), that a modification of the Cartan method of moving frames leads to the coordinate functions of k^r_N.

In general, consider a principal G-bundle $p\colon P \to M$, a left G-space F and the associated bundle $E = P[F] = P \times_G F$. Every $u \in P_x$, $x \in M$, is interpreted as the frame map $\tilde{u}\colon F \to E_x$, $\tilde{u}(a) = \{u, a\}$, $a \in F$. For every section $s\colon M \to E$, the induced map

$$P \to F, \quad u \mapsto \tilde{u}^{-1}\big(s(p(u))\big) \tag{7}$$

is said to be the frame form of s, [16]. If z^a are some local coordinates on F, then the locally defined compositions of (7) with z^a are called the coordinate functions of s.

Further, every left action $l\colon G \times F \to F$ induces the infinitesimal action $\lambda\colon \mathfrak{g} \times F \to TF$. The Maurer-Cartan form $\varphi\colon TG \to \mathfrak{g}$ yields an identification $G \times \mathfrak{g} \approx TG$. This defines an involutive distribution Λ on $G \times F$,

$$\Lambda(g, z) = \big\{\big((g, X), \lambda(X, z)\big); X \in \mathfrak{g}\big\}, \quad g \in G, z \in F,$$

whose integral manifolds determine action l. If we consider some local coordinates z^a on F and a basis of $\mathfrak{g}$, the coordinate expression of λ is of the form

$$dz^a = \eta^a_I(z)\xi^I, \quad (\xi^I) \in \mathfrak{g},\ I = 1, \ldots, \dim G.$$

Then the equations of Λ are

$$dz^a - \eta^a_I(z)\varphi^I = 0, \qquad \varphi = (\varphi^I).$$

They are usually called the equations of the infinitesimal action λ of G on F.

The elements of G_N are said to be zero order frames of N. They are characterized by the property that the image $\tilde{g}(c)$ of $c \in S$ under the frame map $\tilde{g}$ lies in N. So the frame form of k^r_N is a map $G_N \to S^r_n$. Consider the canonical coordinates

$$x^i, x^p, \quad i = 1, \ldots, n,\ p = n+1, \ldots, m$$

on $\mathbb{R}^{n,m-n}$. The induced coordinates on the r-th jet prolongation $J^r\mathbb{R}^{n,m-n}$ are

$$x^p_i, x^p_{ij}, \ldots, x^p_{i_1\ldots i_r}.$$

Choose a local coordinate system x^i, x^p on S centered at c. This identifies locally S^r_n with $J^r_0\mathbb{R}^{n,m-n}$. We write

$$(a^p_i, a^p_{ij}, \ldots, a^p_{i_1\ldots i_r})\colon G_N \to S^r_n \tag{8}$$

for the locally defined coordinate functions of section k^r_N of $G_N[S^r_n]$.

The algorithm for finding (8) by the Cartan-like procedure from [17] is described in [14]. This general approach is based on the use of zero order frames of N. However, the evaluations in zero order frames are top-heavy because of the nontrivial topological character of the classical Grassmann manifolds. Thus, in practice one always uses the first order frames of N. So, also here we restrict ourselves to the first order frames.

Assume that H acts transitively on S^1_n, which is satisfied for all classical Klein spaces. Choose a point $c_n \in S^1_n$, write H_1 for its stability group and S^r_{n1} for the fiber of $S^r_n \to S^1_n$ over c_n. Clearly, S^r_{n1} is an H_1-space. A frame $\tilde{g} \in G_N$ is said to be first order frame of N, if $\tilde{g}(c_n) = T_{gc}N$. Clearly, the space G_{N1} of all first order frames of N is a principal bundle $G_{N1}(N, H_1)$. If we restrict ourselves to the first order frames, the frame form of k^r_N is a map $G_{N1} \to S^r_{n1}$. Assume further that the equations of c_n are $dx^p = 0$. In other words, the jet coordinates of c_n are $x^p_i = 0$. So the first order frames of N are characterized by $a^p_i = 0$. The induced global coordinates on S^r_{n1} are $x^p_{ij}, \ldots, x^p_{i_1 \ldots i_r}$. If we interpret k^r_N as a section of the associated bundle $G_{N1}[S^r_{N1}]$, then its coordinate functions

$$(a^p_{ij}, \ldots, a^p_{i_1 \ldots i_r}) \colon G_{N1} \to S^r_{n1}$$

are globally defined.

The simpliest algorithm appears in the case there exists an Abelian subgroup $K \subset G$ such that $\mathfrak{g}$ is the product $\mathfrak{k} \times \mathfrak{h}$. (But all classical Klein spaces have this property. For example, if A_m is an m-dimensional affine space, we have $G = GA(m)$, $H = GL(m)$ and $K = \mathbb{R}^m \subset GA(m)$ is the Abelian subgroup of all translations on A_m.) We choose a basis of $\mathfrak{g}$

$$e_\alpha, e_\lambda, \qquad \alpha, \beta = 1, \ldots, m, \ \lambda, \mu, \nu = m+1, \ldots, \dim G$$

such that e_λ lie in $\mathfrak{h}$ and e_α is a basis of $\mathfrak{k}$.

This assumption is equivalent to the following relations on the structure constants of G

$$c^\alpha_{\beta\gamma} = 0, \quad c^\lambda_{\alpha\beta} = 0, \quad c^\alpha_{\lambda\mu} = 0. \tag{9}$$

Hence the coordinate form of the structure equations $d\varphi + \frac{1}{2}[\varphi, \varphi] = 0$ of φ is

$$\begin{aligned} d\varphi^\alpha &= c^\alpha_{\lambda\beta} \varphi^\beta \wedge \varphi^\lambda, \\ d\varphi^\lambda &= c^\lambda_{\alpha\mu} \varphi^\mu \wedge \varphi^\alpha - \tfrac{1}{2} c^\lambda_{\mu\nu} \varphi^\mu \wedge \varphi^\nu. \end{aligned} \tag{10}$$

We shall write π^λ for the restriction of φ^λ to H. The bundle projection $G \to S$ identifies locally K with S. So the basis e_α defines local coordinates x^α on S, with $(x^\alpha) = (x^i, x^p)$.

In what follows we shall write φ for the restriction φ_{N1} of φ to G_{N1}, as usual in concrete investigations. So our starting point are the equations

$$\varphi^p = 0.$$

If we substitute them into (10), we obtain

$$0 = c^p_{i\lambda} \varphi^\lambda \wedge \varphi^i.$$

Using the Cartan lemma, we find

$$c^p_{i\lambda}\varphi^\lambda = a^p_{ij}\varphi^j, \qquad a^p_{ij} = a^p_{ji}, \tag{11}$$

where a^p_{ij} are some functions on G_{N1}.

In [14], we deduced

Proposition 3. *a^p_{ij} coincide with the coordinate functions of k^2_N on G_{N1}.*

In particular, (11) implies that the differential equations of H_1 are

$$c^p_{i\lambda}\pi^\lambda = 0.$$

Now we apply exterior differentiation to (11). Using the structure equations, we obtain an expression of the form

$$\left[da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl})\varphi^\lambda\right] \wedge \varphi^j = 0. \tag{12}$$

If we apply Cartan lemma to (12), we obtain

$$da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl})\varphi^\lambda = a^p_{ijk}\varphi^k.$$

(We shall see that a^p_{ijk} are the additional coordinate functions of k^3_N on G_{N1}.)

This procedure can be iterated. Assume by induction that after r-3 steps we have the equations of the infinitesimal action of H_1 on S^{r-2}_{n1}

$$\begin{gathered} dx^p_{ij} - \Phi^p_{ij\lambda}(x^q_{kl})\pi^\lambda = 0, \\ \vdots \\ dx^p_{i_1\dots i_{r-2}} - \Phi^p_{i_1\dots i_{r-2}\lambda}(x^q_{kl},\dots,x^q_{j_1\dots j_{r-2}})\pi^\lambda = 0 \end{gathered}$$

with $c^p_{i\lambda}\pi^\lambda = 0$, and it holds

$$\begin{gathered} c^p_{i\lambda}\varphi^\lambda = a^p_{ij}\varphi^j, \\ \vdots \\ da^p_{i_1\dots i_{r-2}} - \Phi^p_{i_1\dots i_{r-2}\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-2}})\varphi^\lambda = a^p_{i_1\dots i_{r-2}j}\varphi^j. \end{gathered}$$

If we apply exterior differentiation to the last row and use all these equations, we obtain certain relations of the form

$$\left[da^p_{i_1\dots i_{r-2}k} - \Phi^p_{i_1\dots i_{r-2}k\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-1}})\varphi^\lambda\right] \wedge \varphi^k = 0.$$

In [14], we deduced

Proposition 4. *The additional equations of the infinitesimal action of H_1 on S^{r-1}_{n1} are*

$$dx^p_{i_1\dots i_{r-1}} - \Phi^p_{i_1\dots i_{r-1}\lambda}(x^q_{j_1j_2},\dots,x^q_{j_1\dots j_{r-1}})\pi^\lambda = 0 \quad \textit{with } c^p_{i\lambda}\pi^\lambda = 0.$$

The additional coordinate functions $a^p_{i_1\dots i_r}$ of k^r_N on G_{N1} satisfy

$$da^p_{i_1\dots i_{r-1}} - \Phi^p_{i_1\dots i_{r-1}\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-1}})\varphi^\lambda = a^p_{i_1\dots i_r}\varphi^{i_r}.$$

The Cartan method of moving frames is usually used for finding differential invariants of $N \subset S$ and for solving the equivalence problem for N. The fact that the above procedure yields the equations of the infinitesimal action of H_1 on S^r_{n1} was used in [17] for local computations of the geometric objects of N. Our analysis of these algorithms led us to the following conceptual definition, [14]. Let A be an H-space.

Definition 7. A geometric (n,r)-object on S is an H-equivariant map $\mu\colon S^r_n \to A$.

Since μ is an H-map, it induces the associated bundle morphism $\bar\mu\colon K^r_nS \to G[A]$. The map

$$\mu_N = \bar\mu \circ k^r_N\colon N \to G_N[A]$$

is called the value of geometric (n,r)-object μ on N. More generally, let $W \subset S^r_n$ be an H-invariant submanifold. An n-submanifold $N \subset S$ is said to be of type W, if the values of k^r_N lie in $G_N[W]$. We can introduce a geometric object of type W as an H-map $\mu\colon W \to A$. For a submanifold N of type W, $\bar\mu \circ k^r_N$ is the value of μ on N. Very simple examples of W are elliptic, parabolic and hyperbolic contact $(2,2)$-elements on Euclidean 3-space.

The equations of the infinitesimal action can be used, at least locally, for constructing the equivariant maps. A general global result is due to R. Palais, [21]. We refer the reader to [14] for more details concerning the case of contact elements. We underline that the globality of the infinitesimally equivariant maps frequently follows from the geometrical interpretation of the results of evaluations.

In practice, one constructs the geometric objects of N by using the first order frames. If we interpret H as a principal H_1-bundle $H(H/H_1,H_1)$, then S^r_n coincides with the associated bundle $H[S^r_{n1}]$. The left action of H on S^r_n has the form

$$\bar h\{h,y\} = \{\bar h h, y\}, \quad h,\bar h \in H, y \in S^r_{n1}.$$

Let B be an H_1-space. The associated bundle $H[B]$ is an H-space with respect to the action

$$\bar{h}\{h, z\} = \{\bar{h}h, z\}, \quad h, \bar{h} \in H, z \in B.$$

This definition is correct, for $\bar{h}\{hh_1, h_1^{-1}z\} = \{\bar{h}hh_1, h_1^{-1}z\} = \bar{h}\{h, z\}$, $h_1 \in H_1$. For every H_1-map $\nu\colon S^r_{n1} \to B$, the induced map $\bar{\nu}\colon H[S^r_{n1}] \to H[B]$ is H-equivariant. So every H_1-map $\nu\colon S^r_{n1} \to B$ gives rise to a geometric (n, r)-object on S.

We underline that the concept of r-th order geometric object for n-submanifolds of S is of universal character. Its specification to an n-submanifold $N \subset S$ (or to a submanifold of type W) is constructed by means of the contact elements, so that it is independent of parametrizations of N.

The differential invariants of submanifolds are the simpliest example of geometric objects. In this case, $A = \mathbb{R}$ with the identity action of H. Further, if we consider the action of H on $\mathbb{R}$ by means of homotheties, we obtain the so-called relative invariants.

Remark 3. According to the Cartan-like algorithm of this section (see [14] for the use of zero order frames), the geometric objects of a submanifold $N \subset S$ are determined by the restriction φ_N of the Maurer-Cartan form of G over N. This corresponds to the well known role of φ_N in the equivalence problem for N, see [3], [7]. We recall that this role is based on the fact that, for a connected manifold N, two maps f_1, $f_2\colon N \to G$ are congruent, i.e. there exists $g \in G$ such that $f_1(x) = gf_2(x)$ for all $x \in N$, if and only if $\varphi \circ Tf_1 = \varphi \circ Tf_2\colon TN \to \mathfrak{g}$.

5 Submanifolds of Cartan spaces

Consider a Cartan space $\mathcal{S}(M)$ such that H acts transitively on S^1_n. Let $N \subset M$ be an n-submanifold. The elements of Q_N are zero order frames of N, they are characterized by $\tilde{u}(c) \in s_N(N)$. A frame $u \in (Q_N)_x$ is said to be first order frame of N, if $\tilde{u}(c_n) = \Gamma^1_N(x)$. Analogously to Section 4, these frames form a reduction Q_{N1} of Q_N to H_1. In this situation, a frame $u \in Q_N$ is a first order frame of N, iff $\omega^p_N(u) = 0$. We write $\bar{S}^r_n = (\bar{K}^r_n S)_c$. Then $\bar{K}^r_n S$ is an associated bundle $G[\bar{S}^r_n]$. Further, we write $\bar{S}^r_{n1}$ for the fiber $\bar{S}^r_n \to S^1_n$ over c_n.

The r-th absolute contact differential Γ^r_N can be viewed as a section of the associated bundle $Q_{N1}[\bar{S}^r_{n1}]$. By (4), the coordinates x^i, x^p identify $\bar{S}^r_n$ locally with $\bar{J}^r_0\mathbb{R}^{n,m-n}$. Hence the induced coordinates on $\bar{S}^r_{n1}$ are

$$x^p_{ij}, \dots, x^p_{i_1\dots i_r}$$

arbitrary in all subscripts. The coordinate functions of Γ^r_N

$$(b^p_{ij}, \dots, b^p_{i_1\dots i_r})\colon Q_{N1} \to \bar{S}^r_{n1}$$

are globally defined.

Assume the existence of $K \subset G$ as in Section 4. Then we have the following simple procedure for finding the coordinate functions of Γ^r_N, in which the role of the Maurer-Cartan form φ from Section 4 is replaced by the connection form ω. We write ω for the restriction ω_{N1} of ω to Q_{N1}. So our starting point are the equations

$$\omega^p = 0.$$

In [13], we deduced

Proposition 5. *We have*

$$c^p_{i\lambda}\omega^\lambda = b^p_{ij}\omega^j. \tag{13}$$

The algorithm from Section 4 is now modified as follows, [13]. Write formally the relations, with arbitrary x^p_{ij},

$$c^p_{i\lambda}\varphi^\lambda = x^p_{ij}\varphi^j. \tag{14}$$

Applying exterior differentiation to (14), using the structure equations of φ and $\varphi^p = 0$, we find an expression of the form

$$\left[dx^p_{ij} - \Psi^p_{ij\lambda}(x^q_{kl})\varphi^\lambda\right] \wedge \varphi^j = 0.$$

Proposition 6. *The equations of the infinitesimal action of H_1 on $\bar{S}^2_{n1}$ are*

$$dx^p_{ij} - \Psi^p_{ij\lambda}(x^q_{kl})\pi^\lambda = 0 \quad \textit{with } c^p_{i\lambda}\pi^\lambda = 0.$$

This procedure can be iterated. Assume that after r-3 steps we have deduced the equations of the infinitesimal action of H_1 on $\bar{S}^{r-2}_{n1}$

$$\begin{aligned} & dx^p_{ij} - \Psi^p_{ij\lambda}(x^q_{kl})\pi^\lambda = 0, \\ (15) \qquad & \vdots \\ & dx^p_{i_1\dots i_{r-2}} - \Psi^p_{i_1\dots i_{r-2}\lambda}(x^q_{kl},\dots,x^q_{j_1\dots j_{r-2}})\pi^\lambda = 0 \end{aligned}$$

with $c^p_{i\lambda}\pi^\lambda = 0$. Then we write formally the relations, with arbitrary $x^p_{i_1\dots i_{r-2}j}$,

$$dx^p_{i_1\dots i_{r-2}} - \Psi^p_{i_1\dots i_{r-2}\lambda}(x^q_{j_1 j_2},\dots,x^q_{j_1\dots j_{r-2}})\varphi^\lambda = x^p_{i_1\dots i_{r-2}j}\varphi^j. \tag{16}$$

Applying exterior differentiation to (16) with $\varphi^p = 0$, using the structure equations of φ and (15), we obtain an expression of the form

$$\left[dx^p_{i_1\dots i_{r-2}j} - \Psi^p_{i_1\dots i_{r-2}j\lambda}(x^q_{j_1 j_2},\dots,x^q_{j_1\dots j_{r-1}})\varphi^\lambda\right] \wedge \varphi^j = 0.$$

Proposition 7. *The additional equations of the infinitesimal action of H_1 on $\bar{S}^r_{n1}$ are*

$$dx^p_{i_1\dots i_{r-1}} - \Psi^p_{i_1\dots i_{r-1}\lambda}(x^q_{j_1 j_2},\dots,x^q_{j_1\dots j_{r-1}})\pi^\lambda = 0 \quad \textit{with} \quad c^p_{i\lambda}\pi^\lambda = 0.$$

The coordinate functions $b^p_{ij},\dots,b^p_{i_1\dots i_2}$ of Γ^r_N on Q_{N1} satisfy (13) *and*

$$\begin{aligned} db^p_{ij} - \Psi^p_{ij\lambda}(b^q_{kl})\omega^\lambda &= b^p_{ijk}\omega^k, \\ &\vdots \\ db^p_{i_1\dots i_{r-1}} - \Psi^p_{i_1\dots i_{r-1}\lambda}(b^q_{j_1 j_2},\dots,b^q_{j_1\dots j_{r-1}})\omega^\lambda &= b^p_{i_1\dots i_{r-1}j}\omega^j. \end{aligned}$$

Remark 4. We underline that the absolute contact differential of any order of N is determined by the restriction ω_N of the connection form ω over N. This is an important analogy of Remark 3. Clearly, Section 4 can be viewed as a special case, provided we consider S as a flat Cartan space.

Now we generalize the concept of geometric (n,r)-object to Cartan spaces. Let A be an H-space.

Definition 8. A geometric (n,r)-object on $\mathcal{S}(M)$ is an H-equivariant map $\mu\colon \bar{S}^r_n \to A$.

We also say that μ is a semiholonomic (n, r)-object. For $n = 1$ we have $\bar{S}^r_1 = S^r_1$, so that there exist holonomic $(1, r)$-objects only.

So we have the induced bundle morphism $\bar{\mu}\colon Q[\bar{S}^r_n] \to Q[A]$. For a submanifold $N \subset M$, the composition

$$\mu_N = \bar{\mu} \circ \Gamma^r_N \colon N \to Q_N[A]$$

is called the value of μ on N. More generally, if $W \subset \bar{S}^r_n$ is an H-invariant submanifold, then the (n, r)-objects of type W are defined analogously to Section 4. Clearly, one can restrict himself to the first order frames of N in the same way as above.

6 The torsion-free case

For a Cartan space $\mathcal{S}(M)$, Sharpe defines its curvature Ω by

$$d\omega + \tfrac{1}{2}[\omega, \omega] = \Omega, \tag{17}$$

[22]. So Ω is the restriction of the curvature Ω_Γ of Γ to Q. It is well known that Ω_Γ can be interpreted as a map

$$\Omega_\Gamma \colon P \times_M \bigwedge^2 TM \to \mathfrak{g}.$$

Hence we may consider Ω as a map

$$\Omega \colon Q \times_M \bigwedge^2 TM \to \mathfrak{g}. \tag{18}$$

The coordinate form of (17) is

$$d\omega^I + \frac{1}{2} c^I_{JK} \omega^J \wedge \omega^K = R^I_{\alpha\beta} \omega^\alpha \wedge \omega^\beta, \quad I, J, K = 1, \dots, \dim G. \tag{19}$$

In [9] we introduced the following concept in a slightly different, but equivalent way. Write $L = \mathfrak{g}/\mathfrak{h} = T_c S$ and $\psi\colon \mathfrak{g} \to L$ for the factor projection.

Definition 9. The composition $\sigma = \psi \circ \Omega \colon Q \times_M \bigwedge^2 TM \to L$ is called the torsion of $\mathcal{S}$.

The absolute differentiation with respect to Γ identifies T_xM with $T_{s(x)}E_x$. Clearly, L is an H-space and the corresponding associated bundle satisfies

$$Q[L] \approx \bigcup_{x\in M} T_{s(x)}E_x. \tag{20}$$

Hence σ can be interpreted as a section

$$\sigma\colon M \to Q[L \otimes \bigwedge^2 L^*]. \tag{21}$$

By (19), the coordinate expression of σ is

$$R^{\gamma}_{\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta}. \tag{22}$$

This implies that σ coincides with the standard torsion in the classical case of an affine connection on the linear frame bundle of M.

Remark 5. The concept of higher order torsions of Cartan spaces is discussed from a similar point of view in [10].

Our result from [9] can be now formulated as follows. (Another approach to this assertion will be discussed in Section 8.)

Proposition 8. *If the torsion of $\mathcal{S}$ vanishes, then the values of Γ^2_n are holonomic contact $(n,2)$-elements.*

In particular, this is true in the case of a Riemannian manifold (M,g), that is considered as a Cartan space $\mathcal{E}_m(M)$ with respect to the Levi-Civita connection. Thus, from the viewpoint of our approach, the second-order geometric objects on submanifolds of Riemannian spaces are of the same type as in the case of submanifolds of Euclidean spaces.

7 Universal tensor bundles for submanifolds

We present another situation, in which the idea of universal geometric object for submanifolds plays a remarkable role. We start with the case of an arbitrary manifold M. The vertical bundle VN of $N \subset M$ is the factor bundle $(TM)_N/TN$. A section

$$N \to \bigotimes^a TN \otimes \bigotimes^b VN \otimes \bigotimes^c T^*N \otimes \bigotimes^d V^*N$$

will be called a tangent-vertical tensor field on N.

For every $\xi \in (K^1_n M)_x$, we denote by $\tau(\xi) \subset T_x M$ the corresponding n-dimensional subspace and by $\nu(\xi) = T_x M/\tau(\xi)$ the vertical space. Then

$$H^1_n M = \bigcup_{\xi \in K^1_n M} \tau(\xi) \quad \text{and} \quad V^1_n M = \bigcup_{\xi \in K^1_n M} \nu(\xi)$$

are vector bundles over $K^1_n M$ and we have an exact sequence (see also [19])

$$0 \to H^1_n M \hookrightarrow (\pi^1_0)^* TM \to V^1_n M \to 0. \tag{23}$$

Definition 10. The induced bundle over $K^r_n M$

$$H^r_n M = (\pi^r_1)^* H^1_n M \quad \text{or} \quad V^r_n M = (\pi^r_1)^* V^1_n M$$

is called the universal horizontal or vertical (n, r)-bundle over M, respectively.

We define

$$I^{r;a,b}_{n;c,d} M = \bigotimes^a H^r_n M \otimes \bigotimes^b V^r_n M \otimes \bigotimes^c H^{r*}_n M \otimes \bigotimes^d V^{r*}_n M. \tag{24}$$

Every section $\varrho \colon K^r_n M \to I^{r;a,b}_{n;c,d} M$ determines a tangent-vertical tensor field $\varrho \circ k^r_N$ on every n-submanifold N.

For example, consider the m-dimensional affine space A_m and $N \subset A_m$. For a vector $X \in T_x N$, we define $\varphi_x(X) \in V_x N$ as follows. Take a curve $\gamma(t)$ on N such that $\frac{d\gamma(0)}{dt} = X$. In the case of A_m, the acceleration $\frac{d^2\gamma(0)}{dt^2}$ belongs to $T_x A_m$. Its projection into $V_x N$ depends on $\frac{d\gamma(0)}{dt}$ only. This defines a map $\varphi_x \colon T_x N \to V_x N$, that generates a quadratic VN-valued form φ_N on N. Its universal version is a section

$$\varphi \colon K^2_n A_m \to V^2_n A_m \otimes S^2(H^{2*}_n A_m). \tag{25}$$

Definition 11. We say that φ is the universal fundamental form for n-submanifolds of A_m.

For every submanifold $N \subset A_m$, $\varphi \circ k^2_N$ is the fundamental form of N.

We remark that an application of the concept of universal tensor bundles to the calculus of variations on submanifolds is presented in [18]. Another interesting application of this concept can be found in [6].

In the semiholonomic case, we construct the pullbacks

$$\bar{H}^r_n M \to \bar{K}^r_n M \quad \text{and} \quad \bar{V}^r_n M \to \bar{K}^r_n M$$

in the same way as in Definition 10.

8 The reduced torsion and the difference tensor

Consider a submanifold N of a Cartan space $\mathcal{S}(M)$. In the tangent space $T_{s(x)}E_x$, we have an n-dimensional subspace $\tau_N^\Gamma(x)$ corresponding to $\Gamma_N^1(x)$. The factor space

$$\nu_N^\Gamma(x) = T_{s(x)}E_x/\tau_N^\Gamma(x) \tag{26}$$

will be called the vertical space of N at x. Write σ_N for the restriction of σ to Q_N.

Definition 12. The projection $\tilde{\sigma}_N(x)$ of $\sigma_N(x)$ into $\nu_N^\Gamma(x)$ is called the reduced torsion of N at x.

The universal version of the reduced torsion is closely related with the general concept of difference tensor of semiholonomic contact $(n, 2)$-elements. According to [9], every semiholonomic 2-jet $X \in \bar{J}_x^2(M, N)_y$ determines the difference tensor $\Delta X \in T_yN \otimes \bigwedge^2 T_x^*M$. If $(x^i, y^\alpha, y_i^\alpha, y_{ij}^\alpha)$ are the canonical coordinates on $\bar{J}^2(\mathbb{R}^n, \mathbb{R}^m)$, then the coordinate expression of ΔX is

$$\big(x^i, y^\alpha, y^\alpha_{[ij]}\big). \tag{27}$$

So $\Delta X = 0$, iff X is a holonomic 2-jet.

Consider $X \in \operatorname{reg} \bar{T}_n^2 M$ and $\xi = k(X) \in \bar{K}_n^2 M$. The underlying 1-jet $\pi_1^2 X \in \operatorname{reg} T_n^1 X$ identifies $\mathbb{R}^n$ with $\tau(\xi_1)$, $\xi_1 = \pi_1^2 \xi$. The projection of ΔX into $\nu(\xi_1)$ depends on ξ only. This defines

$$\delta(\xi) \in \nu(\xi_1) \otimes \bigwedge^2 \tau(\xi_1)^*, \tag{28}$$

that will be called the difference tensor of ξ. Hence δ is a section

$$\delta\colon \bar{K}_n^2 M \to \bar{V}_n^2 M \otimes \bigwedge^2 \bar{H}_n^{2*} M$$

that is said to be the contact difference tensor. Under the identification (4) of $\tau \bar{K}_n^2 \mathbb{R}^m$ with $\bar{J}^2 \mathbb{R}^{n,m-n}$, both approaches to the difference tensor coincide. Clearly, $\xi \in K_n^2 M$ iff $\delta(\xi) = 0$.

If we analyze $\tilde{\sigma}$ from the viewpoint of semiholonomic $(n,2)$-objects, we realize that it is determined by an H-map

$$\bar{S}^2_n \to (V^1_n S)_c \otimes \bigwedge^2 (H^{1*}_n S)_c, \tag{29}$$

that coincides with the contact difference tensor. This yields

Proposition 9. *We have $\tilde{\sigma}_N(x) = 0$ iff $\Gamma^2_N(x)$ is holonomic.*

Hence Proposition 8 is a direct consequence of Proposition 9.

To illustrate the use of the algorithm from Section 5, we rededuce this assertion by direct evaluation under the additional assumptions on S^1_n and K. In the first order frames of N, (19) and (22) imply

$$0 = c^p_{i\lambda}\omega^i \wedge \omega^\lambda + R^p_{ij}\omega^i \wedge \omega^j, \tag{30}$$

where $R^p_{ij}\omega^i \wedge \omega^j$ is the coordinate expression of $\tilde{\sigma}_N$. The coordinate functions of Γ^2_N satisfy $c^p_{i\lambda}\omega^\lambda = b^p_{ij}\omega^j$. Hence $\tilde{\sigma}_N = 0$ is equivalent to $b^p_{ij}\omega^i \wedge \omega^j = 0$, i.e. $b^p_{ij} = b^p_{ji}$.

As a concrete example, we consider a 2-submanifold $N_2 \subset \mathcal{P}_3$ of a 3-space with projective connection. The projective 3-space P_3 is generated by an affine 4-space A_4 and we write $\{u\} \in P_3$ for the point determined by a nonzero vector $u \in A_4$. We fix a basis u_0, u_1, u_2, u_3 of A_4 and define $c = \{u_0\}$ and c_2 as the linear space in $T_c P_3$ corresponding the 2-plane determined by $\{u_0\}$, $\{u_1\}$, $\{u_2\}$. The Maurer–Cartan form of the projective group $GP(3)$ is (φ^b_a) with $\varphi^a_a = 0$, a, $b = 0,1,2,3$, and we have

$$d\varphi^b_a = \varphi^c_a \wedge \varphi^b_c \qquad \text{with} \quad \varphi^a_a = 0. \tag{31}$$

The differential equations of H are $\varphi^1_0 = \varphi^2_0 = \varphi^3_0 = 0$. One verifies directly that condition (9) is satisfied. Then the relation

$$d\varphi^3_0 = \varphi^0_0 \wedge \varphi^3_0 + \varphi^1_0 \wedge \varphi^3_1 + \varphi^2_0 \wedge \varphi^3_2 + \varphi^3_0 \wedge \varphi^3_3 \tag{32}$$

implies that the additional differential equations of H_1 are $\pi^3_1 = 0$, $\pi^3_2 = 0$.

The restriction (ω^a_b), $\omega^a_a = 0$ of the connection form ω of $\mathcal{P}_3$ to the first order frames of N_2 is characterized by $\omega^3_0 = 0$. If we write $\omega^1_0 = \omega^1$, $\omega^2_0 = \omega^2$, then (13) yields

$$\omega^3_1 = b_{11}\omega^1 + b_{12}\omega^2, \qquad \omega^3_2 = b_{21}\omega^1 + b_{22}\omega^2. \tag{33}$$

Then (30) is of the form

$$0 = \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 + 2R^3_0 \omega^1 \wedge \omega^2. \tag{34}$$

Hence (33) implies $2R^3_0 = b_{21} - b_{12}$. So the equations (14) are of the form

$$\varphi^3_1 = x_{11}\varphi^1_0 + x_{12}\varphi^2_0, \qquad \varphi^3_2 = x_{21}\varphi^1_0 + x_{22}\varphi^2_0. \tag{35}$$

Applying the procedure from Section 5, we obtain the equations of the infinitesimal action of H_1 on $\bar{S}^2_{21}$

$$\begin{aligned} &dx_{11} + x_{11}(\pi^0_0 - 2\pi^1_1 + \pi^3_3) - x_{12}\pi^2_1 - x_{21}\pi^2_1 = 0,\\ &dx_{12} + x_{12}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^1_2 - x_{22}\pi^2_1 = 0,\\ &dx_{21} + x_{21}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^1_2 - x_{22}\pi^2_1 = 0,\\ &dx_{22} + x_{22}(\pi^0_0 - 2\pi^2_2 + \pi^3_3) - x_{12}\pi^1_2 - x_{21}\pi^1_2 = 0. \end{aligned} \tag{36}$$

In particular,

$$d(x_{21} - x_{12}) + (x_{21} - x_{12})(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) = 0, \tag{37}$$

so that R^3_0 is a relative invariant. Further, in the non-parabolic case $x_{12}x_{21} - x_{11}x_{22} \neq 0$ we find

$$d((x_{21} - x_{12})^2/(x_{12}x_{21} - x_{11}x_{22})) = 0.$$

Hence $(R^3_0)^2/(b_{12}b_{21} - b_{11}b_{22})$ is an absolute invariant. Its geometric interpretation was found already by É. Cartan [4].

Remark 6. There exists a natural symmetrization $\mathrm{Sym}\colon \bar{J}^2(M,N) \to J^2(M,N)$ of semiholonomic 2-jets, [11]. In coordinates, one verifies directly that Sym preserves the jet composition

$$\mathrm{Sym}(Y \circ X) = \mathrm{Sym}(Y) \circ \mathrm{Sym}(X), \qquad X \in \bar{J}^2_x(M_1, M_2)_y,\ Y \in \bar{J}^2_y(M_2, M_3)_z.$$

Hence (3) implies that there is an induced symmetrization of contact $(n,2)$-elements

$$\mathrm{Sym}\colon \bar{K}^2_n M \to K^2_n M.$$

We remark that in some geometric constructions $\Gamma^2_N(x)$ enters via its symmetrization $\mathrm{Sym}(\Gamma^2_N(x))$. But this is not the case of the preceding example.

Remark 7. Some general aspects of the holonomicity problem for Γ^r_N in the case $r > 2$ are studied in [12]. The case of $N_2 \subset \mathcal{P}_3$ is treated geometrically in [8].

9 Induced bundles over submanifolds

We point out that the idea of universality can be applied to a wide class of r-th order geometric objects over submanifolds. We write $\operatorname{reg} T^r_n M = P^r_n M$ if we consider it as a principal bundle over $K^r_n M$ with structure group G^r_n. For every n-submanifold $N \subset M$, the map k^r_N induces a bundle $(k^r_N)^* P^r_n M \to N$ that coincides with the r-th order frame bundle $P^r N$. Let B be a G^r_n-space.

Definition 13. The associated bundle $P^r_n M[B] \to K^r_n M$ is called the universal B-bundle of type (n, r) over M.

The map k^r_N induces the associated bundle $(k^r_N)^*(P^r_n M[B]) \approx P^r N[B]$. For $q > r$, we can construct the pullback $(\pi^q_r)^* P^r_n M =: P^{r,q}_n M \to K^q_n M$. Then $k^q_N \colon N \to K^q_n M$ defines $(k^q_N)^*\bigl(P^{r,q}_n M[B]\bigr) \approx P^r N[B]$.

In the case of $S = G/H$, $P^r_n S \to S$ can be interpreted as an associated bundle

$$P^r_n S = G\bigl[(P^r_n S)_c\bigr], \quad u \mapsto \{g, g^{-1}u\},\ g \in G,\ u \in (P^r_n S)_{gc} \tag{38}$$

as well. In general, consider a principal bundle $P(Z, K)$ and a left action of G on P commuting with the right action of K on P, i.e.

$$g(uk) = (gu)k, \quad g \in G,\ u \in P,\ k \in K. \tag{39}$$

If B is a left K-space, we have an induced left action of G on $P[B]$,

$$g\{u, b\} = \{gu, b\}. \tag{40}$$

This is a correct definition, for

$$g\{uk, k^{-1}b\} = \{g(uk), k^{-1}b\} = \{(gu)k, k^{-1}b\} = \{gu, b\}.$$

For example, if $r = 1$ and $B = \mathbb{R}^n$ with the standard action of G^1_n, then $P^{1,q}_n M[\mathbb{R}^n] = H^q_n M$. On the other hand, the vertical bundle $V^q_n M$ is not of this type.

In the case of $P^r_n S(K^r_n S, G^r_n)$ and a G^r_n-space B, we obtain a left action of G on $P^r_n S[B]$. Denote by $\tilde{B} = \bigl(P^r_n S[B]\bigr)_c$ the fiber over $c \in S$. Hence our construction yields a left action of H on $\tilde{B}$. Conversely, given a left action of H on $\tilde{B}$, we have an identification $G[\tilde{B}] \approx P^r_n S[B]$,

$$\bigl\{g, \{u, b\}\bigr\} \mapsto \{gu, b\}.$$

This is a correct definition, for

$$\big\{gh, h^{-1}\{uk, k^{-1}b\}\big\} = \big\{gh, \{h^{-1}uk, k^{-1}b\}\big\} = \big\{gh, \{h^{-1}u, b\}\big\} = \{gu, b\}.$$

Consider another H-space A,

Definition 14. An H-map $\mu\colon \tilde{B} \to A$ is called geometric B-object of type (n, r) over S.

Since $G[\tilde{B}] = P^r_n S[B]$, the map $k^r_N\colon N \to K^r_n S$ induces

$$\begin{gathered} \mu_N\colon P^r N[B] = G_N[B] \to G_N[A], \\ \{u, b\} = \big\{g, \{g^{-1}u, b\}\big\} \mapsto \big\{g, \mu(\{g^{-1}u, b\})\big\}, \end{gathered}$$

that will be called the value of μ on N. This definition is correct. Indeed, if we replace g by gh, we have $\{u, b\} = \big\{gh, \{h^{-1}g^{-1}u, b\}\big\} \mapsto \big\{gh, \mu(\{h^{-1}g^{-1}u, b\})\big\}$ $= \big\{g, \mu(\{g^{-1}u, b\})\big\}$.

There also exists a pullback version of this concept, in which we replace $P^r_n S$ by $P^{r,q}_n S$. In the case $B = pt$ is a singleton, so that $r = 0$, we obtain the concept of (n, q)-object on S introduced in Section 4. Indeed, $P^q_n S[pt] = K^q_n S$.

In particular, a section $\varrho\colon K^r_n S \to I^{r;a,b}_{n;c,d} S$ can be interpreted as a linear morphism

$$\varrho\colon \bigotimes^a H^{r*}_n S \otimes \bigotimes^c H^r_n S \to \bigotimes^b V^r_n S \otimes \bigotimes^d V^{r*}_n S.$$

If we consider the induced action of G on $I^{r;a,b}_{n;c,d} S$ and set

$$B = (\bigotimes^a H^{r*}_n S \otimes \bigotimes^c H^r_n S)_c \quad \text{and} \quad A = (\bigotimes^b V^r_n S \otimes \bigotimes^d V^{r*}_n S)_c,$$

we obtain the concept of invariant section ϱ.

A simple example to Definition 14 is the classical connection on a submanifold N, i.e. a principal connection on $P^1 N$. This is a second order geometric object field on N, [16]. The problem of finding an invariant construction of induced classical connection on a submanifold is important for both affine and projective differential geometries.

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