

Projective-geometric aspects of homogeneous third-order Hamiltonian operators

E.V. Ferapontov¹, M.V. Pavlov^{2,3}, R.F. Vitolo⁴

¹Department of Mathematical Sciences,
Loughborough University,
Loughborough, Leicestershire, LE11 3TU, UK
e.v.ferapontov@lboro.ac.uk

²Department of Mathematical Physics,
Lebedev Physical Institute of Russian Academy of Sciences,
Leninskij Prospekt, 53, Moscow, Russia
m.v.pavlov@lboro.ac.uk

³Laboratory of Geometrical Methods in Mathematical Physics,
Lomonosov Moscow State University,
Leninskie Gory 1, 119991 Moscow, Russia

⁴Department of Mathematics and Physics “E. De Giorgi”,
University of Salento, Lecce, Italy
raffaele.vitolo@unisalento.it

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Abstract

We investigate homogeneous third-order Hamiltonian operators of differential-geometric type. Based on the correspondence with quadratic line complexes, a complete list of such operators with $n \leq 3$ components is obtained.

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Quadratic Complex, Monge Metric, Reciprocal Transformation, Darboux Theorem.

Dedicated to the memory of Professor Yavuz Nutku (1943-2010)

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1 Introduction

First-order homogeneous Hamiltonian operators were introduced in [7] in the study of one-dimensional systems of hydrodynamic type¹. It was demonstrated that these operators are parametrized by flat pseudo-Riemannian metrics. Higher-order operators were subsequently defined in [8]. The structure of homogeneous second-order Hamiltonian operators was investigated in [24, 6], see also [22, 15].

In this paper we address the problem of classification of homogeneous third-order Hamiltonian operators of differential-geometric type [26, 6, 25, 22, 4],

$$P = g^{ij} D^3 + b_k^{ij} u_x^k D^2 + (c_k^{ij} u_{xx}^k + c_{km}^{ij} u_x^k u_x^m) D + d_k^{ij} u_{xxx}^k + d_{km}^{ij} u_{xx}^k u_x^m + d_{kmn}^{ij} u_x^k u_x^m u_x^n. \quad (1)$$

Here u^i , $i = 1, \dots, n$, are the dependent (field) variables, and the coefficients $g^{ij}, \dots, d_{kmn}^{ij}$ depend on u^i only; D stands for the total derivative with respect to x . Homogeneity is understood as follows: the independent

¹they are also known as differential-geometric, or Dubrovin–Novikov brackets

variable x has order -1 , the dependent variables u^i have order 0 , so that the order of u_x^i and D is 1 , etc. The operator P is Hamiltonian if and only if it is formally skew-adjoint, $P^* = -P$, and its Schouten bracket vanishes, $[P, P] = 0$. Equivalently, the corresponding Poisson bracket,

$$\{F_1, F_2\} = \int \frac{\delta F_1}{\delta u^i} P^{ij} \frac{\delta F_2}{\delta u^j} dx,$$

must be skew-symmetric, and satisfy the Jacobi identity. We restrict our considerations to the non-degenerate case, $\det g^{ij} \neq 0$. Operators (1) are form-invariant under point transformations of the dependent variables, $u = u(\tilde{u})$. Under point transformations, the coefficients of (1) transform as differential-geometric objects. For instance, g^{ij} transforms as a $(2, 0)$ -tensor, so that its inverse g_{ij} defines a pseudo-Riemannian metric (that is not flat in general), the expressions $-\frac{1}{3}g_{js}b_k^{si}$, $-\frac{1}{3}g_{js}c_k^{si}$, $-g_{js}d_k^{si}$ transform as Christoffel symbols of affine connections, etc. It was conjectured in [23] that the last connection, $\Gamma_{jk}^i = -g_{js}d_k^{si}$, must be symmetric and flat; this was confirmed in [26], see also [6]. Therefore, there exists a coordinate system (flat coordinates) such that Γ_{jk}^i vanish. These coordinates are determined up to affine transformations. We will keep for them the same notation u^i , note that u^i are nothing but the densities of Casimirs of the corresponding Hamiltonian operator (1). In the flat coordinates the last three terms in (1) vanish, leading to the simplified expression [22],

$$P = D \left(g^{ij} D + c_k^{ij} u_x^k \right) D. \quad (2)$$

This operator is Hamiltonian if and only if the coefficients g^{ij} and c_k^{ij} satisfy the following relations:

$$g_{,k}^{ij} = c_k^{ij} + c_k^{ji}, \quad (3a)$$

$$c_s^{ij} g^{sk} = -c_s^{kj} g^{si}, \quad (3b)$$

$$c_s^{ij} g^{sk} + c_s^{jk} g^{si} + c_s^{ki} g^{sj} = 0, \quad (3c)$$

$$c_{s,m}^{ij} g^{sk} = c_s^{ik} c_m^{sj} - c_s^{ki} c_m^{sj} - c_s^{kj} g_{,m}^{si}. \quad (3d)$$

Here (3a) is equivalent to $P^* = -P$, while (3b)-(3d) are equivalent to $[P, P] = 0$. These conditions are invariant under affine transformations of the flat coordinates. It is useful to rewrite the above system in low indices.

Introducing $c_{ijk} = g_{iq}g_{jp}c_k^{pq}$ one obtains [25]:

$$g_{mn,k} = -c_{mnk} - c_{nmk}, \quad (4a)$$

$$c_{mnk} = -c_{mkn}, \quad (4b)$$

$$c_{mnk} + c_{nkm} + c_{kmn} = 0, \quad (4c)$$

$$c_{mnk,l} = -g^{pq}c_{pml}c_{qnk}. \quad (4d)$$

Our main observation is that the metric g satisfying equations (4) must be the Monge metric of a quadratic line complex. Since complexes of lines belong to projective geometry, equations (4) should be invariant under the full projective (rather than affine) group. We demonstrate that this is indeed the case. Based on the projective classification of quadratic line complexes in \mathbb{P}^3 into eleven Segre types [16], we give a complete list of three-component Hamiltonian operators.

The structure of the paper is as follows. After discussing known examples of third-order homogeneous Hamiltonian operators in Section 2, we summarize our main results in Section 3. In Section 4 we establish a link between homogeneous third-order Hamiltonian operators and Monge metrics/quadratic line complexes. This indicates that the theory is essentially projectively-invariant (Section 5), and leads to the classification results presented in Section 6.

All computations were performed with the software package CDIFF [30] of the REDUCE computer algebra system [27].

2 Examples

To the best of our knowledge, all interesting examples of integrable systems possessing Hamiltonian structures of the form (1) come from the theory of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of 2D topological field theory. These are integrable PDEs of Monge–Ampère type that acquire a Hamiltonian formulation upon transformation into hydrodynamic form [12].

Example 1. [22] *The hyperbolic Monge–Ampère equation, $u_{tt}u_{xx} - u_{xt}^2 = -1$, can be reduced to hydrodynamic form*

$$a_t = b_x, \quad b_t = \left(\frac{b^2 - 1}{a} \right)_x,$$

via the change of variables $a = u_{xx}$, $b = u_{xt}$. It possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = P \begin{pmatrix} \delta H / \delta a \\ \delta H / \delta b \end{pmatrix},$$

with the homogeneous third-order Hamiltonian operator

$$P = D \begin{pmatrix} 0 & D \frac{1}{a} \\ \frac{1}{a} D & \frac{b}{a^2} D + D \frac{b}{a^2} \end{pmatrix} D,$$

and the nonlocal Hamiltonian,

$$H = - \int \left(\frac{1}{2} a (D^{-1} b)^2 + D^{-2} a \right) dx.$$

Note that $\delta H / \delta a = -\frac{1}{2} (D^{-1} b)^2 - \frac{x^2}{2}$, $\delta H / \delta b = D^{-1} (a D^{-1} b)$.

Example 2. [11] The simplest nontrivial case of the WDVV equations is the third-order Monge–Ampère equation, $f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$ [9]. This PDE can be transformed into hydrodynamic form,

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x,$$

via the change of variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$. This system possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = P \begin{pmatrix} \delta H / \delta a \\ \delta H / \delta b \\ \delta H / \delta c \end{pmatrix},$$

with the homogeneous third-order Hamiltonian operator

$$P = D \begin{pmatrix} 0 & 0 & D \\ 0 & D & -Da \\ D & -aD & Db + bD + aDa \end{pmatrix} D,$$

and the nonlocal Hamiltonian,

$$H = - \int \left(\frac{1}{2} a (D^{-1} b)^2 + D^{-1} b D^{-1} c \right) dx.$$

Example 3. Further examples are provided in [17, 18]. One of them is the equation $f_{xxx} = f_{ttx}^2 - f_{ttt}f_{txx}$ which is obtained from the WDVV equation of Example 2 by simply interchanging t and x . Remarkably, the corresponding Hamiltonian formulation is rather different. The change of variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ brings the equation into hydrodynamic form,

$$a_t = b_x, \quad b_t = c_x, \quad c_t = \left(\frac{c^2 - a}{b} \right)_x.$$

This system possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = P \begin{pmatrix} \delta H / \delta a \\ \delta H / \delta b \\ \delta H / \delta c \end{pmatrix},$$

with the homogeneous third-order Hamiltonian operator

$$P = D \begin{pmatrix} -D & 0 & 0 \\ 0 & 0 & D \frac{1}{b} \\ 0 & \frac{1}{b} D & \frac{c}{b^2} D + D \frac{c}{b^2} \end{pmatrix} D,$$

and the nonlocal Hamiltonian,

$$H = \int (cD^{-1}bD^{-1}c + D^{-1}aD^{-1}b) dx.$$

Note that this operator is a direct sum of the one-component operator $-D^3$ (in the variable a), and the two-component operator from Example 1 (in variables b, c). On the contrary, the operator from Example 2 is not reducible.

Two more WDVV-type equations were considered in [17, 18], namely $f_{ttt} + f_{ttt}f_{xxx} - f_{ttx}f_{txx} + f_{ttt}f_{txx} - f_{xtt}^2 + f_{xxx}f_{xtt} - f_{xxt}^2 = 0$, and the corresponding equation obtained by exchanging t and x . Both equations admit homogeneous third-order Hamiltonian structures which are equivalent to the one from Example 3. Operators from Examples 1-3 will feature in the classification results below.

3 Summary of main results

Our first observation (Proposition 1 of Section 4) is that equations (4) can be rewritten in terms of the metric g alone, implying the linear subsystem

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0, \quad (5)$$

along with a more complicated set of nonlinear constraints,

$$g_{m[k,n]l} = -\frac{1}{3}g^{pq}g_{p[l,m]}g_{q[k,n]}, \quad (6)$$

where square brackets denote antisymmetrisation. Any solution to these equations specifies a third-order Hamiltonian operator of the form (2) by setting $c_{nkm} = \frac{1}{3}g_{n[m,k]}$.

Our second remark is that the generic metric $g = g_{ij}du^i du^j$ satisfying the linear subsystem (5) is an arbitrary quadratic expression in du^i and $u^j du^k - u^k du^j$, explicitly,

$$g_{ij}du^i du^j = a_{ij}du^i du^j + b_{ijk}du^i(u^j du^k - u^k du^j) + c_{ijkl}(u^i du^j - u^j du^i)(u^k du^l - u^l du^k), \quad (7)$$

where a_{ij} , b_{ijk} , c_{ijkl} are arbitrary constants.

Since the flat coordinates are defined up to affine transformations, the system (5)-(6) is invariant under point transformations of the form

$$\tilde{u}^i = l^i(\mathbf{u}), \quad \tilde{g} = g,$$

where l^i are arbitrary linear forms in the flat coordinates $\mathbf{u} = (u^1, \dots, u^n)$, and $\tilde{g} = g$ indicates that g transforms as a metric. What is less obvious is that the system (5)-(6) is invariant under the bigger group of projective transformations,

$$\tilde{u}^i = \frac{l^i(\mathbf{u})}{l(\mathbf{u})}, \quad \tilde{g} = \frac{g}{l^4(\mathbf{u})},$$

where l is yet another linear form in the flat coordinates. It will be demonstrated in Section 5 that projective transformations correspond to reciprocal transformations of the Hamiltonian operator (2). Note that the ansatz (7) is invariant under projective transformations indicated above. One can thus formulate two natural classification problems: affine and projective classifications.

Metrics of the form (7) typically arise as Monge metrics of quadratic line complexes. Recall that a quadratic line complex is a $(2n-3)$ -parameter family of lines in the projective space \mathbb{P}^n specified by a single quadratic

equation in the Plücker coordinates. Fixing a point $p \in \mathbb{P}^n$ and taking all lines of the complex that pass through p we obtain a quadratic cone with vertex at p . This field of cones supplies \mathbb{P}^n with a conformal structure (Monge metric) whose general form is given by (7), see Section 4 for more details. The key invariant of a quadratic line complex is its singular variety (which is a hypersurface in \mathbb{P}^n of degree $2n-2$, see [5], Prop. 10.3.2), defined by the equation

$$\det g_{ij} = 0.$$

For $n = 2$ the singular variety is a conic in \mathbb{P}^2 , for $n = 3$ it is the Kummer quartic in \mathbb{P}^3 , etc.

Taking a generic Monge metric (7), bringing it to a suitable normal form via affine/projective transformations, and verifying the remaining nonlinear constraints (6) one can obtain a classification of third-order Hamiltonian operators. Due to the complexity of nonlinear constraints, we only managed to complete this programme in two- and three-component cases (Section 6), note that any one-component operator is equivalent to D^3 . We observe that the singular varieties of Monge metrics corresponding to homogeneous third-order Hamiltonian operators degenerate into double hypersurfaces of degree $n-1$. Our classification results are summarised below (in the two-component case we give both affine and projective classifications, in the three-component situation the affine classification contains too many special cases and moduli, and is omitted):

Two-component case (Theorem 1 of Section 6). *Modulo (complex) affine transformations, the metric of any two-component homogeneous third-order Hamiltonian operator can be reduced to one of the three canonical forms:*

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 \\ -u^1 u^2 & (u^1)^2 \end{pmatrix}, \quad g^{(2)} = \begin{pmatrix} -2u^2 & u^1 \\ u^1 & 0 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The metric $g^{(2)}$ corresponds to the third-order Hamiltonian operator from Example 1 of Section 2. One can verify that the metric $g^{(2)}$ is flat, while $g^{(1)}$ is not flat. The singular varieties of the first two metrics are double lines: $(u^1)^2 = 0$. Applying a projective transformation that sends this line to the line at infinity, one can reduce the first two cases to constant coefficients. This leads to our second result (Theorem 2):

Modulo projective transformations, any two-component homogeneous third-order Hamiltonian operator can be reduced to constant form.

Three-component case (Theorem 3 of Section 6). *Modulo (complex) projective transformations, the metric of any three-component homogeneous*

third-order Hamiltonian operator can be reduced to one of the six canonical forms:

$$\begin{aligned}
g^{(1)} &= \begin{pmatrix} (u^2)^2 + c & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2 u^3 - u^1 \\ 2u^2 & -cu^2 u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix}, \\
g^{(2)} &= \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 & 0 \\ -u^1 u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g^{(4)} &= \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

The corresponding singular varieties, $\det g = 0$, are as follows (see Theorem 3 for explicit formulae):

- $g^{(1)}, g^{(2)}$: double quadric;
- $g^{(3)}, g^{(4)}$: two double planes, one of them at infinity;
- $g^{(5)}, g^{(6)}$: quadruple plane at infinity.

Third-order Hamiltonian operators corresponding to the metrics $g^{(3)}$ and $g^{(4)}$ are direct sums of the two-component operators from Theorem 1, and the one-component operator D^3 (we emphasize that these direct sums can't be transformed to constant form, even by projective transformations). As the correspondence between Monge metrics and Hamiltonian operators (2) respects direct sums, two-component operators are expected to appear in the three-component classification. The metrics $g^{(5)}$ and $g^{(4)}$ correspond to Hamiltonian operators discussed in Examples 2, 3 of Section 2. The metrics $g^{(1)}$ and $g^{(2)}$ give rise to third-order operators which are apparently new. Direct calculations demonstrate that the metrics $g^{(4)}, g^{(5)}, g^{(6)}$ are flat, while $g^{(1)}, g^{(2)}, g^{(3)}$ are not flat (not even conformally flat: they have non-vanishing Cotton tensor).

4 Monge metrics and quadratic line complexes

Our first observation is that system (4) can be rewritten in terms of the metric g alone:

Proposition 1. *The system (4) implies*

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}) = \frac{1}{3}g_{n[m,k]},$$

and the elimination of c results in (5), (6):

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$

$$g_{m[k,n]l} = -\frac{1}{3}g^{pq}g_{p[l,m]}g_{q[k,n]},$$

here square brackets denote antisymmetrisation.

Proof. Taking into account that c is skew-symmetric in the last two indices, the relation (4a) implies

$$c_{mnk} = c_{nkm} - g_{mn,k}, \quad c_{kmn} = c_{nkm} + g_{kn,m}.$$

Substituting this into (4c) we obtain the explicit formula for c ,

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}) = \frac{1}{3}g_{n[m,k]}.$$

With this expression for c , the relations (4a)-(4c) reduce to the linear system (5) for g :

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0.$$

Finally, (4d) gives the nonlinear constraint (6). \square

Note that the linear system (5) can be solved explicitly: any such metric $g = g_{ij}du^i du^j$ is an arbitrary quadratic expression of the form (7) in du^i and $u^j du^k - u^k du^j$:

$$g_{ij}du^i du^j = a_{ij}du^i du^j + b_{ijk}du^i(u^j du^k - u^k du^j) + c_{ijkl}(u^i du^j - u^j du^i)(u^k du^l - u^l du^k),$$

here the coefficients $a_{ij}, b_{ijk}, c_{ijkl}$ are arbitrary constants (without any loss of generality one can impose additional symmetries such as $a_{ij} = a_{ji}$, $b_{ijk} = -b_{ikj}$, $c_{ijkl} = -c_{jikl} = -c_{ijlk}$, etc). The above formula follows from the analogous result for Killing bivectors in pseudo-Euclidean spaces: any Killing bivector is a quadratic expression in Killing vectors. Formula (7) implies that the coefficients g_{ij} are at most quadratic in the flat coordinates u^i , the fact observed previously in [26, 6].

Metrics of the form (7) appear in the theory of quadratic complexes of lines in the projective space \mathbb{P}^n . Let us recall the main construction. Consider two points in \mathbb{P}^n with homogeneous coordinates u^i, v^i , $i = 1, \dots, n+1$. The Plücker coordinates p^{ij} of the line through these points are defined as $p^{ij} = u^i v^j - u^j v^i$. They satisfy a system of quadratic relations of the form $p^{ij}p^{kl} + p^{ki}p^{jl} + p^{jk}p^{il} = 0$, that specify a projective embedding of

the Grassmannian of lines (Plücker embedding). For $n = 3$ we have a single quadratic relation $p^{12}p^{34} + p^{31}p^{24} + p^{14}p^{23} = 0$, known as the Plücker quadric. A quadratic line complex is defined by an additional homogeneous quadratic equation in the Plücker coordinates,

$$Q(p^{ij}) = 0.$$

This specifies a $(2n - 3)$ -parameter family of lines in \mathbb{P}^n . Fixing a point $p \in \mathbb{P}^n$ and taking all lines of the complex that pass through p we obtain a quadratic cone with vertex at p . This family of cones supplies \mathbb{P}^n with a conformal structure (Monge metric) whose explicit form can be obtained as follows. Let us set $v^i = u^i + du^i$. Then the Plücker coordinates take the form $p^{ij} = u^i du^j - u^j du^i$. In the affine chart $u^{n+1} = 1$, $du^{n+1} = 0$, part of the Plücker coordinates simplify to $p^{(n+1)i} = du^i$, and the equation of the complex takes the so-called Monge form:

$$Q(du^i, u^j du^k - u^k du^j) = 0,$$

here $i, j, k = 1, \dots, n$. This is nothing but the general metric (7). What renders the classification of three-component homogeneous third-order Hamiltonian operators possible, is the existing classification of quadratic line complexes in \mathbb{P}^3 [16].

5 Projective invariance and reciprocal transformations

As the flat coordinates u^i are defined up to affine transformations, the system (5)-(6) is invariant under transformations of the form

$$\tilde{u}^i = l^i(\mathbf{u}), \quad \tilde{g} = g,$$

where l^i are linear forms in the flat coordinates $\mathbf{u} = (u^1, \dots, u^n)$, and $\tilde{g} = g$ indicates that g transforms as a metric (with low indices). On the other hand, the relation to quadratic line complexes indicates that our problem is projectively-invariant. Indeed, the system (5)-(6) is invariant under the group of projective transformations of the form

$$\tilde{u}^i = \frac{l^i(\mathbf{u})}{l(\mathbf{u})}, \quad \tilde{g} = \frac{g}{l^4(\mathbf{u})}, \quad (8)$$

where l is yet another linear form in the flat coordinates. Note that the Monge form (7) is also invariant under projective transformations (8).

It turns out that projective transformations (8) correspond to reciprocal transformations of the corresponding Hamiltonian operator (2). We recall that a reciprocal transformation is a nonlocal change of the independent variable x defined as

$$d\tilde{x} = A(\mathbf{u})dx, \quad (9)$$

where $A(\mathbf{u})$ is a function of field variables. Reciprocal transformations of Hamiltonian operators of hydrodynamic type were investigated previously in [10, 14, 1]. In general, transformed operators become nonlocal. It is remarkable that in the special case when $A(\mathbf{u})$ is linear in the flat coordinates, reciprocal transformations preserve the locality of third-order operators (2).

Proposition 2. *The class of homogeneous third-order Hamiltonian operators (2) is invariant under reciprocal transformations of the form (9), where $A(\mathbf{u})$ is linear in the flat coordinates u^i . Reciprocal transformations induce projective transformations (8) of the corresponding Monge metrics.*

Proof. Let us set $A = c_i u^i + c_0$. In the new independent variable \tilde{x} , the Casimir functionals, $\int u^i dx$, take the form $\int \frac{u^i}{A} d\tilde{x}$. Thus, the transformed Casimir densities are $\tilde{u}^i = \frac{u^i}{A}$, which is a particular case of (8). The general case of (8) is obtained by combining the above transformation with arbitrary affine changes of u^i .

The second formula (8) results from the following calculation. Consider two functionals, $F = \int f(\mathbf{u})dx$ and $H = \int h(\mathbf{u})dx$ (for simplicity we restrict to functionals of hydrodynamic type). Their Poisson bracket equals

$$\{F, H\} = \int f_i P^{ij} h_j dx = \int f_i D(g^{ij} D + c_k^{ij} u_x^k) D h_j dx.$$

Using $\int f dx = \int \tilde{f} d\tilde{x}$, $\int h dx = \int \tilde{h} d\tilde{x}$ where $f = A\tilde{f}$, $h = A\tilde{h}$, and making the substitutions $dx \rightarrow \frac{1}{A} d\tilde{x}$, $D \rightarrow A\tilde{D}$ (here $\tilde{D} \equiv D_{\tilde{x}}$), one obtains

$$\{F, H\} = \int (A\tilde{f}_i + c_i \tilde{f}) \underline{A\tilde{D}} (A\tilde{g}^{ij} \tilde{D} + A c_k^{ij} \tilde{u}_{\tilde{x}}^k) \underline{A\tilde{D}} (A\tilde{h}_j + c_j \tilde{h}) \frac{1}{\underline{A}} d\tilde{x}.$$

Cancelling the underlined terms one can observe that, in spite of the explicit presence of \tilde{f} and \tilde{h} in the integrand (which may potentially lead to non-locality of the transformed operator), the fact that they appear with constant coefficients c_i allows one to rewrite the above expression in the form

$$\{F, H\} = \int \tilde{f}_i \tilde{P}^{ij} \tilde{h}_j d\tilde{x},$$

where \tilde{P} is a local homogeneous third-order operator with the leading term $A^4 g^{ij} \tilde{D}^3$. Thus, $\tilde{g}^{ij} = A^4 g^{ij}$, which is equivalent to the second formula (8) (recall that in (8) g denotes the metric with low indices).

In the new Casimirs \tilde{u}^i , the transformed operator \tilde{P} can be computed in the following way. Taking into account that $\tilde{u}^k = \frac{u^k}{A}$, where $A = c_m u^m + c_0 = c_0(1 - c_m \tilde{u}^m)^{-1}$, one obtains

$$\frac{\partial \tilde{u}^k}{\partial u^i} = \frac{\delta_i^k - c_i \tilde{u}^k}{A}.$$

Thus,

$$A \frac{\partial \tilde{f}}{\partial u^i} = (\delta_i^k - c_i \tilde{u}^k) \frac{\partial \tilde{f}}{\partial \tilde{u}^k},$$

so that the bracket

$$\{F, H\} = \int \left(A \frac{\partial \tilde{f}}{\partial u^i} + c_i \tilde{f} \right) \tilde{D}(A g^{ij} \tilde{D} + A c_k^{ij} u_{\tilde{x}}^k) A \tilde{D} \left(A \frac{\partial \tilde{h}}{\partial u^j} + c_j \tilde{h} \right) d\tilde{x}$$

assumes the form

$$\{F, H\} = \int \left((\delta_i^k - c_i \tilde{u}^k) \frac{\partial \tilde{f}}{\partial \tilde{u}^k} + c_i \tilde{f} \right) \tilde{D}(A g^{ij} \tilde{D} + A c_k^{ij} u_{\tilde{x}}^k) A \tilde{D} \left((\delta_j^k - c_j \tilde{u}^k) \frac{\partial \tilde{h}}{\partial \tilde{u}^k} + c_j \tilde{h} \right) d\tilde{x}.$$

Using the identity

$$\tilde{D} \left((\delta_j^k - c_j \tilde{u}^k) \frac{\partial \tilde{h}}{\partial \tilde{u}^k} + c_j \tilde{h} \right) = (\delta_j^k - c_j \tilde{u}^k) \tilde{D} \frac{\partial \tilde{h}}{\partial \tilde{u}^k},$$

one obtains

$$\begin{aligned} \{F, H\} &= \int \frac{\partial \tilde{f}}{\partial \tilde{u}^k} (\delta_i^k - c_i \tilde{u}^k) \tilde{D}(A g^{ij} \tilde{D} + A c_k^{ij} u_{\tilde{x}}^k) A (\delta_j^k - c_j \tilde{u}^k) \tilde{D} \frac{\partial \tilde{h}}{\partial \tilde{u}^k} d\tilde{x} \\ &\quad - \int c_i \tilde{D} \tilde{f} \cdot (A g^{ij} \tilde{D} + A c_k^{ij} u_{\tilde{x}}^k) A (\delta_j^k - c_j \tilde{u}^k) \tilde{D} \frac{\partial \tilde{h}}{\partial \tilde{u}^k} d\tilde{x}. \end{aligned}$$

Since $\tilde{D} \tilde{f} = (\partial \tilde{f} / \partial \tilde{u}^m) \tilde{u}_{\tilde{x}}^m$, one arrives at

$$\{F, H\} = \int \frac{\partial \tilde{f}}{\partial \tilde{u}^m} [(\delta_i^m - c_i \tilde{u}^m) \tilde{D} - c_i \tilde{u}_{\tilde{x}}^m] (A g^{ij} \tilde{D} + A c_k^{ij} u_{\tilde{x}}^k) A (\delta_j^k - c_j \tilde{u}^k) \tilde{D} \frac{\partial \tilde{h}}{\partial \tilde{u}^k} d\tilde{x}.$$

Using the identity

$$(\delta_i^m - c_i \tilde{u}^m) \tilde{D} - c_i \tilde{u}_{\tilde{x}}^m = \tilde{D}(\delta_i^m - c_i \tilde{u}^m),$$

one ultimately arrives at the Hamiltonian operator \tilde{P} written in the transformed Casimirs \tilde{u}^i :

$$\tilde{P}^{ij} = \tilde{D}(\delta_i^m - c_i \tilde{u}^m) A(g^{ij} \tilde{D} + c_k^{ij} u_{\tilde{x}}^k) A(\delta_j^k - c_j \tilde{u}^k) \tilde{D}.$$

This is again a local homogeneous third-order expression of the form (2). \square

6 Classification results

In this section we classify homogeneous third-order Hamiltonian operators with the number of components $n = 1, 2$ and 3 . Our approach is based on the correspondence with Monge metrics and quadratic line complexes. All classification results are obtained modulo (complex) projective transformations as introduced in Section 5. To save space we only present canonical forms for the corresponding Monge metrics rather than Hamiltonian operators themselves.

6.1 One-component case

Any one-component operator can be reduced to D^3 , see [26, 25, 6]. Indeed, in this case system (4) implies $g_{11,1} = c_{111} = 0$. This result goes back to [20, 2, 21] where a complete contact classification of scalar third-order Hamiltonian operators was obtained.

6.2 Two-component case

Here we provide both affine and projective classifications. The main results are summarised below.

Theorem 1. *Modulo (complex) affine transformations, the metric of any two-component homogeneous third-order Hamiltonian operator can be reduced to one of the three canonical forms:*

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 \\ -u^1 u^2 & (u^1)^2 \end{pmatrix}, \quad g^{(2)} = \begin{pmatrix} -2u^2 & u^1 \\ u^1 & 0 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The metric $g^{(1)}$ gives rise to the Hamiltonian operator

$$D \begin{pmatrix} D & D \frac{u^2}{u^1} \\ \frac{u^2}{u^1} D & \frac{(u^2)^2 + 1}{2(u^1)^2} D + D \frac{(u^2)^2 + 1}{2(u^1)^2} \end{pmatrix} D,$$

the metric $g^{(2)}$ corresponds to the Hamiltonian operator from Example 1 of Section 2,

$$D \left(\begin{array}{cc} 0 & D \frac{1}{u^1} \\ \frac{1}{u^1} D & \frac{u^2}{(u^1)^2} D + D \frac{u^2}{(u^1)^2} \end{array} \right) D.$$

Note that these third-order operators are compatible (form a Hamiltonian pair). The singular varieties of the first two metrics are double lines: $(u^1)^2 = 0$. Applying a projective transformation that sends this line to the line at infinity, one can reduce the first two cases to constant form. This leads to

Theorem 2. *Modulo projective transformations, any two-component homogeneous third-order Hamiltonian operator can be reduced to constant coefficient form.*

Proof of Theorem 1. Setting $U = (u^1 du^2 - u^2 du^1, du^1, du^2)$, one can represent a generic two-component Monge metric in the form $g = UQU^t$ where Q is a constant 3×3 symmetric matrix. Setting

$$Q = \begin{pmatrix} r & h_1 & -h_2 \\ h_1 & f_{11} & f_{12} \\ -h_2 & f_{12} & f_{22} \end{pmatrix}$$

one obtains, explicitly,

$$g = r(u^1 du^2 - u^2 du^1)^2 + 2(u^1 du^2 - u^2 du^1)(h_1 du^1 - h_2 du^2) + f_{ij} du^i du^j. \quad (10)$$

Equations (6) impose a single cubic constraint,

$$\det Q = r(f_{11}f_{22} - f_{12}^2) - h_1^2 f_{22} - h_2^2 f_{11} - 2h_1 h_2 f_{12} = 0. \quad (11)$$

We have the following cases.

Case 1: $r \neq 0$. By scaling the dependent variables one can set $r = 1$. By shifting the dependent variables one can also assume that $h_1 = h_2 = 0$, so that the constraint (11) simplifies to $f_{11}f_{22} - f_{12}^2 = 0$. This means that the constant part of the metric (10), $f_{ij} du^i du^j$, is degenerate, and therefore can be reduced to $(du^1)^2$ by an affine transformation. This results in the metric

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 \\ -u^1 u^2 & (u^1)^2 \end{pmatrix}.$$

Case 2: $r = 0$. Modulo affine transformations one can always assume $h_1 = 1$, $h_2 = 0$ (if both h_1 and h_2 vanish we have the constant coefficient case that can be reduced to $g^{(3)}$). Then (11) implies $f_{22} = 0$, and by shifting the dependent variables one can eliminate f_{11} and f_{12} . This results in

$$g^{(2)} = \begin{pmatrix} -2u^2 & u^1 \\ u^1 & 0 \end{pmatrix}.$$

It remains to point out that the metrics $g^{(1)}, g^{(2)}, g^{(3)}$ are not affinely equivalent: the degrees of their coefficients are different. \square

Proof of Theorem 2. One can verify that the projective transformation

$$\tilde{u}^1 = \frac{1}{u^1}, \quad \tilde{u}^2 = \frac{u^2}{u^1}, \quad \tilde{g} = \frac{g}{(u^1)^4}$$

reduces the first and the second metrics to constant coefficient forms $(d\tilde{u}^1)^2 + (d\tilde{u}^2)^2$ and $-2d\tilde{u}^1d\tilde{u}^2$, respectively. Thus, the corresponding third-order Hamiltonian operators can be transformed to constant forms by one and the same reciprocal transformation. This explains their compatibility. \square

6.3 Three-component case

Here we discuss the main result of this paper - projective classification of three-component Hamiltonian operators (affine classification would contain too many cases and moduli). This is achieved by going through the list of normal forms of quadratic line complexes in \mathbb{P}^3 , which fall into eleven Segre types [16], and calculating the nonlinear constraints (6). Let us briefly recall the main setup. Consider the Plücker quadric $p^{12}p^{34} + p^{31}p^{24} + p^{14}p^{23} = 0$, let Ω be the 6×6 symmetric matrix of this quadratic form. A quadratic line complex is the intersection of the Plücker quadric with another homogeneous quadratic equation in the Plücker coordinates, defined by a 6×6 symmetric matrix Q . The key invariant of a quadratic complex is the Jordan normal form of the matrix $Q\Omega^{-1}$. It is labelled by the Segre symbol that carries information about the number and sizes of Jordan blocks. Thus, the symbol [111111] indicates that the Jordan form of $Q\Omega^{-1}$ is diagonal; the symbol [222] indicates that the Jordan form of $Q\Omega^{-1}$ consists of three 2×2 Jordan blocks, etc. We will also use ‘refined’ Segre symbols with additional round brackets indicating coincidences among the eigenvalues of some of the Jordan blocks, e.g., [(11)(11)(11)] denotes the subcase of [111111] with three pairs of coinciding eigenvalues, the symbol [(111)(111)] denotes the subcase with two triples of coinciding eigenvalues, etc.

The Monge metric results from the equation of the complex upon setting $p^{ij} = u^i du^j - u^j du^i$, and using the affine chart $u^4 = 1$, $du^4 = 0$ (in some cases it will be more convenient to use different affine charts, say, $u^1 = 1$, $du^1 = 0$: this will be indicated explicitly where appropriate). The singular surface of a generic quadratic line complex in \mathbb{P}^3 is Kummer's quartic surface, that can be defined as the degeneracy locus of the corresponding Monge metric. For Monge metrics associated to third-order Hamiltonian operators this quartic always degenerates into a double quadric (that may further split into a pair of planes).

Theorem 3. *Modulo (complex) projective transformations, the Monge metric of any three-component homogeneous third-order Hamiltonian operator can be reduced to one of the six canonical forms:*

1. Segre type [(111)111]: we have a one-parameter family of metrics ($c \neq \pm 1$):

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2 u^3 - u^1 \\ 2u^2 & -cu^2 u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix},$$

$\det g^{(1)} = (c+1)(c-1)(u^1 u^2 - u^3)^2$, the singular surface is a double quadric.

2. Segre type [(111)12]:

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$

$\det g^{(2)} = (u^1 u^2 - u^3)^2$, the singular surface is a double quadric.

3. Segre type [11(112)]:

$$g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 & 0 \\ -u^1 u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\det g^{(3)} = (u^1)^2$, the singular surface is a pair of double planes (one of them at infinity).

4. Segre type [(114)]:

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

$\det g^{(4)} = -(u^1)^2$, the singular surface is a pair of double planes (one of them at infinity).

5. Segre type [(123)]:

$$g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$\det g^{(5)} = -1$, the singular surface is a quadruple plane at infinity.

6. Segre type [(222)]:

$$g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\det g^{(6)} = 1$, the singular surface is a quadruple plane at infinity.

Proof of Theorem 3. The classification is achieved by going through the list of Segre types of quadratic complexes and selecting those whose Monge metrics fulfil (6). In what follows we use the notation of [13]; more details on the projective classification of quadratic complexes in \mathbb{P}^3 can be found in [16].

Segre type [111111]. In this case the equation of the complex is

$$\begin{aligned} \lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + \lambda_3(p^{13} + p^{42})^2 - \lambda_4(p^{13} - p^{42})^2 \\ + \lambda_5(p^{14} + p^{23})^2 - \lambda_6(p^{14} - p^{23})^2 = 0, \end{aligned}$$

here λ_i are the eigenvalues of $Q\Omega^{-1}$. The corresponding Monge metric is

$$\begin{aligned} [a_1 + a_2(u^3)^2 + a_3(u^2)^2](du^1)^2 + [a_2 + a_1(u^3)^2 + a_3(u^1)^2](du^2)^2 \\ + [a_3 + a_1(u^2)^2 + a_2(u^1)^2](du^3)^2 + 2[\alpha u^3 - a_3 u^1 u^2]du^1 du^2 \\ + 2[\beta u^2 - a_2 u^1 u^3]du^1 du^3 + 2[\gamma u^1 - a_1 u^2 u^3]du^2 du^3, \end{aligned}$$

where $a_1 = \lambda_5 - \lambda_6$, $a_2 = \lambda_3 - \lambda_4$, $a_3 = \lambda_1 - \lambda_2$, $\alpha = \lambda_5 + \lambda_6 - \lambda_3 - \lambda_4$, $\beta = \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6$, $\gamma = \lambda_3 + \lambda_4 - \lambda_1 - \lambda_2$, notice that $\alpha + \beta + \gamma = 0$. A direct computation shows that the only metrics that satisfy (6) are those for which the eigenvalues fulfil the relation (up to permutations of the λ_i):

$$\lambda_2 = \lambda_3 = \lambda_4 = \frac{\lambda_1 + \lambda_5 + \lambda_6}{3}.$$

Complexes of this type are denoted $[(111)111]$. Without any loss of generality one can set $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_1 = 1$, $\lambda_5 = \frac{c-1}{2}$, $\lambda_6 = -\frac{c+1}{2}$ where c is a parameter (note that one can add a multiple of the Plücker quadric to the equation of the complex). This results in the metric

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2u^3 - u^1 \\ 2u^2 & -cu^2u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix}.$$

Note that although different values of c correspond to projectively non-equivalent complexes, the corresponding singular surface, $\det g^{(1)} = 0$, which is the double quadric $(u^1u^2 - u^3)^2 = 0$, does not depend on c . Non-equivalent complexes with coinciding singular surfaces are called cosingular. It was shown in [3] that varieties of cosingular line complexes are generically curves, with the only exception provided by complexes of Segre type $[(111)111]$, in which case the variety of cosingular complexes is two-dimensional. This explains, in particular, why the cosingular complex of Segre type $[(111)(111)]$, known as ‘special’, does not occur in our classification.

Another, more symmetric choice of representative within the same class, can be obtained if one assumes

$$\lambda_2 = \lambda_4 = \lambda_6 = \frac{\lambda_1 + \lambda_3 + \lambda_5}{3},$$

where without any loss of generality one can set $\lambda_2 = \lambda_4 = \lambda_6 = 0$, $\lambda_1 = p$, $\lambda_3 = q$, $\lambda_5 = r$, which results in the metric

$$\begin{pmatrix} r + p(u^2)^2 + q(u^3)^2 & (r - q)u^3 - pu^1u^2 & (p - r)u^2 - qu^1u^3 \\ (r - q)u^3 - pu^1u^2 & q + p(u^1)^2 + r(u^3)^2 & (q - p)u^1 - ru^2u^3 \\ (p - r)u^2 - qu^1u^3 & (q - p)u^1 - ru^2u^3 & p + q(u^1)^2 + r(u^2)^2 \end{pmatrix},$$

recall that $p + q + r = 0$. In this case

$$\det g^{(1)} = pqr((u^1)^2 + (u^2)^2 + (u^3)^2 + 1)^2,$$

the corresponding singular surface is the double (imaginary) sphere.

Segre type $[11112]$. The equation of the complex is

$$\begin{aligned} \lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + \lambda_3(p^{13} + p^{42})^2 \\ - \lambda_4(p^{13} - p^{42})^2 + 4\lambda_5p^{14}p^{23} + (p^{14})^2 = 0. \end{aligned}$$

The corresponding Monge metric is

$$[\lambda(u^2)^2 + \mu(u^3)^2 + 1](du^1)^2 + [\lambda(u^1)^2 + \mu](du^2)^2 + [\mu(u^1)^2 + \lambda](du^3)^2 \\ + 2[\alpha u^3 - \lambda u^1 u^2]du^1 du^2 + 2[\beta u^2 - \mu u^1 u^3]du^1 du^3 + 2\gamma u^1 du^2 du^3,$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$, $\alpha = -\lambda_3 - \lambda_4 + 2\lambda_5$, $\beta = \lambda_1 + \lambda_2 - 2\lambda_5$, $\gamma = -\alpha - \beta$. A direct computation shows that there are two subcases which satisfy (6). Up to permutations of the eigenvalues λ_i , we have

Subcase [(111)12]:

$$\lambda_2 = \lambda_3 = \lambda_4 = (1/3)(\lambda_1 + 2\lambda_5).$$

Without any loss of generality one can set $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 = -1/2$. This results in the metric

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 - u^3 & 2u^2 \\ -u^1 u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix}.$$

Subcase [11(112)]:

$$\lambda_3 = \lambda_4 = \lambda_5 = (1/2)(\lambda_1 + \lambda_2).$$

Without any loss of generality one can set $\lambda_1 = 1/2$, $\lambda_2 = -1/2$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$. This results in the metric

$$g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1 u^2 & 0 \\ -u^1 u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Segre type [114]. The equation of the complex is

$$\lambda_1(p^{12} + p^{34})^2 - \lambda_2(p^{12} - p^{34})^2 + 4\lambda_3(p^{14}p^{23} + p^{42}p^{13}) \\ + 2p^{14}p^{42} + 4(p^{13})^2 = 0.$$

Setting $p^{ij} = u^i du^j - u^j du^i$ and using the affine chart $u^1 = 1$, $du^1 = 0$ we obtain the associated Monge metric,

$$\lambda(du^2)^2 + [\lambda(u^4)^2 + 4](du^3)^2 + [\lambda(u^3)^2 - 2u^2](du^4)^2 \\ + 2\alpha u^4 du^2 du^3 + 2[u^4 - \alpha u^3]du^2 du^4 - 2\lambda u^3 u^4 du^3 du^4,$$

where $\lambda = \lambda_1 - \lambda_2$, $\alpha = 2\lambda_3 - \lambda_1 - \lambda_2$. A direct computation shows that the only metrics that satisfy (6) are those for which $\lambda_1 = \lambda_2 = \lambda_3$. Complexes of this type are denoted [(114)]. Without any loss of generality one can set $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which results in the metric

$$4(du^3)^2 - 2u^2(du^4)^2 + 2u^4 du^2 du^4 = 0.$$

Setting $u^4 \rightarrow u^1$, $u^3 \rightarrow u^3/2$ we obtain

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Segre type [123]. The equation of the complex is

$$\begin{aligned} -\lambda_1(p^{12} - p^{34})^2 + 4\lambda_2 p^{13} p^{42} + 4(p^{13})^2 \\ + \lambda_3(4p^{14} p^{23} + (p^{12} + p^{34})^2) + 2p^{14}(p^{12} + p^{34}) = 0. \end{aligned}$$

Setting $p^{ij} = u^i du^j - u^j du^i$ and using the affine chart $u^1 = 1$, $du^1 = 0$ we obtain the associated Monge metric,

$$\begin{aligned} \lambda(du^2)^2 + [\lambda(u^4)^2 + 4](du^3)^2 + [\lambda(u^3)^2 + 2u^3](du^4)^2 \\ + 2\alpha u^4 du^2 du^3 + 2[1 - \lambda u^3] du^2 du^4 + 2[\gamma u^2 - \lambda u^3 u^4 - u^4] du^3 du^4, \end{aligned}$$

where $\lambda = \lambda_3 - \lambda_1$, $\alpha = 2\lambda_2 - \lambda_1 - \lambda_3$, $\gamma = \lambda - \alpha$. The only metrics of this form which fulfil (6) are those for which $\lambda_1 = \lambda_2 = \lambda_3$. Complexes of this type are denoted [(123)]. Without any loss of generality one can set $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which results in the metric

$$4(du^3)^2 + 2u^3(du^4)^2 + 2du^2 du^4 - 2u^4 du^3 du^4.$$

Setting $u^4 \rightarrow i\sqrt{2}u^1$, $u^3 \rightarrow u^2/2$, $u^2 \rightarrow -iu^3/\sqrt{2}$ we obtain

$$g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Segre type [222]. Here we have two (projectively dual) subcases, with the equations

$$2\lambda_1 p^{12} p^{34} + 2\lambda_2 p^{13} p^{42} + 2\lambda_3 p^{14} p^{23} + (p^{12})^2 + (p^{13})^2 + (p^{14})^2 = 0,$$

and

$$2\lambda_1 p^{12} p^{34} + 2\lambda_2 p^{13} p^{42} + 2\lambda_3 p^{14} p^{23} + (p^{23})^2 + (p^{24})^2 + (p^{34})^2 = 0,$$

respectively. Setting $p^{ij} = u^i du^j - u^j du^i$ and using the affine chart $u^1 = 1$, $du^1 = 0$ we obtain the associated Monge metrics,

$$(du^2)^2 + (du^3)^2 + (du^4)^2 + 2\alpha u^4 du^2 du^3 + 2\beta u^3 du^2 du^4 + 2\gamma u^2 du^3 du^4,$$

and

$$\begin{aligned} & ((u^3)^2 + (u^4)^2)(du^2)^2 + ((u^2)^2 + (u^4)^2)(du^3)^2 + ((u^2)^2 + (u^3)^2)(du^4)^2 \\ & + 2(\alpha u^4 - u^2 u^3) du^2 du^3 + 2(\beta u^3 - u^2 u^4) du^2 du^4 + 2(\gamma u^2 - u^3 u^4) du^3 du^4, \end{aligned}$$

where $\alpha = \lambda_2 - \lambda_1$, $\beta = \lambda_1 - \lambda_3$, $\gamma = \lambda_3 - \lambda_2$. In both cases the condition (6) implies $\lambda_1 = \lambda_2 = \lambda_3$ (such complexes are denoted [(222)]), however, the second metric becomes degenerate. This is the constant case $g^{(6)}$.

Other Segre types, namely [1113], [1122], [15], [24], [33], [6], do not correspond to homogeneous third-order Hamiltonian operators. Thus, the only allowed Segre types are those for which:

- (a) The Jordan normal form of $Q\Omega^{-1}$ contains at least three Jordan blocks (that is, there are at least three entries in square brackets).
- (b) There are three distinct Jordan blocks with the same eigenvalue λ (that is, there is a round bracket with three entries). According to [16], p. 200, this implies that the singular surface of the corresponding quadratic complex is a double quadric (possibly, reducible).
- (c) The average of the remaining three eigenvalues (outside round brackets) equals λ (without any loss of generality one can set $\lambda = 0$).

As demonstrated above, the only Segre types that satisfy all these conditions are [(111)111], [(111)12], [11(112)], [(114)], [(123)], [(222)]. The types [15], [24], [33], [6] do not satisfy condition (a); the types [1113], [1122] satisfying (a)-(c) lead to degenerate Monge metrics.

□

7 Concluding remarks

The main result of this paper is a complete classification of 3-component homogeneous third-order Hamiltonian operators of differential-geometric type, that was obtained based on the link to Monge metrics and quadratic line complexes. Modulo projective equivalence, we found six types of such operators. This was done by going through the list of normal forms of quadratic

complexes in \mathbb{P}^3 , labelled by their Segre types [16]. We observed that the necessary condition for a quadratic line complex to be associated with a Hamiltonian operator is the degeneration of its singular surface, known as Kummer's quartic, into a double quadric (that itself may split into a pair of planes).

- The main challenge is to extend our classification to the general n -component case $n > 3$. First of all, one can generate new examples of homogeneous Hamiltonian operators with arbitrary number of components by taking direct sums of operators with fewer components. Thus, it is natural to restrict to the classification of irreducible operators. Although the relation to Monge metrics and quadratic line complexes is still available, there exist no reasonable 'normal forms' for quadratic complexes even in \mathbb{P}^4 . All known examples suggest however that singular varieties of quadratic complexes in \mathbb{P}^n corresponding to third-order n -component Hamiltonian operators are not arbitrary, and degenerate into a double hypersurface of degree $n - 1$. We recall that the singular variety of a generic quadratic line complex in \mathbb{P}^n is a hypersurface of degree $2n - 2$. For $n = 3, 4$ these varieties are known as Kummer's quartics in \mathbb{P}^3 [19] and Segre sextic hypersurfaces in \mathbb{P}^4 [28], respectively.
- It would be interesting to construct first-order Hamiltonian operators compatible with the third-order operators found in this paper, and to investigate the corresponding integrable hierarchies. For all Examples from Section 2 this was done in [11, 22, 17, 18], see also [29] for some results in the case of constant third-order operators $\eta^{ij}D^3$.
- Some n -component third-order Hamiltonian operators (2) possess, in addition to the n local Casimirs u^i , another n nonlocal Casimirs of the form $s^i = \psi_j^i(\mathbf{u})D^{-1}u^j$. Changing from the flat coordinates u^i to the nonlocal variables s^i one obtains a first-order constant coefficient operator $\eta^{ij}D$, thus establishing the Darboux theorem. Although this procedure works for all Hamiltonian operators from Examples 1-3 of Section 2 [11, 22, 17, 18], it does not seem to be universally applicable. For general third-order Hamiltonian operators, the Darboux theorem is yet to be established.

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