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Spherical symmetry in classical and quantum Galilei general relativity

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Abstract

In the context of Galilei general relativistic classical and quantum mechanics we find the spherically symmetric exact solution, giving a result of existence and uniqueness. From the classical viewpoint, this solution yields a geometrical interpretation of Newton’s gravitation law in terms of a connection on spacetime with non vanishing curvature. Moreover, within our covariant geometric approach to the quantum theory, we find out that from the physical viewpoint there is only one quantum model for a spinless particle in a spherically symmetric gravitational field.

Résumé

Dans le cadre de la mécanique classique et quantique dans la théorie de la relativité générale galiléen nous trouvons la solution exacte sphériquement symétrique, en donnant un résultat d’existence et d’unicité. Du point de vue classique, cette solution donne une interpretation géométrique de la loi de gravitation de Newton en termes d’une connexion sur l’espace–temps avec une courbure non nulle. En outre, à l’intérieur de notre approche géométrique covariante à la théorie quantique nous trouvons que du point de vue physique il n’y a qu’un modèle quantique d’une particule sans spin dans un champ gravitationnel sphériquement symétrique.

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Introduction

The theory of general relativity, as formulated by Einstein, must be taken as a touchstone for anyone who wants to study gravitation. Hence, it would be desirable to study quantum mechanics on an Einstein's general relativistic background. But the covariant formulation of the quantum theory on a curved spacetime presents several serious difficulties.

In order to tackle this hard problem it may be useful to start with a Galilei curved background. In fact, the Galilei general relativity (even if it is clearly less satisfactory than Einstein's general relativity from the physical viewpoint) has several important features in common with Einstein's theory and, at the same time, it skips some deep difficulties. So, in the Galilei context, we can learn interesting facts to be applied later to the Einstein case.

The Galilei general relativity and the related approach to quantum mechanics has been developed in several different ways by a large number of people; particularly interesting are the works of [1], [4], [5], [11], [14], [15], [18], [19], [22], [23].

Recently, it was presented a book [10] in which this approach is developed in detail in a mathematically rigorous and self–contained way.

Here, spacetime is a manifold fibred over absolute (i.e., observer–independent) time and equipped with a vertical Riemannian metric. The gravitational field is represented by a time–preserving torsion–free connection on spacetime.

The first field equation is expressed by requiring the closure of a two–form induced naturally from the metric and the connection; it turns out from this equation that the connection is metric, but it is not completely determined by the metric due to the degeneracy of the latter.

The second field equation expresses the coupling of the gravitational field with the matter sources, and is given by equating the Ricci tensor of the connection to the energy tensor.

Quantum theory has been developed in [10] for a spinless particle, and then extended to the spin case in [3]. The assumptions needed to describe quantum mechanics of a scalar particle are only two: a complex hermitian line bundle based on spacetime, whose sections will represent quantum histories, and a hermitian universal connection (i.e., a family of hermitian connections on the above bundle parametrised by observers on spacetime) such that its curvature is proportional to the natural two–form on spacetime. This connection yields a natural quantum Lagrangian, the generalised Schroedinger equation, and the quantum operators.

It is interesting to find exact solutions of the classical and quantum theory in this context. This paper is devoted to the study of spherically symmetric solutions (with respect to a worldline of a particle) of classical Galilei general relativistic spacetime and of the quantum connection.

The classical part of our work is carried on in the spirit of what was done in Einstein's general relativity by D. Birkhoff. His theorem [7] yields the uniqueness of the Schwarzschild's solution under hypotheses of spherical symmetry given on the metric.

In Galilei general relativity we need conditions of spherical symmetry both on the metric and on the connection. We would like our spacetime, metric structure and gravitational field to be spherically symmetric around the worldline of a particle; intuitively, this can be expressed by requiring the non–existence of distinguished points (other than points on the selected worldline), or distinguished directions (other than radial spacelike directions). Moreover, being the selected particle the unique source of the gravitational field, we must allow the field to have a singularity on the worldline.

Following such guidelines, we express spherical symmetry conditions on the metric by means of techniques of Riemannian geometry that we have developed for this purpose (see Appendix 2). For spacetime connection, we express conditions of spherical symmetry by means of requirements on the observed trajectory of test particles; we think that, among all possible approaches, this is the most intuitive and physically significant one.

These two groups of hypotheses imply the existence and the uniqueness of a spherically symmetric gravitational field. Indeed, for this field the equation of particle motion turns out to be equivalent to Newton's law of gravitation. Also, as a by–product, we find a unique observer which has a suitable spherical symmetry with respect to the trajectories of test particles. From the geometrical viewpoint, our spherical symmetry conditions imply that spacetime is topologically trivial; we stress that this fact is not assumed *a priori*.

Finally, we analyse spherical symmetry on the quantum bundle and the quantum connection. Spherical symmetry of the quantum theory is not obtained by spherical symmetry of the classical theory, but just reflects the latter. We prove that our quantum potential can always be seen as the standard Newtonian one, after a change of gauge.

The first section is mainly concerned with the minimal necessary background of Galilei general relativistic classical and quantum theory. In the second section we add spherical symmetry assumptions to the model and obtain the main results. In Appendix 2, we present the main facts of Riemannian geometry and develop our definition of spherically symmetric Riemannian manifold. Finally, in Appendix 2 we give some elements of the theory of positive spaces, developed in [10], that provides a way to deal with units of measurement that clarifies the independence of the theory from the choice of scales.

We emphasise the fact that our procedure will be always covariant; this is guaranteed by the use of intrinsic techniques of calculus on manifolds. Moreover, we will use jet spaces as a fundamental tool, that provides the unique framework in which it is possible to deal with derivatives of maps in an intrinsic way.

Before starting the discussion of the problem, we need some mathematical preliminaries.

Manifolds will be always $\mathcal{C}^\infty$, second countable and connected. All maps are intended to be $\mathcal{C}^\infty$, unless otherwise specified. Among the main mathematical tools will be fibred manifolds, and among these, bundles. We will follow the conventions of [2] for main definitions. Finally, we will use intensively the theory of connections; a standard reference is [13], while a non–standard approach in terms of jet bundles is developed in [17].

1 Galilei general relativistic classical and quantum mechanics

In this section we recall just a few basic facts of Galilei general relativistic classical and quantum mechanics, which will suffice for our aims. The reader will find a more detailed account in [9], [10].

Classical theory

Assumption G.1 We assume the *spacetime* to be a fibred manifold

$$t : \boldsymbol{E} \to \boldsymbol{T} ,$$

where $\boldsymbol{E}$ is an orientable 4–dimensional manifold and $\boldsymbol{T}$ is a 1–dimensional oriented affine space associated with the vector space $\mathbb{T}$.

The *time form* is defined to be the form $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

A *time unit of measurement* is defined to be a positively–oriented element $u_0 \in \mathbb{T}$, or, equivalently, $u^0 \in \mathbb{T}^*$. We will denote by (x^0, y^i), $1 \leq i \leq 3$, a chart of $\boldsymbol{E}$ adapted to the fibring t and to the affine structure on $\boldsymbol{T}$; this means that x^0 coincides with the pullback of an affine coordinate on $\boldsymbol{T}$.

In coordinate expressions latin indices $i, j, \ldots$ will run from 1 to 3 and will label fibre coordinates, and the index 0 will label the coordinate on $\boldsymbol{T}$; greek indices $\lambda, \mu, \ldots$ will be used to label coordinates both on the fibre and on the base, hence $0 \leq \lambda, \mu, \dots \leq 3$.

We will denote by (∂_λ) and (d^λ) the local bases of vector fields and one–forms on $\boldsymbol{E}$ induced by an adapted chart. Moreover, $\dot\partial_\lambda$ will denote the local base of sections of the vector bundle $TT\boldsymbol{E} \to T\boldsymbol{E}$.

The check $(\check{\ })$ will denote vertical restriction. Accordingly, $(\check{d}^i)$ will denote the local base of sections of $V^*\boldsymbol{E} \to \boldsymbol{E}$.

As an example, it is easy to see that the coordinate expression of the time form is $dt = u_0 \otimes d^0$.

We will deal with the jet bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$; its fibre coordinates will be denoted by (y^i_0). We have the natural fibred affine monomorphism over $\boldsymbol{E}$

$$(1) \qquad \mathit{Л} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} ,$$

whose coordinate expression is $\mathit{Л} = u^0 \otimes (\partial_0 + y^j_0 \partial_j)$; $\mathit{Л}$ makes $J_1\boldsymbol{E}$ an affine subbundle of $\mathbb{T}^* \otimes T\boldsymbol{E}$ constituted by all elements that project on $1_\mathbb{T} \in \mathbb{T}^* \otimes \mathbb{T}$. It is also remarkable the complementary epimorphism $\vartheta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$, which has the coordinate expression $\vartheta = \vartheta^j \otimes \partial_j = (d^j - y_0^j d^0) \otimes \partial_j$. A detailed and complete treatment of the subject can be found in [17].

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$, and its *absolute velocity* is defined to be the section $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$ of the bundle $J_1\boldsymbol{E} \to \boldsymbol{T}$. We stress that it is not possible to speak of zero absolute velocity of a motion, due to the affine structure of $J_1\boldsymbol{E}$.

An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}\,.$$

Let $o : \boldsymbol{E} \to J_1\boldsymbol{E}$ be an observer. A local section $s : \boldsymbol{I} \subset \boldsymbol{T} \to \boldsymbol{E}$ is said to be a *local integral section* if $o \circ s = Ts$. The solutions of this differential equation yield a flow of local fibred isomorphisms of $\boldsymbol{E} \to \boldsymbol{T}$. Conversely, a flow of local fibred isomorphisms of $\boldsymbol{E} \to \boldsymbol{T}$ determines a unique observer; thus, we have a bijective correspondence between (local) flows and observers. We say an observer o to be *complete* if its flow is defined on $\mathbb{T} \times \boldsymbol{E}$.[2]

Remark 1.1 It is easy to prove that, if $\boldsymbol{E} \to \boldsymbol{T}$ is a bundle, then there is a bijective correspondence between complete observers and splittings of $\boldsymbol{E}$ into a product bundle. In fact, the existence of a complete observer is due to the theorem that states triviality of bundles over a contractible base [20, p.53].

It can also be proved that $\boldsymbol{E} \to \boldsymbol{T}$ is a bundle if and only if there exists a complete observer on $\boldsymbol{E}$.

Assumption G.2 We assume spacetime to be equipped with a *scaled vertical Riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{A} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})\,,$$

where $\mathbb{A}$ is a positive space (see Appendix 2) which represents area units.

It is clear that for all $\tau \in \boldsymbol{T}$ the fiber $\boldsymbol{E}_\tau := t^{-1}(\tau)$ is endowed with a scaled Riemannian metric, hence g determines a Riemannian connection on each fibre of $\boldsymbol{E}$. We will also denote by $\overline{g}$ the contravariant form of g.

The coordinate expressions of g and $\overline{g}$ are:

$$g = g_{ij}\check{d}^i \otimes \check{d}^j,\ g_{ij} \in \mathcal{C}^\infty(\boldsymbol{E}, \mathbb{A} \otimes \mathbb{R}), \qquad \overline{g} = g^{ij}\partial_i \otimes \partial_j,\ g^{ij} \in \mathcal{C}^\infty(\boldsymbol{E}, \mathbb{A}^* \otimes \mathbb{R}).$$

We define a *spacetime connection* to be a linear connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}$ on the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$ such that the following conditions hold:

1. K has vanishing torsion;
2. K is dt-preserving, i.e. $\nabla dt = 0$.

[2]We stress that in the standard relativity literature the word 'observer'usually denotes just a single time–like curve, instead of a time like flow.

It turns out that the coordinate expression of K is of the type

$$K = d^{\lambda} \otimes \left(\partial_{\lambda} + \left(K_{\lambda j}{}^{i} \dot{y}^{j} + K_{\lambda 0}{}^{i} \dot{x}^{0} \right) \dot{\partial}_{i} \right) , \qquad 0 \leq \lambda \leq 3 ;$$

here, the above two conditions on K are equivalent respectively to $K_{\lambda\mu}{}^{0} = 0$ and $K_{\lambda\mu}{}^{i} = K_{\mu\lambda}{}^{i}$, for $0 \leq \lambda, \mu \leq 3$.

It can be shown (see [9], [10]) that there is a natural bijection between spacetime connections and torsion free affine connections

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}$$

on the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ (see [9] for a general definition of torsion). Such a bijection is characterised by the equation $\Gamma_{\lambda\mu}{}^{i} = K_{\lambda\mu}{}^{i}$.

Assumption G.3 We assume $\boldsymbol{E}$ to be endowed with a spacetime connection K.

We say K[3] to be the *gravitational field*.

Now that we have introduced the main objects of our theory, we have to establish relations between them, namely the field equations. We will state these equations by means of the following natural maps:

$$\gamma := Д \circ \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E} ,$$
$$\Omega := \nu_{\Gamma} \overline{\wedge} \vartheta : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{A}) \otimes \overset{2}{\wedge} T^* J_1\boldsymbol{E} ,$$

where γ is a connection on the bundle $J_1\boldsymbol{E} \to \boldsymbol{T}$, ν_{Γ} is the vertical projection complementary to Γ, and $\overline{\wedge}$ stands for wedge product followed by a metric contraction. The coordinate expression of γ and Ω turns out to be:

$$\gamma = u^0 \otimes \left(\partial_0 + y_0^i \partial_i + \gamma^i \partial_i^0 \right) , \quad \text{with} \quad \gamma^i = K_{hk}{}^{i} y_0^h y_0^k + 2K_{h0}{}^{i} y^h + K_{00}{}^{i}$$
$$\Omega = g_{ij} u^0 \otimes \left(d_0^i - \gamma^i d^0 - \left(K_{hk}{}^{i} y_0^k + K_{h0}{}^{i} \right) \vartheta^h \right) \wedge \vartheta^j$$

It can be proved that Ω is the unique scaled two–form on $J_1\boldsymbol{E}$ induced naturally by g and Γ (see [8]).

Assumption G.4 We assume that the following *first field equation* holds on $\boldsymbol{E}$ for g and K:

$$d\Omega = 0 .$$

[3] In this paper we will not consider the electromagnetic field, because it yields no nontrivial contribution to our search for exact solutions. Henceforth, we will not use the superscript ♮ to label the gravitational field and related objects, as is done in [10].

It can be seen [10] that the first field equation is equivalent to the system:

$$\nabla^K \overline{g} = 0\,, \qquad R^{i\ j}_{\lambda\ \mu} = R^{j\ i}_{\mu\ \lambda}\,.$$

The first condition reads in coordinates as

$$K_{hik} = -\frac{1}{2}(\partial_h g_{ik} + \partial_k g_{ih} - \partial_i g_{hk})\,, \qquad K_{0ij} + K_{0ji} = -\partial_0 g_{ij}\,,$$

and implies that the restriction of K on each fibre to coincide with the Riemannian connection induced by g.

The above assumption can be interpreted also through the observers as follows. Let o be an observer, and $\Phi = 2o^*\Omega$. Then [9], the first field equation is equivalent to the system

$$\nabla^{K'} \overline{g} = 0\,, \qquad d\Phi = 0\,.$$

The second condition can be interpreted as the existence of a local potential $a : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*\boldsymbol{E}$ for Φ, i.e. $\Phi = 2da$. We remark that, if (x^0, y^i) is an adapted chart in which the coordinate expression of the observer is $o = d^0 \otimes \partial_0$ (such a chart always exists), then we have:

$$\Phi = -2d^0 \otimes (K_{0j0} d^0 \wedge d^j + K_{ij0} d^i \wedge d^j)\,.$$

Our next assumptions express the coupling of the gravitational field with the matter sources; this is done by comparing the gravitational Ricci tensor with the energy tensor.

We define the *energy tensor* to be a given section

$$\mathrm{T} : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}\,.$$

The energy tensor can often be expressed in terms of a coupling constant. Namely, if we set $\mathbb{M}$ to be the positive space that represents mass units, then we assume that a *gravitational coupling constant* $\mathrm{K} \in \mathbb{T}^{-2} \otimes \mathbb{A}^{\frac{3}{2}} \otimes \mathbb{M}^{-1}$ is given.

Assumption G.5 We assume the following *second field equation* to hold on $\boldsymbol{E}$ for g, K and T:

$$r = \mathrm{T}\,, \tag{2}$$

where r is the Ricci tensor of K.

Remark 1.2 Just as an example, in the case in which the matter source is constituted by an incoherent fluid, the energy tensor is given by

$$\mathrm{T} := \mathrm{K}\boldsymbol{\mu}\,,$$

where and $\boldsymbol{\mu} : \boldsymbol{E} \to \mathbb{A}^{-\frac{3}{2}} \otimes \mathbb{M}$ is a *mass density*.

Anyway, we are interested in exact solutions in the vacuum, so we will take $\mathrm{T} = 0$. In this case, the vertical restriction of equation (2) yields fibres of $\boldsymbol{E}$ to be Ricci flat Riemannian manifolds, and hence flat, due to the fact that the dimension of the fibres is 3.

Assumption G.6 *(Generalised Newton's law of motion)* We assume the *law of motion* for a particle, whose motion is $s : \boldsymbol{T} \to \boldsymbol{E}$, to be the equation

$$\nabla^\gamma j_1 s = 0 \,.$$

Accordingly, a motion s is said to be *Newtonian* if it fulfills the above assumption. It is important to note that a motion s is Newtonian if and only if $\nabla_{Ts} Ts = 0$, exactly as for Einstein general relativity's law of motion.

In coordinates, a motion s is Newtonian if the following equation holds:

$$\partial^2_{00} s^i - (K_{hk}{}^i \circ s)\partial_0 s^h \partial_0 s^k - 2(K_{0h}{}^i \circ s)\partial_0 s^h - (K_{00}{}^i \circ s) = 0 \,.$$

Quantum theory

Now, we give the assumptions for the quantum theory of a scalar charge–free particle of given mass $m \in \mathbb{M}$.

Assumption G.7 We assume the *quantum bundle* to be a complex line–bundle $\boldsymbol{Q} \to \boldsymbol{E}$ endowed with a Hermitian fibre metric h.

Quantum histories are represented by *quantum sections* $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$.

A typical (complex) linear normal chart on the fibres of $\boldsymbol{Q}$ will be denoted by z, and the corresponding base by b. The coordinate expression of a quantum section turns out to be $\Psi = \psi b$.

We assume the universal unit of measurement of quantum theory to be the given *Planck's constant* $\boldsymbol{\hbar} \in (\mathbb{T}^+)^* \otimes \mathbb{A} \otimes \mathbb{M}$. If $u_0 \in \mathbb{T}^+$, then we set $\hbar := \boldsymbol{\hbar}(u_0)$.

The only object that we need to postulate on $\boldsymbol{Q}$ is the *quantum connection* Ч. Let us denote by И the *Liouville form* of $\boldsymbol{Q}$, i.e.

$$\text{И} : \boldsymbol{Q} \to V\boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \,.$$

Assumption G.8 We assume that a connection Ч on the bundle

$$\boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$$

is given with the following properties:

1. Ч is linear Hermitian;
2. Ч is *universal* (see [17] for the definition);
3. the curvature of Ч fulfills the equation:

$$R_{\text{Ч}} = i\frac{m}{\boldsymbol{\hbar}}\Omega \otimes \text{И} : \boldsymbol{Q}^\uparrow \to \overset{2}{\wedge} T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} \boldsymbol{Q}^\uparrow \,,$$

The requirement of universality of $Ч$ is equivalent to the statement that $Ч$ can be seen as a *system of connections* $\xi_Ч : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$ on $\boldsymbol{Q} \to \boldsymbol{E}$, i.e. a family of connections on $\boldsymbol{Q} \to \boldsymbol{E}$ parametrised by observers.

It turns out (see [10]) that, chosen an observer o and a normal base on $\boldsymbol{Q}$, we have the coordinate expression with respect to a chart adapted to o:

$$Ч = d^\lambda \otimes \partial_\lambda + d_0^i \otimes \partial_i^0 + i\frac{m}{\hbar}\left(-\frac{1}{2}g_{ij}y_0^i y_0^j d^0 + g_{ij}y_0^i d^j + a_\lambda d^\lambda\right) \otimes И\,,$$

where $a = a_\lambda u^0 \otimes d^\lambda : \boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E}$ is a distinguished choice (determined by $Ч$) of a (local) potential of Φ, that is dependent only on the observer o, and not on the adapted chart.

Hence, the components of $Ч$ are given by $Ч_0 = -H/\hbar$, $Ч_j = p_j/\hbar$ and $Ч_j^0 = 0$. Here we have set $H = \frac{1}{2}g_{ij}y_0^i y_0^j - m a_0$ and $p_j = m g_{ij} y_0^i + m a_j$. H and p are the classical observer–dependent energy and momentum of the particle; they are also dependent on the choice of a. Equation $Ч_j^0 = 0$ just expresses the universality of the connection.

The quantum connection allows us to perform covariant derivatives of quantum sections by means of pull–back. Accordingly, if Ψ is a quantum section, then we define the two natural maps

$$\begin{aligned}
\overset{\circ}{\mathcal{L}}_\Psi &:= \frac{1}{2}\left(h\left(\Psi, i\overset{\circ}{\nabla}\Psi\right) + h\left(i\overset{\circ}{\nabla}\Psi, \Psi\right)\right)\nu \;:\; J_1\boldsymbol{E} \to \mathbb{L}^{\frac{3}{2}} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}\,,\\
\check{\mathcal{L}}_\Psi &:= \frac{1}{2}\frac{\boldsymbol{\hbar}}{m}\left((\bar{g} \otimes h)\left(\check{\nabla}\Psi, \check{\nabla}\Psi\right)\right)\nu \;:\; J_1\boldsymbol{E} \to \mathbb{L}^{\frac{3}{2}} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}\,.
\end{aligned}$$

Here, $\overset{\circ}{\nabla}$ and $\check{\nabla}$ are, respectively, the horizontal and vertical projection of the covariant derivative induced by $Ч$, and

$$\nu := \sqrt{|g|}u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 \;:\; \boldsymbol{E} \to \mathbb{T} \otimes \mathbb{L}^{\frac{3}{2}} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}$$

is the canonical scaled volume form on $\boldsymbol{E}$.

In order to obtain a Lagrangian density defined on $J_1\boldsymbol{Q}$ from the above two canonical maps, it can be proved that the linear combination $\mathcal{L}_\Psi = \overset{\circ}{\mathcal{L}}_\Psi - \check{\mathcal{L}}_\Psi$ is the unique one (up to a scalar factor) that projects on $\boldsymbol{E}$. So, $\mathcal{L}_\Psi$ induces the fibred morphism over $\boldsymbol{E}$

$$\mathcal{L} \;:\; J_1\boldsymbol{Q} \to \mathbb{L}^{\frac{3}{2}} \otimes \overset{4}{\wedge} T^*\boldsymbol{E}\,, \tag{3}$$

whose coordinate expression is

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}\Bigg(-\frac{\hbar}{m}g^{ij}\bar{z}_i z_j - i\left(\bar{z}_0 z - \bar{z}z_0\right) + \\
&+ i a^i\left(\bar{z}_i z - \bar{z}z_i\right) + \frac{m}{\hbar}\left(2a_0 - a_i a^i\right)\bar{z}z\Bigg)\sqrt{|g|}d^0 \wedge d^1 \wedge d^2 \wedge d^3\,.
\end{aligned}$$

We assume $\mathcal{L}$ to be the Lagrangian density of quantum theory. The corresponding Euler–Lagrange equation turns out to be a generalised Schroedinger equation for a (spinless and chargeless) quantum particle of given mass m, in a curved spacetime with absolute time, under the action of a given gravitational field.

We have a distinguished Lie algebra of functions defined on $J_1\boldsymbol{E}$, namely the *algebra of quantisable functions*, constituted by a certain kind of polynomials of second order in the velocities. The quantum connection yields a Lie algebra isomorphism of this algebra into a distinguished algebra of vector fields on $\boldsymbol{Q}$, namely the *algebra of quantum vector fields*. Then, it is possible to introduce quantum operators, and, among them, the Schroedinger operator. In this way, we have a full implementation of the correspondence principle in a covariant formulation involving a curved spacetime. It is out of the aims of this paper to go into details in this direction; the interested reader can consult [3], [9], [10] for a complete treatment of the subject.

2 Spherically symmetric exact solutions

In this section we introduce a definition of spherical symmetry in Galilei relativity that is clear and unambiguous from the mathematical viewpoint, and also coincides with the intuitive physical idea of spherical symmetry.

Classical theory

Assumption S.1 We assume spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ to be a bundle.

Thus, fibres turn out to be all diffeomorphic, and (see 1.1) the family of complete observers on $\boldsymbol{E}$ is not empty.

Assumption S.2 We assume that a section $c : \boldsymbol{T} \to \boldsymbol{E}$ is given.

From the physical viewpoint, the above section represents the worldline of a particle, which will play the role of center of symmetry.

Assumption S.3 For all $\tau \in \boldsymbol{T}$ each geodesic in $\boldsymbol{E}_\tau$ starting from $c(\tau)$ admits a parametrisation in $I\!R$.

As a first consequence, each fibre of $\boldsymbol{E}$ turns out to be a complete, and hence inextendable, Riemannian manifold (see Appendix 2).

We define the *distance function from* $c(\boldsymbol{T})$ to be the map

$$\boldsymbol{r} : \boldsymbol{E} \to \mathbb{L} : e \mapsto r(e) := d_{t(e)}(c(t(e)), e) ,$$

where $d_{t(e)}$ is the distance function induced by the Riemannian metric in $\boldsymbol{E}_{t(e)}$ (see Appendix 2), and $\mathbb{L} := \mathbb{A}^{\frac{1}{2}}$ is the positive space that represents length units. Accordingly, a *length unit of measurement* is defined to be an element $l_0 \in \mathbb{L}$.

As it is known (see Appendix 2), r is a continuous function on each fibre of $\boldsymbol{E}$. But there could be fibres of $\boldsymbol{E}$ on which r is bounded. Such fibres would be geodesic spheres, and on each one of this fibers there would also be a point where r would have a maximum. So, our intuitive view of spherical symmetry would be preserved, in this case, if, for example, such fibres be diffeomorphic to the sphere S^3. In fact, there would be the same extremal point of the distance function along all directions.

Anyway, we would like to devote ourselves to the study of the simplest and most physically significant cases of spherically symmetric spacetimes; this is the reason why we introduce the following assumptions.

Assumption S.4 We assume that, for all $\tau \in \boldsymbol{T}$, $(\boldsymbol{E}_\tau, g_\tau)$ is spherically symmetric with respect to the point $c(\tau)$.

Remark 2.1 [4] The above assumption, together with the flatness of fibres required by the second field equation (see 1.2), imply that the type fibre of $\boldsymbol{E}$ is non compact. In fact, it is known that any compact 3–manifold which admits an action of $SO(3)$ where at least one orbit is a sphere has a finite cover diffeomorphic to S^3 or $S^2 \times S^1$. Thus its universal cover is not diffeomorphic to $\mathbb{R}^3$. On the other hand, each fibre of $\boldsymbol{E}$ is endowed with a flat Riemannian metric and hence (see [13]) its universal cover is diffeomorphic to $\mathbb{R}^3$. Hence, a compact fibre would lead to a contradiction.

As a direct consequence (see Appendix 2) the fibres of $\boldsymbol{E}$ turn out to be isometric to $\mathbb{R}^n$.

Due to the above remark, we are searching for a natural vector bundle structure for $\boldsymbol{E}$.

We define the *vertical bundle of* $\boldsymbol{E}$ *along* c to be the pullback bundle $c^*V\boldsymbol{E} \to \boldsymbol{T}$. We can endow this bundle with the scaled fibre metric $c^*g : \boldsymbol{T} \to \mathbb{A} \otimes (c^*V\boldsymbol{E} \underset{\boldsymbol{T}}{\otimes} c^*V\boldsymbol{E})$, turning it into a Riemannian vector bundle. Due to the canonical isomorphism $Vc^*V\boldsymbol{E} \simeq c^*V\boldsymbol{E} \underset{\boldsymbol{T}}{\times} c^*V\boldsymbol{E}$, we can also see c^*g as a scaled vertical metric on $c^*V\boldsymbol{E} \to \boldsymbol{T}$.

For all $\tau \in \boldsymbol{T}$, we can introduce the exponential map relatively to the Riemannian manifold $\boldsymbol{E}_\tau$:

$$\exp_{c(\tau)} : V_{c(\tau)}\boldsymbol{E} \to \boldsymbol{E}_\tau \,;$$

it turns out from assumptions (S.3), (S.4), that $\exp_{c(\tau)}$ is a diffeomorphism. Indeed, we have the following stronger result:

Theorem 2.1 *The map exp* $: c^*V\boldsymbol{E} \to \boldsymbol{E}$ *is a bundle isomorphism over* $\mathrm{Id}_{\boldsymbol{T}}$ *that preserves the vertical metrics.*

Proof. In fact, the fibres of $\boldsymbol{E}$ must be flat (see 1.2). Corollary 2.5 ensures that exp restricts to an isometry on each fibre.

We have only to test the differentiability of the map exp. Indeed, exp is the restriction of the flow of the *bundle geodesic spray* [16] induced by the metric g. So, exp is differentiable because locally it represents the solution of a system of second-order ordinary differential equations dependent on a parameter in $\boldsymbol{T}$. $\square$

[4] The author kindly thanks the anonymous referee, who suggested this remark.

Corollary 2.1 *The map* $\exp|_{c(\boldsymbol{T})}$ *endows* $\boldsymbol{E} \to \boldsymbol{T}$ *with the structure of a vector bundle with a scaled fiber metric* $c^* g : \boldsymbol{T} \to \mathbb{A} \otimes (\boldsymbol{E}^* \underset{\boldsymbol{T}}{\otimes} \boldsymbol{E}^*)$*, in which the section* c *turns out to be the zero section.*

Moreover, the map $\boldsymbol{r} : \boldsymbol{E} \to \mathbb{L}$ is $\mathcal{C}^\infty$ on $\boldsymbol{E}' := \boldsymbol{E} \setminus c(\boldsymbol{T})$.

The trivialisation theorem for bundles over contractible bases [20, p.53] tells us that trivialisations preserve structures; more precisely, we can state the following result.

Corollary 2.2 *Let* $(\boldsymbol{P}, h)$ *be a typical fibre of* $\boldsymbol{E} \to \boldsymbol{T}$ *(*h *is a Euclidean metric).*

Then, there exists a family of complete observers o *such that the induced isomorphisms* $\phi^o : \boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{P}$ *restricts to isometries on each fibre.*

A (complete) observer is said to be *isometric* if it yields a (global) metric-preserving splitting of $\boldsymbol{E}$.

Remark 2.2 It is easy to prove that an observer o is isometric if and only if o fulfills the equation

$$L_o \bar{g} = 0 \, ,$$

that in coordinates adapted to o reads as $\partial_0 g^{ij} = 0$.

In the last part of this section we give spherical symmetry assumptions on the spacetime connection.

We would like to allow our solution to have a singularity on $c(\boldsymbol{T})$; henceforth we weaken assumption G.3 in the following way.

Assumption S.5 We assume the spacetime connection K to be defined on the subbundle $\boldsymbol{E}' = \boldsymbol{E} \setminus c(\boldsymbol{T})$ of $\boldsymbol{E}$.

Now, we show a canonical splitting of the space $\boldsymbol{E}'$.

For each $\tau \in \boldsymbol{T}$ we have the '$I\!R^+$–projective'equivalence relation in $\boldsymbol{E}'_\tau$; namely, for all $e, f \in \boldsymbol{E}'_\tau$ we set $e \sim f$ if and only if there exists $k \in I\!R^+$ such that $e = kf$. The quotient set $\boldsymbol{S} := \boldsymbol{E}'/\sim$ can be endowed with the unique bundle structure (over $\boldsymbol{T}$) such that for all $l \in \mathbb{L}$ the canonical inclusions $i_l : \boldsymbol{S} \hookrightarrow \boldsymbol{E}'$ turn out to be bundle morphisms.

Proposition 2.1 *The bundle* $\boldsymbol{E}'$ *splits canonically as follows:*

$$\boldsymbol{E}' \to \mathbb{L} \times \boldsymbol{S} \, . \tag{4}$$

The above splitting can also be seen as the fibred splitting $\mathbb{L} \underset{\boldsymbol{T}}{\times} \boldsymbol{S}$ by regarding $\mathbb{L}$ as the trivial bundle $\boldsymbol{T} \times \mathbb{L} \to \boldsymbol{T}$.

Accordingly, we have the further remarkable splitting:

$$J_1 \boldsymbol{E}' \to (\mathbb{T}^* \otimes T\mathbb{L}) \times J_1 \boldsymbol{S} \, . \tag{5}$$

We stress that the bundle $\boldsymbol{S}$ is a trivial bundle, but it is not equipped with a canonical splitting [20].

As for the vertical metric g, we remark that the splitting (4) is clearly orthogonal. In fact, we have the following intrinsic objects:

1. $g_{\mathbb{L}} : T\mathbb{L} \underset{\mathbb{L}}{\times} T\mathbb{L} \to \mathbb{A} \otimes I\!\!R : (u, v) \mapsto uv$ (see Appendix 2) is the canonical scaled metric on $\mathbb{L}$;
2. $g_{\boldsymbol{S}} : V\boldsymbol{S} \underset{\boldsymbol{S}}{\times} V\boldsymbol{S} \to I\!\!R$ is the unique vertical Riemannian metric such that for all $l \in \mathbb{L}$ the canonical inclusions $i_l : \boldsymbol{S} \hookrightarrow \boldsymbol{E}'$ turn out to be fibred isometric immersions with respect to the scaled vertical Riemannian metrics $l^2 g_{\boldsymbol{S}}$ and g.

It turns out that the metric g splits as:

$$g = g_{\mathbb{L}} + \boldsymbol{r}^2 g_{\boldsymbol{S}} \,.$$

We would like to characterise the form of complete isometric observers with respect to the splitting (4). Let $\mathcal{I}$ be the family of complete isometric observers.

Proposition 2.2 *Let $o \in \mathcal{I}$. Then, with respect to the splittings* (4), (5), *o can be written as:*

$$o = (o_{\mathbb{L}}, o_{\boldsymbol{S}}) \,,$$
$$o_{\mathbb{L}} : \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes T\mathbb{L} \,, \qquad o_{\boldsymbol{S}} : \boldsymbol{S} \to J_1\boldsymbol{S} \,,$$

and the components of o fulfill:

$$o_{\mathbb{L}} = 0 \,, \qquad\qquad L_{o_{\boldsymbol{S}}} \bar{g}_{\boldsymbol{S}} = 0 \,.$$

Proof. In fact, o has a flow of fibred isometries; but such isometries must be constituted by the identity on $\mathbb{L}$ and by a flow of time–dependent isometries of $\boldsymbol{S}$. □

Remark 2.3 Any observer $o \in \mathcal{I}$ gives rise to a splitting $\phi^o : \boldsymbol{E}' \to \boldsymbol{T} \times \boldsymbol{P}'$, where $\boldsymbol{P}' := \boldsymbol{P} \setminus 0$. Moreover, we can define $S := \boldsymbol{P}'/\!\sim$ as in (4), and we obtain the splitting

$$\boldsymbol{P}' \to \mathbb{L} \times S \,. \tag{6}$$

Hence, it turns out from the above proposition that ϕ^o preserves the splitting (4), yielding a splitting of $\boldsymbol{S}$:

$$\phi^o = (\phi^o_{\mathbb{L}}, \phi^o_{\boldsymbol{S}}) \,, \tag{7}$$
$$\phi^o_{\mathbb{L}} = \mathrm{Id}_{\mathbb{L}} \times \mathrm{Id}_{\boldsymbol{T}} \,, \qquad \phi^o_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{T} \times S \,. \tag{8}$$

In coordinate expressions we will make use of the splitting of $\boldsymbol{E}$ induced by an observer $o \in \mathcal{I}$, employing adapted charts of the type $(x^0; r, y^\alpha)$. In particular, we will use on $\mathbb{L}$ the canonical coordinate r, obtained by composing the function $\boldsymbol{r} : \boldsymbol{E}' \to \mathbb{L}$ with a standard isomorphism $\mathbb{L} \to I\!\!R^+$ given by the choice of a length unit of measurement $l_0 \in \mathbb{L}$. Moreover, (x^0, y^α) will be an adapted coordinate system on $\boldsymbol{S}$, with α, β running from 1 to 2. For convenience we will choose as (y^α) spherical coordinates; in this way the Christoffel symbols of the metric part of K will take the usual form.

Now, we are looking for a spherical symmetry assumption on gravitational fields. Such an assumption can be given only through an observer; indeed, we could characterise a spherically symmetric field only by means of the trajectory of test particles (i.e., particles whose mass is so small that it does not perturb the gravitational field). But this makes sense only with respect to a splitting of $\boldsymbol{E}$ induced by an observer. Also, due to the splitting (5), while it is possible to speak of zero velocity along $\mathbb{L}$ of a test particle, this makes no longer sense in the case of velocities along $\boldsymbol{S}$ (i.e., radial velocities). Moreover, this splitting tells us that there is no way to isolate components in K that are responsible of rotation of test particles.

We have at our disposal the distinguished family of complete isometric observers $\mathcal{I}$; this family actually is just the family of all complete observers that preserve the structure that we have on the spacetime $\boldsymbol{E}$, namely the Riemannian vector bundle structure. So, it is natural to postulate next assumption by means of observers in $\mathcal{I}$.

We introduce two categories of observers in $\mathcal{I}$; such observers will be characterised by means of symmetries of test particle motions. Indeed, requiring the existence of such observers is an assumption on the gravitational field K.

Definition 2.1 An observer $o \in \mathcal{I}$ is said to be *radially symmetric* if, with respect to the splitting $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$ induced by o, the Newtonian motion z with initial conditions

$$z(\tau_0) = (\tau_0, l_0, s_0) \in \boldsymbol{T} \times \mathbb{L} \times S \quad \text{and} \quad j_1 z(\tau) = v_0 \in \mathbb{T}^* \otimes T_{l_0}\mathbb{L}$$

fulfills the following requirements:

1. z is defined on the whole $\boldsymbol{T}$;
2. with respect to the splittings (5), (6), we have

$$z : \boldsymbol{T} \to \boldsymbol{T} \times \mathbb{L} \times S : (\tau) \mapsto (\tau, l(\tau), s_0)\,, \quad j_1 z : \boldsymbol{T} \to \mathbb{T}^* \otimes T\mathbb{L}\,,$$

   where $l : \boldsymbol{T} \to \mathbb{L}$ does not depend on the initial condition $s_0 \in S$.

Definition 2.2 An observer $o \in \mathcal{I}$ is said to be *rotationally symmetric* if, with respect to the splitting $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$ induced by o (see (8)), the motions z, z' with initial conditions respectively

$$\begin{aligned} z(\tau_0) &= (\tau_0, l_0, s_0) \in \boldsymbol{T} \times \mathbb{L} \times S \qquad & j_1 z(\tau) &= v_0 \in \mathbb{T}^* \otimes T_{s_0}S \\ z'(\tau_0) &= (\tau_0, l_0, s'_0) \in \boldsymbol{T} \times \mathbb{L} \times S \qquad & j_1 z'(\tau) &= v'_0 \in \mathbb{T}^* \otimes T_{s'_0}S \end{aligned}$$

and such that $\|v_0\| = \|v'_0\|$ fulfill the following requirement: any isometry $I : S \to S$ such that $TI_\tau(s, v_0) = (s', v'_0)$ maps the motion z into the motion z', i.e.:

$$z' = (\mathrm{Id}_{\mathbb{L}}, I) \circ z\,.$$

The spacetime connection K is said to be *spherically symmetric* if there exists an observer $o \in \mathcal{I}$ that is both radially and rotationally symmetric. Accordingly, such an observer is said to be *spherically symmetric.*

Assumption S.6 We assume that the spacetime connection K is spherically symmetric.

The analysis of the coefficients of K yields one of the main result.

We denote by K^o the flat connection on $\boldsymbol{E}$ with respect to the affine structure induced by an observer $o \in \mathcal{I}$ on $\boldsymbol{E}$.

Theorem 2.2 *Let o be an observer with the properties required in Assumption S.6. Then K is uniquely determined (up to a time-dependent factor); namely, we have:*

$$\begin{gathered} K = K^o + dt \otimes dt \otimes N\,, \\ N\,:\boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes T\mathbb{L} \end{gathered} \tag{9}$$

where:

$$N = \frac{\boldsymbol{k}}{\boldsymbol{r}^2}, \qquad \boldsymbol{k}\,:\boldsymbol{T} \to \mathbb{T}^{-2} \otimes \mathbb{L}^3$$

Proof. The first field equations (1) imply that $K_{0j}^{\ i} = -K_{0i}^{\ j}$, hence we have $K_{0r}^{\ r} = K_{0\alpha}^{\ \alpha} = 0$.

Moreover, being $K_{rr}^{\ r} = 0$ the equation of motion along $T\mathbb{L}$ takes the form

$$\partial^2_{00} l^r - K_{00}^{\ r} = 0$$

for a motion l with initial velocity in $T\mathbb{L}$, hence $K_{00}^{\ r}$ is defined on $\boldsymbol{T} \times \mathbb{L}$ due to Assumption S.6. The equation of motion along TS turns out to be:

$$2K_{0r}^{\ \alpha}\partial_0 l^r = -K_{00}^{\ \alpha}\,.$$

At each point the above equation must hold for all values of $\partial_0 l^r$, yielding $K_{0r}^{\ \alpha} = K_{0\alpha}^{\ r} = K_{00}^{\ \alpha} = 0$.

Let us come back to the first field equations. We see that $d\Phi = 0$ is equivalent to the system

$$\begin{gathered} 2\partial_i K_{0\alpha\beta} d^i \wedge d^\alpha \wedge d^\beta = 0 \\ \partial_0 K_{0\alpha\beta} = 0 \end{gathered}$$

where the first equation yields $\partial_r K_{0\alpha\beta} = 0$.

Now, we use the second field equation in order to complete our information.

The component r_{0r} turns out to be identically 0; moreover, we have

$$\begin{gathered} r_{0\theta} = \partial_\phi K_{0\theta}^{\ \phi} = 0\,, \\ r_{0\phi} = \partial_\theta K_{0\phi}^{\ \theta} - \sin\theta\cos\theta K_{0\phi}^{\ \theta} = 0\,. \end{gathered}$$

This system has the general solution $K_{0\phi}^{\theta} = qe^{\frac{1}{2}\sin^2\theta}$, where q is a real constant. From Assumption S.6 (rotational symmetry of o) we can deduce $q = 0$.

Finally,

$$r_{00} = \partial_r K_{00}^{r} + \frac{2}{r} K_{00}^{r} = 0$$

has the general solution $K_{00}^{r} = \frac{k}{r^2}$, with $k : \boldsymbol{T} \to I\!R$.

The result is obtained by setting $\boldsymbol{k} = k(u^0)^2 \otimes (l_0)^3$. □

Remark 2.4 It turns out that K is a *Newtonian connection* in the sense of [10]. It is easy to see that, in this case, the form Ω splits as follows:

$$\Omega = \Omega^o + \Omega^N \,, \tag{10}$$

$$\Omega^o := \nu_{\Gamma^o} \overline{\wedge} \vartheta \,, \qquad \Omega^N := -N \overline{\wedge} \vartheta \,. \tag{11}$$

Moreover, we obtain

$$\Omega^N = 2o^*\Omega = \Phi \,:\, \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \overset{2}{\wedge} T^*(\boldsymbol{T} \times \mathbb{L}) \,.$$

We have a stronger characterisation of a spherically symmetric connection. Indeed, next result shows the uniqueness of the spherically symmetric observer.

Corollary 2.3 *Let K be a spherically symmetric connection, and $o \in \mathcal{I}$ a corresponding spherically symmetric observer. Then*

1. *K^o does not depend on the choice of o;*
2. *there exists a unique spherically symmetric observer.*

Proof.

1. From the identity $K^o = K - N$ we deduce that the connection K^o is observer–independent.
2. Let us denote by $K^{\|}$ the unique flat connection associated with a spherically symmetric connection K. There is a unique affine structure on $\boldsymbol{E}$ such that $K^{\|}$ is the induced canonical flat connection. Due to the fact that each spherically symmetric observer induces a different affine structure on $\boldsymbol{E}$, by the first part of the proof we deduce that there exists exactly one spherically symmetric observer.

□

Remark 2.5 We stress that the connection K is not flat; in fact, we have (up to a pull–back):

$$R = \frac{\boldsymbol{k}}{\boldsymbol{r}^3} \left(2dt \otimes dt \wedge \mathrm{Id}_{T\mathbb{L}} - dt \otimes dt \wedge \mathrm{Id}_{TS}\right) \neq 0 \,.$$

With respect to o, the law of motion for a motion $s : \boldsymbol{T} \to \boldsymbol{E}$ takes the form:

$$\nabla^{K^o}_{Ts} Ts = N \circ s \, .$$

The above equation can be seen as a reinterpretation of the classical Newton's gravitational force in terms of a non–vanishing curvature on a topologically trivial spacetime.

Even if our section c is the zero section of $\boldsymbol{E}$, it makes no sense to ask if c is a Newtonian motion, because K is not defined on $c(\boldsymbol{T})$.

A confrontation with the classical Newton's law of gravitation suggests that, if we assume for the particle a mass $m \in \mathbb{M}$, then we can assume

$$\boldsymbol{k} = \mathrm{K} m \, .$$

Remark 2.6 We could prove that there exists a unique *freely falling observer* that is determined by the radial velocity of freely falling particles starting from the infinity with radial velocity 0 and approaching c. It turns out that this observer is no longer isometric, even if it is radially and rotationally symmetric.

The final result concerns the existence of spherically symmetric connections.

Theorem 2.3 *There exists a spherically symmetric connection K.*

PROOF. In fact, $\mathcal{I}$ is a non–empty family (see [20]). If $o \in \mathcal{I}$, then we can define the field K^o, choose a function $\boldsymbol{k}$ in order to define the field N, and we are done by means of an obvious pull–back. □

We have found that, under the simplest and more intuitive hypotheses of spherical symmetry, spacetime is trivial from the topological viewpoint. Moreover, we have found that there is a unique spherically symmetric gravitational field, together with a unique observer with respect to which motions have spherically symmetric trajectories. The gravitational field is a geometrical version of the classical Newton's law of gravitation; the 'force'of the field is seen as the non vanishing curvature of the connection.

Remark 2.7 By means of the objects that we have found, we can build a Lorentz metric on $\boldsymbol{E}'$. In fact, let us consider the following potential of the 2–form Φ:

$$a = -(\mathrm{K} m)/\boldsymbol{r} \, : \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{T}^* \, . \tag{12}$$

We define the Lorentz metric g_l on $\boldsymbol{E}'$ to be the map

$$g_l = (\theta^* g) \circ o + 2a \otimes dt \; ;$$

here, θ is the epimorphism complementary to $\varPi$ (see equation (1).

In adapted coordinates, the Christoffel symbols of the spherically symmetric connection K are the same of the corresponding ones of the Levi–Civita connection induced by g_l. Moreover, the Einstein tensor G_l induced by g_l turns out to be

$$G_l = \frac{1}{r^2} g_{\mathbb{L}} - \frac{1}{4} g_{\boldsymbol{S}} .$$

Thus, we can interprete the Galilei spherically symmetric spacetime as an Einstein spacetime with the Lorentz metric g_l and the energy tensor $\mathrm{T} = \frac{1}{r^2} g_{\mathbb{L}} - \frac{1}{4} g_{\boldsymbol{S}}$. See [22], [23] for a deeper analysis.

Quantum theory

The assumptions given for the classical theory are not sufficient to provide spherical symmetry conditions for the quantum theory. In this section, we introduce hypotheses of spherical symmetry on the quantum bundle and the quantum connection by means of the following criterion: we assume that the quantum bundle and the quantum connection split as the spacetime in (4) and the gravitational field in (9). In this way, we give spherical symmetry conditions in the quantum theory by requiring for the quantum structures the same kind of symmetry found for the classical structures. Such hypotheses are given in practice by adding to assumptions G.7, G.8 respectively the hypotheses of Lemma 2.1 and Lemma 2.2.

The next step is the analysis of the isomorphism classes of spherically symmetric quantum bundles and connections; we find out that there is only the trivial class, hence there is exactly one spherically symmetric quantum theory from the physical viewpoint.

Lemma 2.1 *Let us suppose that, in agreement with the splitting* (4) *of* $\boldsymbol{E}'$*, the restriction* $\boldsymbol{Q}'$ *of the quantum bundle* $\boldsymbol{Q} \to \boldsymbol{E}$ *to the space* $\boldsymbol{E}'$ *be the product*

$$\boldsymbol{Q}' = \boldsymbol{Q}_{\mathbb{L}} \times \boldsymbol{Q}_{\boldsymbol{S}}$$

of two Hermitian complex line bundles $\boldsymbol{Q}_{\mathbb{L}} \to \mathbb{L}$ *and* $\boldsymbol{Q}_{\boldsymbol{S}} \to \boldsymbol{S}$.

Then, there is exactly one class of Hermitian complex line bundle isomorphisms of $\boldsymbol{Q}$*, and there is exactly one class of Hermitian complex line bundle isomorphisms of* $\boldsymbol{Q}_{\mathbb{L}}$.

Proof. In fact, being $\boldsymbol{E}$ and $\mathbb{L}$ contractible spaces, $\boldsymbol{Q}$ and $\boldsymbol{Q}_{\mathbb{L}}$ are trivial bundles. □

From the physical viewpoint, a global trivialisation of $\boldsymbol{Q}$ can be interpreted as a quantum gauge. Choosing the trivial bundle $\boldsymbol{Q} := \boldsymbol{E} \times \boldsymbol{V}$ (where $\boldsymbol{V}$ is a complex Hermitian line) would mean to choose a distinguished quantum gauge, but we have neither physical nor mathematical motivation to assume a priori such a distinguished gauge.

We remark that the bundle $\boldsymbol{Q}_{\mathbb{L}}$ can be seen also as a bundle on $\boldsymbol{T} \times \mathbb{L}$ by means of a pull–back; hence, in what follows, analogously to G.7, $\boldsymbol{Q}^{\uparrow}_{\mathbb{L}}$ will denote the bundle $\mathbb{T}^* \otimes T\mathbb{L} \underset{\boldsymbol{T}\times\mathbb{L}}{\times} \boldsymbol{Q}_{\mathbb{L}} \to \mathbb{T}^* \underset{\boldsymbol{T}\times\mathbb{L}}{\otimes} T\mathbb{L}$.

We define the *kinetic energy* and the *momentum* of the particle with respect to the spherically symmetric observer o to be, respectively, the maps

$$G := \frac{1}{2} m g \circ (\nabla_o, \nabla_o) \; : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{A} \otimes \boldsymbol{M} \; ,$$
$$P := \theta^* \lrcorner (m g^\flat \circ \nabla_o) \; : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \boldsymbol{M} \otimes T^*\boldsymbol{E} \; ,$$

where $\nabla_o : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ is the covariant derivative induced by the observer, defined as $\nabla_o \circ j_1 s = j_1 s - o \circ s$ for each section $s : \boldsymbol{T} \to \boldsymbol{E}$. Their coordinate expressions turn out to be

$$G = \frac{1}{2} m g_{ij} y^i_0 y^j_0 \; , \qquad P = -m g_{ij} y^i_0 y^j_0 d^0 + m g_{ij} y^j_0 d^i \; .$$

Lemma 2.2 *Suppose that the quantum connection* Ч *splits as follows*

$$Ч = Ч^{\parallel} + C^N \; ,$$

where $Ч^{\parallel}$ *is a quantum connection associated with the spacetime connection* $K^{\parallel}$*, and* C^N *is a section*

$$C^N \; : \boldsymbol{Q}^{\uparrow}_{\mathbb{L}} \to T^* \left(\boldsymbol{T} \times \mathbb{L} \times \mathbb{T}^* \otimes T\mathbb{L} \right) \otimes \boldsymbol{Q}^{\uparrow}_{\mathbb{L}} \simeq T^* J_1(\boldsymbol{T} \times \mathbb{L}) \otimes \boldsymbol{Q}_{\mathbb{L}}$$

Then it turns out that C^N *is a Hermitian form; moreover, for any global quantum gauge,* Ч *can be expressed as*

$$Ч = Ч^0 + i \frac{1}{\hbar} (G + P + m a^{\parallel} + m a^N)$$

where $Ч^0$ *is the trivial connection induced by the gauge,* $a^{\parallel}$ *is a potential of the two–form* $\Phi^{\parallel} = 2 o^* \Omega^{\parallel}$*, and* a^N *is a potential of the type* $a^N : \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*(\boldsymbol{T} \times \mathbb{L})$ *of the form* Φ *associated with* o*.*

Thus, in any chart adapted to the gauge and to o*, the coordinate expression of the above* C^N *is of the type*

$$C^N = i \frac{m}{\hbar} (a_0 d^0 + a_r d^r) \otimes И \; ,$$

where $a_0, a_r : \boldsymbol{T} \times \mathbb{L} \to I\!R$ *and* $\partial_0 a_r - \partial_r a_0 = \Phi_{0r}$*.*

Conversely, given a global quantum gauge, any two global potentials $a^{\parallel}$ *of* $\Phi^{\parallel}$ *and* $a^N : \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*(\boldsymbol{T} \times \mathbb{L})$ *of* Φ *yield a quantum connection* Ч *through the above construction.*

PROOF. It follows from a computation in coordinates. □

Our next task is to prove that for any two such quantum connections there exists an automorphism of the quantum bundle that maps one connection into the other.

Lemma 2.3 *Assume the same hypotheses of Lemma 2.1, and choose a global quantum gauge. Let* Ч_1 *and* Ч_2 *be two quantum connections with the properties of Lemma 2.2. Let* $a^{\parallel}_1$, $a^{\parallel}_2$ *be the corresponding global potentials of* $\Phi^{\parallel}$, *and* $a^N_1, a^N_2 : \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*(\boldsymbol{T} \times \mathbb{L})$ *the corresponding global potentials of* Φ.

Then

$$a^{\parallel}_2 = a^{\parallel}_1 - \frac{\boldsymbol{\hbar}}{m}\theta^{\parallel}\,, \qquad a^N_2 = a^N_1 - \frac{\boldsymbol{\hbar}}{m}\theta^N$$

where $\theta^{\parallel} : \mathbb{L} \times \boldsymbol{S} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*(\mathbb{L} \times \boldsymbol{S})$ *and* $\theta^N : \boldsymbol{T} \times \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes T^*(\boldsymbol{T} \times \mathbb{L})$ *are two closed forms.*

Proof. It follows from an easy computation in coordinates. □

Theorem 2.4 *Let* Ч_1, Ч_2 *be two quantum connections with the properties of Lemma 2.2 Then there is a global automorphism of Hermitian complex line bundle* f *of* $\boldsymbol{Q}$ *over* $\boldsymbol{E}'$ *such that* $f^*\text{Ч}_1 = \text{Ч}_2$.

Proof. With the same notation of the above lemma, we see that the function $e^{i\alpha^{\parallel}+i\alpha^N}$, where $d\alpha^{\parallel} = \theta^{\parallel}$ and $d\alpha^N = \theta^N$ gives rise by complex multiplication to a global automorphism f of $\boldsymbol{Q}$ over $\boldsymbol{E}'$, i.e. to a global change of gauge with respect to which we have:

$$a^{\parallel}_2 + a^N_2 = a^{\parallel}_1 + a^N_1 - \frac{\boldsymbol{\hbar}}{m}(\theta^{\parallel} + \theta^N)\,.$$

□

Thus, we have proved the existence of a unique class of physically equivalent quantum structures $(\boldsymbol{Q}, \text{Ч})$ of the above type. It is natural to define any representative of this class as *spherically symmetric quantum structure*. The simplicity of the above results lies on the splittings that we have required for the quantum bundle and the quantum connection, and on the topological triviality of the manifolds $\boldsymbol{E}$ and $\boldsymbol{T} \times \mathbb{L}$.

Of course, we can exhibit the distinguished representative given by the quantum bundle

$$\boldsymbol{Q} = \mathbb{L} \times \boldsymbol{S} \times \mathbb{C}$$

and the quantum potential

$$a = -(\text{K}m)/\boldsymbol{r} \,:\, \mathbb{L} \to \mathbb{T}^* \otimes \mathbb{A} \otimes \mathbb{T}^*\,.$$

Indeed, this spherically symmetric solution of the quantum structure is nothing but the standard one, but within a formulation involving a curved spacetime.

Appendix

Spherically symmetric Riemannian manifolds

The aim of this appendix is to provide a mathematical definition of spherically symmetric Riemannian manifold. This definition turns out to be particularly useful in analysing geometrical properties of a spherically symmetric spacetime.

The proof of the main results relies on some basic facts of Riemannian geometry, that are well-known to mathematicians. Our sources are [13, 6, 12]; we especially used [12] as a textbook on geodesics, and [13] for what concerns isometry groups.

First of all we present an overview of the standard mathematical background together with some useful results.

Throughout this section manifolds are smooth and connected, as in the whole paper.

Let $(\boldsymbol{M}, g)$ be a Riemannian manifold. We recall that the group $\mathcal{I}(\boldsymbol{M})$ of the isometries of $\boldsymbol{M}$ is a Lie group that acts effectively on $\boldsymbol{M}$ on the left [13, p. 239]. The isotropy subgroup $\mathcal{I}_{\surd}(\boldsymbol{M})$ at $p \in \boldsymbol{M}$ turns out to be a compact Lie subgroup of $\mathcal{I}(\boldsymbol{M})$ [13, p. 49].

It is important to note that, given $p \in \boldsymbol{M}$, there is a $\rho \in \mathbb{R}^+$ such that the orthogonal group $O(g_p)$ acts freely (but not necessarily isometrically) on the geodesic spheres centered at p in $\boldsymbol{M}$ of radius $r < \rho$ (*strongly convex neighbourhood*, see [12]). This is due to the fact that there exists a $\rho \in \mathbb{R}^+$ such that the exponential map $\exp_p : T_p\boldsymbol{M} \to \boldsymbol{M}$ induces a diffeomorphism between the balls centered at 0 in in $T_p\boldsymbol{M}$ of radius $r < \rho$ and the geodesic spheres centered at p in $\boldsymbol{M}$ of radius $r < \rho$, but this diffeomorphism needs not to be extendable to the whole $\boldsymbol{M}$.

We are mainly concerned with global properties of the exponential mapping exp.

A Riemannian manifold $(\boldsymbol{M}, g)$ is said to be *complete* if each geodesic admits a geodesic parametrisation on the whole $\mathbb{R}$. It is clear that if $\boldsymbol{M}$ is complete, then exp is defined on the whole $T\boldsymbol{M}$; in general, this map is not injective, particularly in compact manifolds where the existence of nontrivial homotopy groups provides a topological obstruction [12].

It can be seen (see [12]) that the existence of critical points of the map exp is strictly related to the problem of non–uniqueness of minimising geodesics (i.e., geodesics that minimise the canonical length functional).

It is possible to provide a maximal open subset $U_p \subset T_p\boldsymbol{M}$ for each given $p \in \boldsymbol{M}$ that is star–shaped with respect to 0_p and in which $\exp_p$ is a diffeomorphism with an open submanifold $\boldsymbol{M}$ [12], [6], and the unique geodesic connecting p with $q \in \exp_p(U_p)$ turns out to be a minimising geodesic.

The set $C(p) := \exp_p(\partial U_p)$ is defined to be the *cut locus* of $p \in \boldsymbol{M}$. Sard's theorem implies that $C(p)$ has zero measure, but little is known about its geometrical structure, especially in the case of $\boldsymbol{M}$ non–compact. Anyway, it can be easily proved that $\boldsymbol{M}$ consists of the disjoint union of $\exp_p(U_p)$ and $C(p)$ [6, p. 101].

In the non–compact case we can establish the following property [6, p. 91]: for each $p \in \boldsymbol{M}$ there exists a geodesic $c : [0, +\infty) \to \boldsymbol{M}$ starting from p that minimises the distances between p and all points of c. Such a geodesic is called a *ray*.

Finally, we introduce our notion of spherically symmetric (connected) Riemannian manifold.

Definition 2.3 Let $(\boldsymbol{M}, g)$ be a complete Riemannian manifold, and $p \in \boldsymbol{M}$.

We say $\boldsymbol{M}$ to be *spherically symmetric with respect to* p if the local action of $O(g_p)$ on $\boldsymbol{M}$ through the map $\exp_p$ is globally extendable to an action by isometries.

In other words, $\boldsymbol{M}$ is spherically symmetric with respect to p if and only if the exponential mapping yields a group isomorphism $O(g_p) \to \mathcal{I}_{\surd}(\boldsymbol{M})$.

A very interesting property arises when $\boldsymbol{M}$ is non–compact.

Proposition 2.3 *Let $(\boldsymbol{M}, g)$ be a complete, spherically symmetric (with respect to $p \in \boldsymbol{M}$), non–compact Riemannian manifold. Then $C(p) = \emptyset$.*

Proof. Let $c : [0, +\infty) \to \boldsymbol{M}$ be a ray.

Each isometry takes c into another ray; moreover, our hypotheses imply that there is a bijective correspondence between directions in $T_p\boldsymbol{M}$ and geodesic rays, and this yields the result. □

Corollary 2.4 *Assume the same hypotheses of the above proposition. Then* $\exp_p$ *is a diffeomorphism.*

Thus, a complete non–compact Riemannian manifold that is spherically symmetric with respect to one of its points is diffeomorphic to $\mathbb{R}^n$, but not necessarily isometric; anyway, we have a necessary and sufficient condition by which $\exp_p$ be an isometry.

Corollary 2.5 *Assume the same hypotheses of the above proposition. Then* $\exp_p$ *is an isometry if and only if $\boldsymbol{M}$ is flat.*

Proof. In fact, if $\exp_p$ is an isometry, then $\boldsymbol{M}$ is obviously flat.

Conversely, being $\boldsymbol{M}$ flat and diffeomorphic to $T_p\boldsymbol{M}$, $\boldsymbol{M}$ must be isometric to $(\mathbb{R}^n, e)$, where e is the Euclidean metric [6, p.135].

The exponential mapping commutes with isometries [13, p.161]; this yields the map $\exp_0 : T_0\mathbb{R}^n \to \mathbb{R}^n$ that turns out to be the identity map, and hence an isometry. So, the map $\exp_p$ turns out to be an isometry. □

Note also that in the above case, $\boldsymbol{M}$ turns out to be spherically symmetric with respect to all of its points.

Positive spaces

In this section we recall some basic facts on positive spaces. This theory has been developed in [9], [10] in order to make the independence of classical and quantum mechanics from scales explicit.

A *positive space* is defined to be an abelian semigroup $\mathbb{U}$, with $0 \notin \mathbb{U}$, endowed with a scalar multiplication by $\mathbb{R}^+$ such that

$$(r+s)u = ru + su\,, \qquad (rs)u = r(su)\,, \qquad r(u+v) = ru + rv\,, \qquad 1u = u\,,$$

for all $r, s \in \mathbb{R}^+$ and $u, v \in \mathbb{U}$, and $\mathbb{U}$ can be generated over $\mathbb{R}^+$ by a single element.

It can be given the definition of tensor product of positive spaces (over $\mathbb{R}^+$) in a natural way by means of a universal property. Moreover, it can be defined the tensor product (over $I\!R^+$) of a positive space and a vector space; it can be shown that the resulting space has a natural real vector space structure.

From the geometrical viewpoint, a positive space $\mathbb{U}$ turns out to be a manifold diffeomorphic to $I\!R^+$, with tangent space

$$T\mathbb{U} \simeq \mathbb{U} \times (\mathbb{U} \otimes I\!R) .$$

Another interesting feature is that it is possible to give the definition of *square root* of a positive space $\mathbb{U}$. Namely, this is defined to be a positive space $\sqrt{\mathbb{U}}$ together with a quadratic map $q : \sqrt{\mathbb{U}} \to \mathbb{U}$ such that an obvious universal property holds (see [10]). It can be shown that such square roots exist and are unique up to canonical isomorphisms.

More generally, we can define n^{th} roots of a positive space $\mathbb{U}$, and also integer powers (and hence rational powers) by means of tensorial product setting $\mathbb{U}^{-1} := \mathbb{U}^*$. In particular, we have:

$$\mathbb{U}^{\frac{p}{q}} := \mathbb{U}^p \otimes \mathbb{U}^{*q} .$$

We will use the following notational conventions for tensor product and dual: If $\mathbb{U}$ and $\mathbb{V}$ are two unit spaces, and $u \in \mathbb{U}$, $v \in \mathbb{V}$, then we will write

$$uv := u \otimes v \ , \qquad \frac{1}{u} := u^* \ .$$

These conventions will give to all formulae a similar aspect to the corresponding ones used by physicists.

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