Ordinary differential equations described by their Lie symmetry algebra*

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Published in *J. Geom. Phys.*,
85 (2014), 2–15.

Abstract

The theory of Lie remarkable equations, *i.e.*, differential equations characterized by their Lie point symmetries, is reviewed and applied to ordinary differential equations. In particular, we consider some relevant Lie algebras of vector fields on $\mathbb{R}^k$ and characterize Lie remarkable equations admitted by the considered Lie algebras.

Keywords: Symmetries, ordinary differential equations.

MSC 2010 classification: 58J70, 58A20.

1 Introduction

In the context of the geometric theory of symmetries of (systems of) differential equations (DEs) [8, 9, 15, 30, 31], a natural problem is to see when a DE, either partial (PDE) or ordinary (ODE), is uniquely determined by its Lie algebra of point symmetries. The core of this paper is to investigate the inverse problem in the context of ODEs: given a Lie algebra $\mathfrak{s}$ of vector fields, how to construct ODEs having $\mathfrak{s}$ as a Lie point symmetry subalgebra and satisfying some specific properties, which will be clarified below, that ensure the uniqueness of such DE. The idea of describing DEs admitting a given Lie algebra of symmetries dates back at least to S. Lie, who stated that $u_{xx} = 0$ is the unique scalar 2^{nd} order ODE, up to point transformations, admitting an 8-dimensional Lie algebra of symmetries. Of course, a similar idea also applies to PDE: for instance, in [34] (see also [35, 36]), the author proved that the only scalar 2^{nd} order PDE, with an unknown function and two independent variables, admitting the Lie algebra of projective vector fields of $\mathbb{R}^3$ as Lie point symmetry subalgebra is the Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 0$. The above idea plays a central role also in gauge theories, where one wants to obtain information on differential operators possessing a prescribed algebra of symmetries. The results of [24] go in this direction: 2^{nd} order field equations possessing translational and gauge symmetries and the corresponding conservation laws (via Noether theorem) are always derivable from a variational principle.

The standard procedure (also used in [34]) for obtaining a scalar DE admitting a prescribed Lie algebra of symmetries is that of computing the differential invariants of its prolonged action, under some regularity hypotheses; the invariant DE is then described by the vanishing of an arbitrary function of such invariants. If the prolonged action is not regular, invariant DEs can be obtained by a careful study of the singular set of the aforementioned action. An efficient method for obtaining invariant scalar ODEs in the latter case is that of using Lie determinants [32], which we shall employ for our purposes. See [7, 14, 18, 33] for more approaches to the problem. In general, DEs do not possess a sufficient number of independent Lie point symmetries able to characterize them (among the others we recall KdV equation, Burgers' equation, Kepler's equations). In this case, one can ask if they can be characterized by a more general algebra of symmetries. A possible generalization of the concept of Lie remarkable equations is that suggested in [18, 34]: this amounts to extending the category of symmetries used in the definitions of Lie remarkable equations to contact symmetries. For instance, the minimal surface equation of $\mathbb{R}^3$

*Work supported by GNFM, GNSAGA, Università di Messina, Università di Perugia, Università del Salento.

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is completely determined by its contact symmetry algebra [34]. Also, an example of high-order Lie remarkable equation in this 'extended' sense is

$$10u_{(3)}^3u_{(7)} - 70u_{(3)}^2u_{(4)}u_{(6)} - 49u_{(3)}^2u_{(5)}^2 + 280u_{(3)}u_{(4)}^2u_{(5)} - 175u_{(4)}^4 = 0, \tag{1}$$

where $u_{(k)} = d^k u/dx^k$, which possesses a 10-dimensional Lie algebra of contact symmetries (see [12, 32]). Sometimes, in order to completely characterize a given DE, one should also consider non-local symmetries. This is the situation discussed in [17], where the idea of complete symmetry group was proposed and exploited in order to characterize uniquely Kepler's equation. This idea was subsequently exploited by several authors in different ways for characterizing many differential equations [3, 4, 5, 6, 19, 25, 26].

Following the terminology introduced in [21, 22, 23, 28, 29], we call *Lie remarkable* a DE which is completely characterized by its Lie algebra of point symmetries. Of course, this concept needs some cares and comments, which we will give below. Thus, before giving a mathematical definition of it, we have to analyze all the requirements that can make a DE unique, also by means of simple examples. It is well known that, locally, any r^{th} order differential equation $\mathcal{E}$ with n independent variables and m dependent ones can be interpreted as a submanifold of the r-jet $J^r(n,m)$ of the trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Let us denote by $\text{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of $\mathcal{E}$. Thus, when saying that $\mathcal{E}$ is uniquely determined by $\text{sym}(\mathcal{E})$, one should fix, as data of the problem, the number of independent and dependent variables, the order of the DE and its dimension as submanifold. For instance, (see also Section 4.1), the unique 5^{th} order ODE admitting the algebra of projective vector fields of $\mathbb{R}^2$ is equation of item 5 of Theorem 4, but also $u_{xx} = 0$ is the unique 2^{nd} order ODE admitting the algebra of projective vector fields of $\mathbb{R}^2$ as Lie algebra of point symmetries. Remaining in the realm of projective algebra, the system $\{y_{xx} = 0, u_{xx} = 0\}$ is uniquely determined by the 15-dimensional projective Lie algebra of $\mathbb{R}^3$, but, as we already said, also the Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 0$ admits the same 15-dimensional Lie algebra of vector fields as Lie algebra of point symmetries. Both the system $\{y_{xx} = 0, u_{xx} = 0\}$ and $u_{xx}u_{yy} - u_{xy}^2 = 0$ are, in their own class, the only DEs admitting the projective algebra of $\mathbb{R}^3$ as Lie algebra of point symmetries. As the last consideration, we observe that if an equation $\mathcal{E}$ admits a Lie algebra of point symmetries, also an open submanifold of $\mathcal{E}$ admits the same Lie algebra of symmetries, so that when speaking about Lie remarkable equations one should think of them up to inclusion. Bringing all the above observations together, below we formulate a more precise definition of Lie remarkable equations.

Notations and conventions: Throughout the paper, we will use the Einstein summation convention, unless otherwise specified. We will always use the word "symmetry" for "infinitesimal point symmetry". When we speak about a Lie algebra we always mean a *Lie algebra of vector fields* of finite dimension, unless otherwise specified. Finally, if $\mathfrak{s}$ and $\mathfrak{g}$ are Lie algebras, $\mathfrak{s} \leq \mathfrak{g}$ means that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$.

Definition 1. An l-dimensional r^{th} order equation $\mathcal{E} \subset J^r(n,m)$ is called *Lie remarkable* if it is the only l-dimensional r^{th} order equation in $J^r(n,m)$, up to inclusion and up to point transformations, admitting $\text{sym}(\mathcal{E})$ as a Lie symmetry subalgebra.

Below we will shed light on the above definition by means of a simple example. Equation

$$\mathcal{E}_1 : \; u_{xx} = \frac{1}{2}u_x + e^{-2x}u_x^3$$

is not Lie-remarkable. In fact $\text{sym}(\mathcal{E}_1)$ is linearly generated by

$$\partial_u\,, \quad \partial_x + u\partial_u\,, \quad u\partial_x + \frac{u^2}{2}\partial_u \tag{2}$$

but also the equation

$$\mathcal{E}_2 : \; u_{xx} = \frac{1}{2}u_x$$

admits $\text{sym}(\mathcal{E}_1)$ as a Lie subalgebra of its Lie symmetry algebra. Indeed, $\text{sym}(\mathcal{E}_1) \lneq \text{sym}(\mathcal{E}_2)$, as $\text{sym}(\mathcal{E}_2)$ is isomorphic to the projective Lie algebra of $\mathbb{R}^2$. Thus, $\mathcal{E}_1$ and $\mathcal{E}_2$ are not equivalent. To conclude, $\mathcal{E}_1$ is not Lie-remarkable, whereas $\mathcal{E}_2$ is.

Of course, an abstract Lie algebra can be realized, in terms of vector fields, in different non-equivalent ways. For instance, S. Lie [20] investigated the possible realizations of the non-commutative Lie algebra of dimension 2 (i.e., the Lie algebra spanned by two elements X and Y such that $[X,Y] = X$) as Lie algebra of vector fields on $\mathbb{R}^2$. He showed that, almost every point of $\mathbb{R}^2$ has a neighborhood on which there are coordinates x, u in which

$$1.\, \{X, Y\} = \{e^u\partial_u\,, -\partial_u\} \;\text{ or }\; 2.\, \{X, Y\} = \{\partial_u\,, \partial_x + u\partial_u\}. \tag{3}$$

Of course, realizations (3) are not equivalent, as the orbits of the first realization are 1-dimensional whereas the orbits of the second one are 2-dimensional.

Definition 2. We say that an l-dimensional r^{th} order equation $\mathcal{E} \subset J^r(n,m)$ is *associated* with a Lie algebra of vector fields $\mathfrak{s}$ if it is the only l-dimensional r^{th} order equation in $J^r(n,m)$, up to inclusion and up to point transformations, admitting $\mathfrak{s}$ as a Lie subalgebra of $\text{sym}(\mathcal{E})$.

We would like to stress that, in Definition 2, choosing a realization of the Lie algebra in terms of vector fields is crucial, otherwise the definition of a DE associated with a Lie algebra would not be well posed. Indeed, non-equivalent realizations of the same (abstract) Lie algebra lead, in general, to different DEs. For instance, let us we consider the non-commutative Lie algebra of dimension 2 and its realizations 1. and 2. of (3). The most general 2^{nd} order ODE having 1. of (3) as a Lie symmetry subalgebra is

(4) $$u_{xx} = f(x)u_x + u_x^2$$

whereas, if we consider 2. of (3), we obtain

(5) $$u_{xx} = g\left(\frac{u_x}{u}\right) u.$$

Finally we note that equations (4) are point-equivalent, for any $f \in C^\infty(\mathbb{R})$, to $u_{xx} = 0$ as they possess an 8-dimensional Lie algebra of point symmetries (more directly, one can easily check that the Liouville-Cartan invariant vanishes), whereas equations of type (5) are not point equivalent each other. In fact, equation (5) with $g\left(\frac{u_x}{u}\right) = \left(\frac{u_x}{u}\right)^4$ cannot be linearizable as the only 2^{nd} order ODEs which can have this property are of type $u_{xx} = h(x,u,u_x)$ with $h_{u_x u_x u_x u_x} = 0$. Thus, we can say that equation $u_{xx} = 0$ is associated with Lie algebra of vector fields 1. of (3), whereas there are no 2^{nd} order ODEs associated with Lie algebra of vector fields 2. of (3).

A first consequence of Definition 2 is the following obvious proposition.

Proposition 3. *If the equation $\mathcal{E}$ is associated with $\mathfrak{s} \leq \text{sym}(\mathcal{E})$, then it is associated with any subalgebra $\widetilde{\mathfrak{s}}$ of $\text{sym}(\mathcal{E})$ such that $\mathfrak{s} \leq \widetilde{\mathfrak{s}}$.*

We remark that DEs of different order can be associated to the same Lie algebra of vector fields. From the above discussions it is clear that a Lie remarkable equation needs a Lie algebra of point symmetries of suitable dimension: in Section 3.3 we show that, in the case of scalar ODEs, this leads also to the existence of first integrals.

In the present paper we shall construct, in an algorithmic way, (system of) ODEs associated with relevant Lie algebras of vector fields on $\mathbb{R}^k$ by using sufficient conditions contained in Section 3 (more precisely, Propositions 7, 9 and 10). As first step, we obtain scalar Lie remarkable ODEs by means of the local classification of primitive Lie algebras of vector fields on $\mathbb{R}^2$ (a list of such Lie algebras of vector fields can be found in [32]). Note that they include the euclidean, affine, special conformal and projective Lie algebra of $\mathbb{R}^2$. Then we concentrate on the computations of Lie remarkable systems of ODEs. Below we give the main theorems.

Theorem 4. *Lie remarkable scalar ODEs associated with primitive Lie algebras of vector fields on $\mathbb{R}^2$ are listed below (we refer to table* (13)*) :*

1. *There are no Lie remarkable scalar ODEs associated with Lie algebras **I**, **II** and **III**;*
2. *with algebras **IV** or **V** it is associated the equation of straight lines $u_{xx} = 0$;*
3. *with Lie algebra **VI** it is associated the equation $u_{xx} = 0$ and the equation of the vanishing affine curvature $3u_{xx}u_{xxxx} - 5u_{xxx}^2 = 0$;*
4. *with Lie algebra **VII** it is associated the equation the equation of circles $(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2 = 0$;*
5. *with Lie algebra **VIII** it is associated the equation $u_{xx} = 0$ and the equation of conic sections $9u_{xxxxx}u_{xx}^2 + 40u_{xxx}^3 - 45u_{xx}u_{xxx}u_{xxxx} = 0$.*

Equation of item 3 is also known as *generalized Kummer-Schwartz equation* (see [19] for a discussion of this topic). Equation of item 4 can be realized as the vanishing of total derivative of the euclidean curvature $u_{xx}(1+u_x^2)^{-\frac{3}{2}}$ of the curve $u = u(x)$. As regard to equation of item 5, it was somehow expectable to obtain it: indeed **VIII** is the projective Lie algebra of $\mathbb{R}^2$ and a projective transformation sends a conic curve into a conic curve.

For what concerns systems of ODEs, by means of the methods described above, we found Lie remarkable systems of ODEs in 2 dependent variables associated with euclidean, affine, conformal and projective Lie algebra of $\mathbb{R}^3$. We summarize our results in the theorem below.

Theorem 5. *Lie remarkable systems of ODEs in 2 dependent variables associated with isometry, affine, special conformal and projective Lie algebra of the euclidean space $\mathbb{R}^3$ are, respectively, listed below*

1. *with the isometry Lie algebra it is associated the system of straight lines* $\{u^k_{xx} = 0,\ k = 1, 2\}$ *;*
2. *with the affine Lie algebra it is associated the system* $\{u^k_{xx} = 0,\ k = 1, 2\}$ *and the system appearing in section 4.2.2;*
3. *with the conformal Lie algebra it is associated the system of circles* $\{(1 + \sum_j (u^j_x)^2)u^k_{xxx} = 3u^k_{xx} \sum_j u^j_x u^j_{xx}\ ,\ k = 1, 2\}$*;*
4. *with the projective Lie algebra it is associated the system* $\{u^k_{xx} = 0,\ k = 1, 2\}$ *and the system appearing in section 4.2.4.*

We see that the equation/system of lines in the euclidean space appears many times in the Theorems above in view of Proposition 3. We underline that it is known that a system of ODEs possessing a Lie symmetry algebra of maximal dimension (i.e. $m^2 + 4m + 3$) is point-equivalent to the system of lines [13]. In this respect, the result of item 1 of Theorem 5 is somehow unexpected: note that, in the scalar case, a 2^{nd} order ODE which admits the 3-dimensional Lie algebra of infinitesimal isometries of the euclidean space as Lie symmetry subalgebra is not necessarily the equation $u_{xx} = 0$ (see the discussion contained in Section 4.2.1). Anyway in Section 4.2.1 we prove the result of item 1 for an arbitrary number of dependent variables. We observe that the result of item 3 holds also for a number of dependent variables less or equal than four. The higher-order case appearing in item 2 is discussed more in detail in Section 4.2.2.

All computations are performed through the use of the computer algebra package ReLie [27], a REDUCE program developed by one of us (F.O.).

2 Preliminaries

In the whole paper, all manifolds and maps are supposed to be C^∞. Here we recall some basic facts regarding jet spaces (for more details, see [9, 31]). In what follows, λ and μ run from 1 to n whereas i and j run from 1 to m. By $J^r(n,m)$ we denote the r^{th} order jet space of the trivial projection $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Note that $J^0(n,m) = \mathbb{R}^n \times \mathbb{R}^m$. A system of coordinates (x^λ, u^i) on $\mathbb{R}^n \times \mathbb{R}^m$ induces a system of coordinates (x^λ, u^i_σ) on $J^r(n,m)$, where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{N}^n_0$ such that $|\sigma| := \sum_i \sigma_i \leq r$, in the following way. For each local map $s\colon I \subset \mathbb{R}^n \to \mathbb{R}^m$, we define its jet prolongation $j_r s\colon I \to J^r(n,m)$ in such a way that

$$u^i_\sigma \circ j_r s = \frac{\partial^{|\sigma|}(u^i \circ s)}{(\partial x^1)^{\sigma_1} \cdots (\partial x^n)^{\sigma_n}}\ ,\ |\sigma| \leq r.$$

In the case $n = 1$, i.e. one independent variable, we also denote x^1 by x and $u^i_\sigma = u^i_{(\sigma_1)}$ both by $u^i_{\sigma_1}$ and $u^i_{\underbrace{xx\ldots x}_{\sigma_1-times}}$.

On $J^r(n,m)$ there is the (higher) contact distribution which is generated by the vector fields

$$D_\lambda \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\lambda} + u^j_{\sigma,\lambda}\frac{\partial}{\partial u^j_\sigma} \quad \text{and} \quad \frac{\partial}{\partial u^j_\tau},$$

where $0 \leq |\sigma| \leq r-1$, $|\tau| = r$ and σ, λ denotes the multi-index $(\sigma_1, \ldots, \sigma_\lambda+1, \ldots, \sigma_n)$. We note that the contact distribution is spanned by tangent vectors to all submanifolds of $J^r(n,m)$ of the type $j_r s(I)$; conversely, an integral n-dimensional manifold of the contact distribution which projects surjectively on I under the canonical map $(x^\lambda, u^i_\sigma) \mapsto (x^\lambda)$ is locally of the form $j_r s(I)$.

Any vector field X on $J^0(n,m)$ can be lifted to a vector field $X^{(r)}$ on $J^r(n,m)$ by lifting its local flow: such vector field preserves the contact distribution. In coordinates, if $X = X^\lambda \partial/\partial x^\lambda + X^i \partial/\partial u^i$ is a vector field on $J^0(n,m)$, then its r-lift $X^{(r)}$ has the following coordinate expression

$$X^{(r)} = X^\lambda \frac{\partial}{\partial x^\lambda} + X^i_\sigma \frac{\partial}{\partial u^i_\sigma}, \tag{6}$$

whose components are iteratively defined by $X^j_{\tau,\lambda} = D_\lambda(X^j_\tau) - u^j_{\tau,\mu} D_\lambda(X^\mu)$ with $|\tau| < r$. In the case $n = 1$, i.e. one independent variable, we also denote $X^i_\sigma = X^i_{(\sigma_1)}$ by $X^i_{\sigma_1}$.

An r^{th} *order differential equation (DE)* $\mathcal{E}$ *with* n *independent variables and* m *unknown functions (or dependent variables)* is a submanifold of $J^r(n,m)$.

A solution is an n-dimensional submanifold of $J^0(n,m)$ which projects surjectively on $\mathbb{R}^n$ and such that its r-prolongation is contained in $\mathcal{E}$. An *infinitesimal point symmetry* of $\mathcal{E}$ is a vector field X on $J^0(n,m)$ such that its r-prolongation $X^{(r)}$ is tangent to $\mathcal{E}$: they transform solutions into solutions. We denote by $\mathrm{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$.

Let $\mathcal{E}$ be locally described by $\{F^i = 0\}$, $i = 1 \dots k$ with $k < \dim J^r(n,m)$. Then finding point symmetries amounts to solve the system

$$X^{(r)}\left(F^i\right) = 0 \quad \text{whenever} \quad F^i = 0.$$

The problem of determining the Lie algebra $\mathrm{sym}(\mathcal{E})$ is said to be the *direct Lie problem*. Conversely, given a Lie subalgebra $\mathfrak{s}$ of the Lie algebra of the vector fields on $J^0(n,m)$, we consider the *inverse Lie problem*, *i.e.*, the problem of characterizing equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathfrak{s} \subset \mathrm{sym}(\mathcal{E})$.

In the present paper we mainly deal with (system of) ordinary differential equations (ODEs), and we assume that they always can be put in normal forms. Thus, by definition, *an* r^{th} *order ODE is the image of a section of the bundle* $J^r(1,m) \to J^{r-1}(1,m)$. Let $\mathfrak{s}$ be a Lie algebra of vector fields on $J^0(1,m)$. Let $\{X_a\}_{1\le a\le k}$ be a basis of $\mathfrak{s}$. We denote by $\mathcal{M}_{\mathfrak{s}^{(r)}}$ the $k \times (1 + mr + m)$ matrix of the components, w.r.t. the basis $\{\partial_{x^1}, \partial_{u^i_{\sigma_1}}\}_{\substack{1\le i\le m \\ 0\le\sigma_1\le r}}$, of the prolongations $X_k^{(r)}$ to $J^r(1,m)$ of each X_k. Namely, if, according with (6), $X_k^{(r)} = {X_k}^1 \partial/\partial x^1 + {X_k}^i_\sigma \partial/\partial u^i_\sigma$, then matrix $\mathcal{M}_{\mathfrak{s}^{(r)}}$ is

$$\mathcal{M}_{\mathfrak{s}^{(r)}} = \begin{pmatrix} {X_1}^1 & {X_1}^1_0 & \cdots\cdots & {X_1}^m_0 & {X_1}^1_1 & \cdots\cdots & {X_1}^m_1 & \cdots\cdots & {X_1}^1_r & \cdots\cdots & {X_1}^m_r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {X_k}^1 & {X_k}^1_0 & \cdots\cdots & {X_k}^m_0 & {X_k}^1_1 & \cdots\cdots & {X_k}^m_1 & \cdots\cdots & {X_k}^1_r & \cdots\cdots & {X_k}^m_r \end{pmatrix} \tag{7}$$

We omit the dependency of the above matrix on the basis of $\mathfrak{s}$ as the computations we have performed and which made use of matrix (7) are independent of the chosen basis. In fact we shall mainly deal with the rank of (7).

3 Sufficient conditions for Lie remarkability and relationship with first integrals

3.1 Sufficient conditions we shall use for constructing Lie remarkable ODEs

The definition of Lie remarkable equations leads naturally to some sufficient conditions for determining them. To start with, we observe that with any Lie algebra $\mathfrak{s}$ of vector fields on a manifold M it is associated an involutive distribution $\mathcal{D}^{\mathfrak{s}}$ (generally, of non-constant rank) defined by

$$p \in M \mapsto \mathcal{D}^{\mathfrak{s}}_p := \{X_p \mid X \in \mathfrak{s}\} \subset T_pM. \tag{8}$$

In view of Frobenius theorem, involutive distributions on a smooth manifold M are integrable, i.e. through each point of M there is a unique maximal leaf, provided they are of constant rank. For involutive distributions of non-constant rank, there exist sufficient conditions which assure their integrability (see for instance, Theorem 3.25 of [16]). For distributions coming from Lie algebra actions, i.e. of type (8), it holds the following theorem.

Theorem 6 ([2]). *Let $\mathfrak{s}$ be a finite-dimensional Lie algebra. Then distribution $\mathcal{D}^{\mathfrak{s}}$ defined by* (8) *is integrable.*

We note that, with any Lie symmetry algebra $\mathrm{sym}(\mathcal{E})$ of a differential equation $\mathcal{E} \subset J^r(n,m)$ of order r, one can associate the distribution $\mathcal{D}^{\mathrm{sym}(\mathcal{E})}$ on the r^{th} order jet space. The following inequality holds:

$$\dim \mathrm{sym}(\mathcal{E}) \geq \dim \mathcal{D}^{\mathrm{sym}(\mathcal{E})}_{\theta}, \ \forall \theta \in J^r(n,m),$$

where $\dim \mathrm{sym}(\mathcal{E})$ is the dimension, as real vector space, of the Lie algebra $\mathrm{sym}(\mathcal{E})$ of the infinitesimal point symmetries of $\mathcal{E}$. An integral submanifold of $\mathcal{D}^{\mathrm{sym}(\mathcal{E})}$ is, in general, an equation in $J^r(n,m)$. By construction, such equation admits all elements in $\mathrm{sym}(\mathcal{E})$ as infinitesimal point symmetries. This leads to the following proposition, which we shall use in Section 4 for computing Lie remarkable equations starting from distinguished Lie algebras of vector fields on $J^0(n,m)$.

Proposition 7. *Let $\mathcal{E} \subset J^r(n,m)$. If $\dim \mathcal{D}^{\mathrm{sym}(\mathcal{E})}_{\theta} > \dim \mathcal{E}$ $\forall \theta \in J^r(n,m) \setminus \{\mathcal{E} \cup \mathcal{F}\}$, where $\mathcal{F}$ is either an empty set or a finite union of submanifolds of dimension less than $\dim \mathcal{E}$, then $\mathcal{E}$ is Lie remarkable.*

Remark 8. The hypotheses of Proposition 7 are motivated by Theorem 6, which we shed light by an example. Let us consider the Lie algebra of vector fields $\mathfrak{s}$ on $\mathbb{R}^3 = (x,y,z)$ linearly generated by

$$x\partial_x + y\partial_y\,, \quad -y\partial_x + x\partial_y\,, \quad z\partial_z$$

In this case, distribution $\mathcal{D}^{\mathfrak{s}}$ is of non-constant rank: indeed it has rank 3 on $\mathbb{R}^3$ minus the algebraic variety described by $z(x^2+y^2) = 0$. The hyperplane S described by $z = 0$ is the unique 2-dimensional submanifold such that $\mathcal{D}^{\mathfrak{s}}_p \subseteq T_pS$, $\forall p \in S$. Note that, outside S, the rank of $\mathcal{D}^{\mathfrak{s}}$ is equal to 3 except on the two lines $\{x=0, y=0, z>0\}$ and $\{x=0, y=0, z<0\}$, where the rank is equal to 1.

Proof of Proposition 7. Let $\widetilde{\mathcal{E}} \subset J^r(n,m)$ be a DE (of the same dimension as $\mathcal{E}$) such that $\mathrm{sym}(\mathcal{E}) \leq \mathrm{sym}(\widetilde{\mathcal{E}})$. Let us suppose that $\widetilde{\mathcal{E}} \neq \mathcal{E}$, so that there exists at least a point of $\widetilde{\mathcal{E}}$ which does not belong to $\mathcal{E}$. Let us denote such a point by θ. As a first case, let us assume that $\theta \in \widetilde{\mathcal{E}} \setminus \{\mathcal{E} \cup \mathcal{F}\}$. Since $\mathrm{sym}(\mathcal{E})$ is a Lie subalgebra of the Lie symmetry algebra of $\widetilde{\mathcal{E}}$, then $\mathcal{D}^{\mathrm{sym}(\mathcal{E})}_{\theta} \subseteq T_\theta \widetilde{\mathcal{E}}$. This inclusion implies that $\dim(\mathcal{D}^{\mathrm{sym}(\mathcal{E})}_{\theta}) \leq \dim(\widetilde{\mathcal{E}}) = \dim(\mathcal{E})$, which contradicts the hypothesis.

Let us now assume that $\theta \in (\widetilde{\mathcal{E}} \setminus \mathcal{E}) \cap \mathcal{F}$. This implies that $\widetilde{\mathcal{E}} \cap \mathcal{F} \neq 0$. For dimensional reasons, $\widetilde{\mathcal{E}} \neq \mathcal{F}$, so that there exists a point in $\widetilde{\mathcal{E}} \setminus \mathcal{F}$ which, in its turn, does not belong to $\mathcal{E}$. Then it is enough to apply the reasoning of the previous case. □

From now on we shall concentrate only on ODEs since they are the target of our investigation.

We would like to underline that, if an ODE $\mathcal{E} \subset J^r(1,m)$ satisfies Proposition 7 and it can be put in normal form (i.e. $\mathcal{E}$ is the image of a section of the bundle $J^r(1,m) \to J^{r-1}(1,m)$, in particular, it is a determined system), then also the k-prolongation $\mathcal{E}^{(k)}$ of $\mathcal{E}$, with $k \in \mathbb{N}$ arbitrary, is Lie remarkable. Indeed, in this case, $\dim \mathcal{E}^{(k)} = \dim \mathcal{E}$ and the rank of the distribution spanned by the symmetries cannot decrease.

Proposition 7 suggests that, in order to construct an l-dimensional Lie remarkable equation, one can start from an $(l+1)$-dimensional Lie algebra of vector fields on $J^0(n,m) = \mathbb{R}^{n+m}$. This leads, in the case of computation of scalar Lie remarkable ODEs of r^{th} order, for dimensional reasons, to consider the *Lie determinant* associated to an $(r+2)$-dimensional Lie algebra of vector fields (see the beginning of Section 4). Actually, in some cases, an l-dimensional ODE can be uniquely determined by an l-dimensional Lie algebra, as the content of the next section shows.

3.2 Lie remarkable ODEs determined by a lower dimensional Lie algebra of vector fields: foliations of equivalent ODEs and (pseudo)-stabilization order

We have already seen that Lie algebra of vector fields 1. of (3) is sufficient to completely characterize the ODE $u_{xx} = 0$, up to point transformations (see the discussion after Definition 2). In this section we investigate other situations in which it is possible to construct Lie remarkable ODEs starting from a Lie algebra of vector fields of lower dimension w.r.t. that of Proposition 7.

Consider realization (2) (in terms of vector fields on $\mathbb{R}^2$) of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Second order ODEs admitting vector fields (2) as Lie point symmetries are

$$u_{xx} = \frac{1}{2}u_x + Ke^{-2x}u_x^3\,, \quad K \in \mathbb{R}. \tag{9}$$

Equation (9) and vector fields (2) appeared in [10] in the context of the local classification of projective structures on a 2-dimensional manifold. A priori, DEs belonging to the above 1-parametric family could be all point-equivalent, so that one could consider the equation (9) with $K = 0$ as its representative, that is the only equation (up to point transformations) admitting the above Lie algebra of vector fields as a subalgebra of Lie point symmetries. A deeper study shows that it is not the case. Indeed, even if the change $x \to x + c$, for some suitable constant c, allows to say that all equations (9) with $K > 0$ (respectively, $K < 0$) are point equivalent, for $K = 0$ equation (9) admits the 8-dimensional Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ as Lie algebra of point symmetries, so that it is not point-equivalent with any equations (9) with $K \neq 0$. We stress that the sufficient criterion given in Proposition 7 is not fulfilled for $K \neq 0$, the above equation being a 3-dimensional submanifold of the 4-dimensional jet space $J^2(1, 1)$. On the other hand, for $K = 0$, the criterion is fulfilled if we consider the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ and the equation $u_{xx} = \frac{1}{2}u_x$ is Lie remarkable (note that it is equivalent to $u_{xx} = 0$ as it admits a 8-dimensional Lie symmetry algebra). From a theoretical viewpoint, obtaining a foliation of point-equivalent equations is possible. For instance, the most general scalar 2^{nd} order PDE admitting the Lie algebra $\mathfrak{s}$ linearly generated by $\{\partial_x, \partial_u, x\partial_u\}$ as a Lie symmetry subalgebra is $u_{xx} = K$, $K \in \mathbb{R}$, as u_{xx} is the only 2^{nd} order differential invariant of $\mathfrak{s}$ (up to functional dependence). All these equations are point equivalent to $u_{xx} = 0$, as they admit the 8-dimensional projective Lie algebra $\mathfrak{sl}(3, \mathbb{R})$ as Lie point symmetry algebra. Then we can say that the Lie algebra of vector fields $\mathfrak{s}$ uniquely determines equation $u_{xx} = 0$ up to point transformations, i.e. it is associated with $\mathfrak{s}$ in the sense of Definition 2. Thus, we have the following proposition:

Proposition 9. *Let $\mathfrak{s}$ be a Lie algebra of vector fields on $J^0(1, 1)$. Let us suppose that there exists some $r \in \mathbb{N}$ such that the rank of $\mathcal{D}^{\mathfrak{s}^{(r)}}$ is of codimension 1 almost everywhere. Let F be the unique function on $J^r(1, 1)$ (up to functional dependence) such that $\mathcal{D}^{\mathfrak{s}^{(r)}}(F) = 0$. Then if the equations $F = k$, $k \in \mathbb{R}$ are point equivalent each other, then equation $F = 0$ is associated with $\mathfrak{s}$.*

Proof. Since F is, by hypothesis, the only differential invariant of r^{th} order of the Lie algebra $\mathfrak{s}$, $F = k$, with $k \in \mathbb{R}$, are all and only the r^{th} order DEs admitting $\mathfrak{s}$ as a Lie symmetry subalgebra. Now, if all these DEs are equivalent each other, they will be equivalent, in particular, to equation $F = 0$. □

Below we explain another way of obtaining Lie remarkable ODEs starting from a Lie algebra of vector fields of lower dimension w.r.t. that of Proposition 7. This is based on the fact that some r^{th} order ODE can be uniquely constructed by a foliation of $(r-1)^{\text{th}}$ order ODEs. Below we shall be more precise by considering a concrete example. The equation $\mathcal{E} := \{u_{xx} = 0\} \subset J^2(1, 1)$ can be uniquely constructed starting from the foliation of $J^1(1, 1)$ given by $\mathcal{F}_k = 0$, where $\mathcal{F}_k := u_x - k$, $k \in \mathbb{R}$. Indeed, we have that $\mathcal{E} = \bigcup_k \mathcal{F}_k^{(1)}$. Even though we can construct the equation $u_{xx} = 0$ starting from such a foliation of $J^1(1, 1)$, the two Lie point symmetries ∂_x and ∂_u of $u_x = k$ are not enough to determine the equation $u_{xx} = 0$ in the Lie remarkable sense (i.e. there are many 2^{nd} order ODEs associated with the Lie algebra of translations) although they determine uniquely the foliation $\mathcal{F}_k = 0$ of $J^1(1, 1)$. To completely determine equation $u_{xx} = 0$ in the Lie remarkable sense, the foliation $u_x = k$ given by a Lie algebra of vector fields should live in $J^2(1, 1)$ rather than in $J^1(1, 1)$ (then, in this case, we need 3 symmetries rather than 2). Indeed, such a foliation and equation $u_{xx} = 0$ represent, essentially, the same object in $J^2(1, 1)$. This trivial observation leads to an interesting consequence. If we find a 3-dimensional Lie algebra of vector fields such that its orbits in $J^2(1, 1)$ form the foliation $u_x = k$, then such algebra completely determines the equation $u_{xx} = 0$. This Lie algebra of vector fields exists: indeed, the 3 infinitesimal homotheties of $\mathbb{R}^2$ (two translations and the stretching) are Lie point symmetries of $u_x = k$ for any k. This 3-dimensional Lie algebra is enough to completely determine $u_{xx} = 0$ in the Lie remarkable sense. So, we constructed a Lie remarkable equation of dimension 3 starting from a 3-dimensional Lie algebra. In more precise words, we have the following proposition.

Proposition 10. *Let $\mathfrak{s}$ be a Lie algebra of vector fields on $J^0(1, 1)$. Let us suppose that there exists an integer r such that the system $\mathfrak{s}^{(r)}(F) = 0$ has a unique solution F (up to functional dependence). If $F \in C^\infty(J^{r-1}(1, 1))$ but $F \notin C^\infty(J^{r-2}(1, 1))$, then equation $D_x(F) = 0$ is Lie remarkable and it is associated with $\mathfrak{s}$.*

Proof. Since $\mathfrak{s}^{(r)}(F) = 0$, we have that $\mathfrak{s} \subset \text{sym}(\{F = k\})\ \forall k \in \mathbb{R}$, that implies $\mathfrak{s} \subset \text{sym}(\{D_x(F) = 0\})$. Thus, on one hand $\mathfrak{s}^{(r)}$ spans almost everywhere a codimension 1 distribution on $J^r(1, 1)$ and, on the other hand, it is tangent to the hypersurface $\{D_x(F) = 0\}$. Since the only solution to $\mathfrak{s}^{(r)}(F) = 0$ is an F of order $(r-1)$ but not $(r-2)$, $D_x(F)$ is of order r (but not $(r-1)$), so that the only possibility for $\mathfrak{s}$ to be a subalgebra of Lie point symmetries of $\{D_x(F) = 0\}$ is that such equation is contained in the set

$$\text{sing}(\mathfrak{s}^{(r)}) := \{\text{points of } J^r(1, 1) \text{ where the rank of } \mathfrak{s}^{(r)} \text{is not maximal}\}.$$

By hypothesis, the rank of $\mathfrak{s}^{(r)}$ is maximal almost everywhere and it is equal to $\dim J^r(1,1)-1=r+1$. Below we see that the set $\mathrm{sing}(\mathfrak{s}^{(r)})$ contains only a hypersurface of order r which is decribed by $\{D_x(F)=0\}$. This comes from the fact that $\mathrm{sing}(\mathfrak{s}^{(r)})$ is given by a system $\{D_x(F)G_i=0\}$, where G_i are smooth functions on $J^{r-1}(1,1)$. In fact, the set $\mathrm{sing}(\mathfrak{s}^{(r)})$ is described by the system formed by all determinants of $(r+1)\times(r+1)$ submatrices of $\mathcal{M}_{\mathfrak{s}^{(r)}}$ (see (7) for the definition) equating to zero. Since in the last column of $\mathcal{M}_{\mathfrak{s}^{(r)}}$ the highest order derivatives appears at first degree, determinants of submatrices containing elements of the last column are polynomial of first degree in the highest order derivatives. Since $D_x(F)$ is exactly of order r (the highest), functions G_i cannot be of highest order. We conclude that equation $\{D_x(F)=0\}$ is the only hypersurface of order r contained in $\mathrm{sing}(\mathfrak{s}^{(r)})$.

In view of the above reasonings, any other r^{th} order scalar ODE $\mathcal{E}$ admitting $\mathfrak{s}$ as a Lie symmetry subalgebra is such that $\mathcal{E}\subset\mathrm{sing}(\mathfrak{s}^{(r)})$. Being $\mathcal{E}$ a hypersurface of order r, in view of the above conclusion we have $\mathcal{E}=\{D_x(F)=0\}$. □

Remark 11. To satisfy the hypotheses of Proposition 10, a necessary condition is that, almost everywhere, $\mathrm{rank}(\mathcal{M}_{\mathfrak{s}^{(r-1)}})=r$ and $\mathrm{rank}(\mathcal{M}_{\mathfrak{s}^{(r)}})=r+1$. Lie algebras which *pseudo-stabilizes* in the sense specified in [32] satisfies such condition.

Remark 12. The fact that $\mathfrak{s}^{(r)}(F)=0$ has a unique solution which, in its turn, is exactly of $(r-1)^{\mathrm{th}}$ order, implies also that $\mathfrak{s}^{(r-1)}(F)=0$. One can ask if we can weaker such hypothesis by assuming only that $\mathfrak{s}^{(r-1)}(F)=0$ has a unique solution of $(r-1)^{\mathrm{th}}$ order. In this case there are examples showing that $D_x(F)=0$ is not Lie remarkable. We construct such an example immediately after (14).

Example 13. Let us consider the Lie algebra $\mathfrak{s}$ linearly generated by the following vector fields on $J^0(1,1)$:

$$\partial_x,\quad \partial_u,\quad x\partial_u,\quad x\partial_x+2u\partial_u\,.$$

We have that

$$\mathcal{M}_{\mathfrak{s}^{(3)}}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&x&1&0&0\\x&2u&u_x&0&-u_{xxx}\end{pmatrix}.$$

The above matrix has rank equal to 4 outside the hypersurface $u_{xxx}=0$. We also note that the only solution to system $\mathfrak{s}^{(3)}(F)$ is $F=F(u_{xx})$. Thus, in view of Proposition 10, we obtain that $u_{xxx}=0$ is the only 3^{rd} order ODE associated with $\mathfrak{s}$.

The phenomenon described above does not appear in the case of PDEs as, in general, one needs several foliations of 1^{st} order PDEs to reconstruct a 2^{nd} order PDE. In fact, the above construction, for PDEs, corresponds to the existence of intermediate integrals, as, by definition, an intermediate integral of an r^{th} order PDE $\mathcal{E}$ is a function f on the jet space of order $r-1$ such that all solutions of the family $f=c$, $c\in\mathbb{R}$ are also solutions of $\mathcal{E}$. For instance, if a Monge-Ampère equation with two independent variables admits two special intermediate integrals, then it can be reconstructed starting from them (see for instance [1]).

3.3 Lie remarkability and existence of first integrals

In view of all that we said so far, it is expectable that Lie remarkable DEs must have a suitable number of symmetries in order to be uniquely determined. This implies, in the case of scalar ODEs, also the existence of first integrals, as the following proposition shows.

Proposition 14. *Let us consider the following ODE*

$$u_{(n)}=f(x,u,u_{(1)},\dots,u_{(n-1)})\,,\quad \text{where } u_{(k)}:=\frac{d^ku}{dx^k} \tag{10}$$

Let $X_1,\dots,X_m$, $m\geq n$, be (point) symmetries of (10)*. Then,*

$$\frac{X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega}{X_{j_1}\lrcorner X_{j_2}\lrcorner\cdots\lrcorner X_{j_n}\lrcorner Z\lrcorner\Omega}, \tag{11}$$

where

$$Z:=\partial_x+u_{(1)}\partial_u+\cdots+u_{(n-1)}\partial_{u_{(n-2)}}+f\partial_{u_{(n-1)}}\,,\ \Omega=dx\wedge du\wedge du_{(1)}\wedge\cdots\wedge du_{(n-1)},$$

is a first integral of (10).

Proof. We observe that

$$L_Z\left(\frac{1}{X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega}\,\Omega\right)=0.$$

Indeed,

$$L_Z\left(\frac{1}{X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega}\,\Omega\right)=-\sum_k\frac{X_{i_1}\lrcorner\cdots\lrcorner X_{i_{k-1}}\lrcorner[Z,X_{i_k}]\lrcorner X_{i_{k+1}}\lrcorner\cdots\lrcorner Z\lrcorner\Omega}{(X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega)^2}\Omega$$
$$-\frac{X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner L_Z\Omega}{(X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega)^2}\Omega+\frac{1}{X_{i_1}\lrcorner X_{i_2}\lrcorner\cdots\lrcorner X_{i_n}\lrcorner Z\lrcorner\Omega}L_Z\Omega=0.$$

We obtained the last equality in view of the following facts. Since X_{i_k} are symmetries of (10), $[Z,X_{i_k}]$ is proportional to Z for any $1\leq i_k\leq m$, so that

$$X_{i_1}\lrcorner\cdots\lrcorner X_{i_{k-1}}\lrcorner[Z,X_{i_k}]\lrcorner X_{i_{k+1}}\lrcorner\cdots\lrcorner Z\lrcorner\Omega=0.$$

Moreover, $L_Z\Omega=\mathrm{div}(Z)\Omega$. □

Corollary 15. *Lie remarkable equations constructed by means of Proposition 7 or 10 possess first integrals of type* (11).

Remark 16. Proposition 14 says that if we have a sufficient number of point symmetries we can construct first integrals. In the case we have exactly n symmetries, (11) is a constant, so it is a trivial first integral. The above construction starts to produce non-trivial first integrals when we have at least $n+1$ symmetries.

Below, as an example of computation, we use Proposition 14 to construct first integrals of equations (9). We recall that Lie point symmetries of (9) are (2) so that the following integrals are obtained:

$$I_1=\frac{\det\begin{pmatrix}1 & u_x & \frac{1}{2}u_x+Ke^{-2x}u_x^3\\ 0 & 1 & 0\\ 1 & u & u_x\end{pmatrix}}{\det\begin{pmatrix}1 & u_x & \frac{1}{2}u_x+Ke^{-2x}u_x^3\\ 0 & 1 & 0\\ u & \frac{1}{2}u^2 & u_xu-u_x^2\end{pmatrix}}=\frac{2Ke^{-2x}u_x^2-1}{u(2Ke^{-2x}u_x^2-1)+2u_x},$$

and

$$I_2=\frac{\det\begin{pmatrix}1 & u_x & \frac{1}{2}u_x+Ke^{-2x}u_x^3\\ 0 & 1 & 0\\ 1 & u & u_x\end{pmatrix}}{\det\begin{pmatrix}1 & u_x & \frac{1}{2}u_x+Ke^{-2x}u_x^3\\ 1 & u & u_x\\ u & \frac{1}{2}u^2 & u_xu-u_x^2\end{pmatrix}}=2\frac{2Ke^{-2x}u_x^2-1}{u^2(2Ke^{-2x}u_x^2-1)+4u_x(u-u_x)}.$$

4 Proof of Theorem 4 and Theorem 5

Let us consider the euclidean space $\mathbb{R}^{1+m}$. From now on, we interpret such a space as the space of 1 independent variable and m dependent variables, *i.e.*, $\mathbb{R}^{1+m}=J^0(1,m)$. In this section, by considering Propositions 3, 7, 10 and 9, we construct Lie remarkable ODEs by starting from a given Lie algebra $\mathfrak{s}$ of vector fields on $\mathbb{R}^{1+m}$. More precisely, we start by taking into account primitive Lie algebras of vector fields on $\mathbb{R}^2$ and construct (when possible) the corresponding associated Lie remarkable scalar ODEs. Then we consider the isometric, special conformal, affine and projective algebra of $\mathbb{R}^{1+m}$ in order to obtain Lie remarkable systems of ODEs.

The dimension of the Lie algebras of vector fields we are going to consider is a function of the number of the dependent variables, so that, as a first step, we ask ourselves the following question: by using Proposition 7 or 10, which combinations of integer values of r and m can produce a Lie remarkable system of m ODEs of order r associated with the given Lie algebra of vector fields $\mathfrak{s}$? We realize that such systems will be found as m-codimensional submanifolds of the jet space $J^r(1,m)$ such that

(12) $$\dim\mathfrak{s}\geq\dim J^r(1,m)-m=rm+1.$$

After this dimensional estimation, we obtain the Lie remarkable system $\mathcal{E}$ associated with the considered Lie algebra of vector fields $\mathfrak{s}$ by imposing on $\mathcal{E}$ the conditions of Proposition 7 or 10. In particular, in order to satisfy conditions of Proposition 7, we find the set $\mathrm{sing}_k(\mathfrak{s}^{(r)})$ of points of $J^r(1,m)$ where the rank of the distribution associated with the prolonged Lie algebra $\mathfrak{s}^{(r)}$ of the chosen Lie algebra of vector fields $\mathfrak{s}$ is at most equal to k, where $k \in \mathbb{N}$ is smaller than the maximal rank of $\mathfrak{s}^{(r)}$. The Lie remarkable equation, whether it exists, is one of this singular set as the vector fields in $\mathfrak{s}^{(r)}$ are tangent to the set $\mathrm{sing}_k(\mathfrak{s}^{(r)})$, provided that the latter is a submanifold of $J^r(1,m)$ (see also Theorem 6.

The computation of the singular set leads to algebraic computations that we solve with the help of computer algebra [27]. The rank condition of Proposition 7 is enforced as a system of equations obtained from the vanishing of determinants of all square submatrices (minors), of suitable dimension, of the matrix of prolonged generators $\mathcal{M}_{\mathfrak{s}^{(r)}}$ (see (7)). We underline that, in the scalar case and for the considered Lie algebras, the method described above reduces to that of Lie determinant, i.e. Lie remarkable equations of order at most r associated with an $(r+2)$-dimensional Lie algebra of vector fields are described by the vanishing of the determinant of the $(r+2)\times(r+2)$ matrix $\mathcal{M}_{\mathfrak{s}^{(r)}}$.

4.1 Primitive Lie algebras and Lie remarkable scalar ODEs: proof of Theorem 4

Let us set $m=1$. Starting from the primitive (non-singular) Lie algebras of vector fields of $\mathbb{R}^2$, we construct Lie remarkable scalar ODEs taking Propositions 7, 10 and 9 into account. For the convenience of the reader, below we report the list of the afore-mentioned Lie algebras, which can be found in [32].

(13)

	Generators	Structure
I	$\partial_x, \partial_u, u\partial_x - x\partial_u + \alpha(x\partial_x + u\partial_u)$, $\alpha \in \mathbb{R}$	$\mathbb{R} \ltimes \mathbb{R}^2$
II	$\partial_x, x\partial_x + u\partial_u, (x^2-u^2)\partial_x + 2xu\partial_u$	$\mathfrak{sl}(2)$
III	$u\partial_x - x\partial_u, (1+x^2-u^2)\partial_x + 2xu\partial_u, 2xu\partial_x + (1-x^2+u^2)\partial_u$	$\mathfrak{so}(3)$
IV	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u$	$\mathbb{R}^2 \ltimes \mathbb{R}^2$
V	$\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u$	$\mathfrak{sa}(2)$
VI	$\partial_x, \partial_u, x\partial_x, u\partial_u, u\partial_x, x\partial_u$	$\mathfrak{a}(2)$
VII	$\partial_x, \partial_u, u\partial_x - x\partial_u, x\partial_x + u\partial_u, (x^2-u^2)\partial_x + 2xu\partial_u, 2xu\partial_x + (u^2-x^2)\partial_u$	$\mathfrak{so}(3,1)$
VIII	$\partial_x, \partial_u, x\partial_x, u\partial_u, u\partial_x, x\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	$\mathfrak{sl}(3)$

In this case, since the Lie algebra **I** is 3-dimensional, Proposition 7 provides sufficient conditions for the existence of a 1st order Lie remarkable ODE associated to **I**, whereas Proposition 10 for the existence of a 2nd order one. A direct computation shows that such ODEs do not exist. In fact, the matrix of 2-prolongations of **I** is

$$\mathcal{M}_{\mathbf{I}^{(2)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u+\alpha x & -x+\alpha u & -1-u_x^2 & -u_{xx}(3u_x+\alpha) \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{\mathbf{I}^{(1)}} & \begin{matrix} 0 \\ 0 \\ -u_{xx}(3u_x+\alpha) \end{matrix} \end{pmatrix},$$

so that the rank of the submatrix of $\mathcal{M}_{\mathbf{I}^{(1)}}$ is 3-dimensional everywhere, which implies that there are no 1st order Lie remarkable ODEs associated with Lie algebra **I**. We cannot construct Lie remarkable equation by means of Proposition 10 as the necessary conditions contained in Remark 11 are not satisfied. Anyway, for our purposes, we write the only solution to system $\mathcal{M}_{\mathbf{I}^{(2)}}(F)=0$, up to functional dependency:

$$F(u_x, u_{xx}) := \frac{e^{-\alpha \arctan(u_x)} u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}, \ \alpha \in \mathbb{R}, \tag{14}$$

Proposition 9 is not satisfied. Indeed $F(u_x, u_{xx})$ is a differential invariant of Lie algebra **I** and the most general 2nd order scalar ODEs admitting **I** as subalgebra of symmetries is $F(u_x, u_{xx}) = k$, $k \in \mathbb{R}$, but previous equations are not all equivalent each other as for $k=0$ we have an equation belonging to the class of projective connections, (i.e. equations $u_{xx} = P(x,u,u_x)$ where P is a polynomial of third degree in u_x), which is closed w.r.t. point transformations, whereas for $k \neq 0$ the equation does not belong to this class. Now we answered a question posed in Remark 12. To this aim, note that equations of the family $D_x(F)=0$, which coincides with $(1+u_x^2)u_{xxx} = u_{xx}^2(\alpha+3u_x)$, are not all equivalent each other since for $\alpha=0$ we obtain a Lie remarkable equation with a 6-dimensional symmetry algebra (see also below, when discussing the case of Lie algebra **VII**) whereas for $\alpha \neq 0$ we do not have any of these properties. Taking into account that the only solution to system $\mathcal{M}_{\mathbf{I}^{(2)}}(F)=0$ is (14), we answered negatively to the question posed in Remark 12.

Note that $ds = e^{\alpha \arctan(u_x)}(1+u_x^2)^{\frac{1}{2}}dx$ is an invariant 1-form; a function of $F(u_x, u_{xx})$ and of its derivatives with respect to ds is, in its turn, a differential invariant. For $\alpha = 0$ we find the classical result of the euclidean geometry that any (euclidean) differential invariant is a function of the curvature and of its derivatives. Similar results can be obtained also for the other Lie algebras: differential invariants of degree 0 and 1 play the same role as the curvature and the arc-length element for the Klein geometries associated to algebras (13). We do not insist here on this aspect even if it is somehow related to the topic of this paper.

By using the same reasonings we adopted for the algebra **I**, we can prove that there are no Lie remarkable equations associated with Lie algebras **II** and **III**. In fact, according to Proposition 7, for dimensional reasons such equations should be of 1st order, but there are no hypersurfaces of $J^1(1,1)$ of order 1 (i.e. 1st order ODEs) where the rank of $\mathcal{M}_{\mathbf{II}^{(1)}}$ (resp. $\mathcal{M}_{\mathbf{III}^{(1)}}$) drops. Both Lie algebras **II** and **III** do not satisfy the hypotheses of Proposition 10 as they do not satisfy necessary conditions of Remark 11. In fact, the rank of both $\mathcal{M}_{\mathbf{II}^{(1)}}$ and $\mathcal{M}_{\mathbf{III}^{(1)}}$ is almost everywhere equal to 3. For dimensional reason, this is the only case to be considered. This concludes the proof of item 1 of Theorem 4. Moreover, both Lie algebras **II** and **III** do not satisfy Proposition 9. Indeed, the unique 2nd order differential invariant of **II** (resp. **III**) up to functional dependence, is $(1+u_x^2)^{-\frac{3}{2}}(u_{xx}u + 1 + u_x^2)$ (resp. $(1+u_x^2)^{-\frac{3}{2}}(1+x^2+u^2)u_{xx} + 2(1+u_x^2)^{-\frac{1}{2}}(u - xu_x)$) and our assertion follows by applying the same reasoning as for the Lie algebra **I**.

By continuing in our analysis, we see that the only Lie remarkable equation associated with algebras **IV** or **V** is $u_{xx} = 0$. In fact

$$\mathcal{M}_{\mathbf{IV}^{(2)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & u & 0 & -u_{xx} \\ u & -x & -1-u_x^2 & -3u_x u_{xx} \end{pmatrix} \tag{15}$$

and the only hypersurface of $J^2(1,1)$ where the rank of (15) is less than 4 is $u_{xx} = 0$. For dimensional reasons, according to Proposition 10, Lie algebra **IV** could produce a Lie remarkable ODE of order at most 3, but necessary conditions contained in Remark 11 are not satisfied. In fact, in view of (15), the rank of $\mathcal{M}_{\mathbf{IV}^{(r)}}$ is, almost everywhere, equal to $r+2$ for $r \leq 2$. A similar reasoning applies also to Lie algebra **V**. Furthermore, both Lie algebras **IV** and **V** do not satisfy Proposition 9. Indeed the unique 3rd (resp. 4th) order differential invariant of **IV** (resp. **V**) up to functional dependence, is $\mathcal{I} = u_{xx}^{-2}((1+u_x^2)u_{xxx} - 3u_x u_{xx}^2)$ (resp. $\mathcal{J} = u_{xx}^{-\frac{8}{3}}(3u_{xx}u_{xxxx} - 5u_{xxx}^2)$). Now, equations $\mathcal{I} = k$ with $k \in \mathbb{R}$ cannot be all equivalent each other as for $k = 0$ we have we have a 6-dimensional Lie symmetry algebra whereas for $k \neq 0$ a 5-dimensional one. Also equation $\mathcal{J} = k$ with $k \in \mathbb{R}$ cannot be all equivalent as the prolongation of a point transformation is always and affine transformation in all derivatives except for the first ones in relation to which it is a projective transformation, so that it is not possible to transform equation $\mathcal{J} = 0$ into equation $\mathcal{J} = k$ for any $k \neq 0$. Since the full symmetry algebra of $u_{xx} = 0$ is the projective algebra $\mathfrak{sl}(3)$, any subalgebra of $\mathfrak{sl}(3)$ containing either the Lie algebra **IV** or **V** leads to the Lie remarkable equation $u_{xx} = 0$ (see also Proposition 3). For instance, starting from affine algebra **VI** we obtain again the Lie remarkable equation $u_{xx} = 0$.

By considering the Lie algebras **VI**, **VII** and **VIII** we can construct, by means of Proposition 7 and following the same reasoning adopted so far, respectively equation of items 3, 4 and 5 of Theorem 4. We note that none of these Lie algebra satisfies necessary conditions of Remark 11. Moreover, also Proposition 9 cannot apply: it is sufficient to look at the differential invariants of such Lie algebras (for instance one can consult the list provided in [32]) and arguing as in the case of Lie algebra **V**.

This concludes the proof of Theorem 4.

4.2 Lie remarkable systems of ODEs: proof of Theorem 5

Here we consider the isometry $\mathcal{I}(\mathbb{R}^3)$, affine $\mathcal{A}(\mathbb{R}^3)$, conformal $\mathcal{C}(\mathbb{R}^3)$ and projective $\mathcal{P}(\mathbb{R}^3)$ Lie algebra of $\mathbb{R}^3$. For the convenience of the reader we recall that

$$\dim \mathcal{I}(\mathbb{R}^3) = 6, \quad \dim \mathcal{A}(\mathbb{R}^3) = 12, \quad \dim \mathcal{C}(\mathbb{R}^3) = 10, \quad \dim \mathcal{P}(\mathbb{R}^3) = 15.$$

In the case of the Lie algebra of infinitesimal isometries $\mathcal{I}(\mathbb{R}^{1+m})$, we prove that, for any dimension, system $\{u_{xx}^k = 0\,,\ k = 1, \ldots, m\}$ is associated with $\mathcal{I}(\mathbb{R}^{1+m})$. This leads to some computations involving the study of a system of PDEs, which can be solved by using some tricks (see next section). In general, similar systems are very hard to treat, so that the algorithmic procedure described in the beginning of Section 4 provides as efficient tools for describing Lie remarkable systems associated with a given Lie algebra of vector fields. We follow the latter methods in Section 4.2.2, 4.2.3 and 4.2.4.

4.2.1 Systems of ODEs associated with Lie algebra $\mathcal{I}(\mathbb{R}^{1+m})$

Generators of infinitesimal isometries of $\mathbb{R}^{1+m}$ are

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u^i}, \quad -u^i\frac{\partial}{\partial x}+x\frac{\partial}{\partial u^i}, \quad u^i\frac{\partial}{\partial u^j}-u^j\frac{\partial}{\partial u^i}, \quad i\neq j, \quad i,j=1\ldots m. \tag{16}$$

There are no 1st order systems ODEs admitting (16) as Lie symmetry subalgebra. Indeed, the most general system of 1st order ODEs admitting ∂_x and ∂_{u^i} as Lie symmetries is of the form $\{u^k_x=c^k\}_{k=1\ldots m}$, $c^k\in\mathbb{R}$. By imposing that the third set of vector fields (16) are symmetries of the above system, we obtain the equation $\delta^{ik}+c^ic^k=0\;\forall\, i,k$, so that if $i=k$ we obtain the contradiction $(c^k)^2+1=0$.

Going to higher order systems, the most general system of 2nd order ODEs admitting ∂_x and ∂_{u^i} as Lie symmetries is of the form

$$\{u^k_{xx}=F^k(u^1_x,\ldots,u^m_x)\}, \quad k=1\ldots m. \tag{17}$$

By imposing that the remaining vector fields of (16) are symmetries of (17), we obtain the following system of PDEs:

$$\begin{cases} -F^k_{u^i_x}-\sum_{h=1}^m u^i_x u^h_x F^k_{u^j_x}+2F^ku^i_x+F^iu^k_x=0, \\ \\ -u^i_xF^j_{u^j_x}+u^j_xF^k_{u^i_x}+F^i\delta^{kj}-F^j\delta^{ki}=0, \end{cases} \qquad \forall\, i,j,k. \tag{18}$$

By integrating the system formed by equations of the first line of (18) with $k=i$, we obtain that

$$F^k=c^k\left(1+\sum_{j=1}^m(u^j_x)^2\right)^{\frac{3}{2}}. \tag{19}$$

Let us observe that in the case $m=1$ the second set of equations of (18) is always satisfied, so that (19) reduces to (14) with $\alpha=0$, which turns out the most general scalar ODE (euclidean curvature equal to a constant) admitting the euclidean algebra of $\mathbb{R}^2$ as a Lie symmetry subalgebra.

In the case $m\geq 2$, by substituting (19) into the second set of equations of (18) with $k=i\neq j$ gives $c^j=0$ for $j\neq k$. In view of arbitrariness of k, we obtain $c^k=0\;\forall\, k$, that is system of item 1 of Theorem 5. This proves item 1 of Theorem 5.

4.2.2 Systems of ODEs associated with Lie algebra $\mathcal{A}(\mathbb{R}^3)$

The generators of this algebra are

$$\frac{\partial}{\partial a}, \quad a\frac{\partial}{\partial b},$$

for all $a,b\in\{x,u^i\}$. We can prove, by using the reasonings contained in the beginning of Section 4, that when $r=5$ there exists the following Lie remarkable equation (u and v denote the dependent variables):

$$\begin{aligned} u_{xxxxx}=&(5(-u_{xx}^4v_{xxxx}^3+6u_{xx}^3u_{xxx}v_{xxx}v_{xxxx}^2+3u_{xx}^3u_{xxxx}v_{xx}v_{xxxx}^2\\ &+36u_{xx}^3u_{xxxx}v_{xxx}^2v_{xxxx}-6u_{xx}^2u_{xxx}^2v_{xx}v_{xxxx}^2-72u_{xx}^2u_{xxx}^2v_{xxx}^2v_{xxxx}-\\ &84u_{xx}^2u_{xxx}u_{xxxx}v_{xx}v_{xxx}v_{xxxx}+72u_{xx}^2u_{xxx}u_{xxxx}v_{xxx}^3-3u_{xx}^2u_{xxxx}^2v_{xx}^2v_{xxxx}\\ &-36u_{xx}^2u_{xxxx}^2v_{xx}v_{xxx}^2+144u_{xx}u_{xxx}^3v_{xx}v_{xxx}v_{xxxx}+48u_{xx}u_{xxx}^2u_{xxxx}v_{xx}^2v_{xxxx}\\ &-144u_{xx}u_{xxx}^2u_{xxxx}v_{xx}v_{xxx}^2+78u_{xx}u_{xxx}u_{xxxx}^2v_{xx}^2v_{xxx}+u_{xx}u_{xxxx}^3v_{xx}^3\\ &-72u_{xxx}^4v_{xx}^2v_{xxxx}+72u_{xxx}^3u_{xxxx}v_{xx}^2v_{xxx}-42u_{xxx}^2u_{xxxx}^2v_{xx}^3))\\ &/(144(u_{xx}^3v_{xxx}^3-3u_{xx}^2u_{xxx}v_{xx}v_{xxx}^2+3u_{xx}u_{xxx}^2v_{xx}^2v_{xxx}-u_{xxx}^3v_{xx}^3)),\\ v_{xxxxx}=&(5(-u_{xx}^3v_{xx}v_{xxxx}^3+42u_{xx}^3v_{xxx}^2v_{xxxx}^2-78u_{xx}^2u_{xxx}v_{xx}v_{xxx}v_{xxxx}^2\\ &-72u_{xx}^2u_{xxx}v_{xxx}^3v_{xxxx}+3u_{xx}^2u_{xxxx}v_{xx}^2v_{xxxx}^2-48u_{xx}^2u_{xxxx}v_{xx}v_{xxx}^2v_{xxxx}\\ &+72u_{xx}^2u_{xxxx}v_{xxx}^4+36u_{xx}u_{xxx}^2v_{xx}^2v_{xxxx}^2+144u_{xx}u_{xxx}^2v_{xx}v_{xxx}^2v_{xxxx} \end{aligned}$$

$$
\begin{aligned}
&+ 84u_{xx}u_{xxx}u_{xxxx}v_{xx}^2v_{xxx}v_{xxxx} - 144u_{xx}u_{xxx}u_{xxxx}v_{xx}v_{xxx}^3 - 3u_{xx}u_{xxxx}^2v_{xx}^3v_{xxxx} \\
&+ 6u_{xx}u_{xxxx}^2v_{xx}^2v_{xxx}^2 - 72u_{xxx}^3v_{xx}^2v_{xxx}v_{xxxx} - 36u_{xxx}^2u_{xxxx}v_{xx}^3v_{xxxx} \\
&+ 72u_{xxx}^2u_{xxxx}v_{xx}^2v_{xxx}^2 - 6u_{xxx}u_{xxxx}^2v_{xx}^3v_{xxx} + u_{xxxx}^3v_{xx}^4)) \\
&/(144(u_{xx}^3v_{xxx}^3 - 3u_{xx}^2u_{xxx}v_{xx}v_{xxx}^2 + 3u_{xx}u_{xxx}^2v_{xx}^2v_{xxx} - u_{xxx}^3v_{xx}^3)).
\end{aligned}
$$

A direct computation proves the following property.

Proposition 17. *The expressions that define the above two equations are differential invariants of the affine Lie algebra.*

As it is well-known [32] the number of independent differential invariants for space curves is 2. In this sense the above system generalizes the result in item 3 of Theorem 4, where the affine algebra uniquely determines the vanishing of the only differential invariant of plane curves. It is reasonable to conjecture that an analogous result could hold in higher dimensional euclidean spaces.

4.2.3 Systems of ODEs associated with Lie algebra $\mathcal{C}(\mathbb{R}^3)$

Conformal vector fields of the euclidean space $\mathbb{R}^n$ form a Lie algebra of dimension $\frac{1}{2}(n+1)(n+2)$ except for the case $n=2$. In fact, in such a case, the dimension of the Lie algebra of conformal vector fields is infinite, as the latter are essentially identified with holomorphic functions of one complex variable. More precisely, if $X_1 + iX_2$ is a holomorphic function, then $X_1\partial_x + X_2\partial_y$ is a conformal vector field of the euclidean metric. If $n \geq 3$, the conformal algebra is formed by translations, rotations, the dilatation and special conformal vector fields. The local flow of such vector fields consists of special conformal transformations: such a transformation can be understood as an inversion followed by a translation and again followed by an inversion. Since it is well known that there are no systems of ODEs admitting an infinite dimensional Lie algebra of point symmetries (provided that such algebra has no zero-dimensional orbits), for the case $n=2$, instead of considering the full conformal Lie algebra, we shall consider the Lie subalgebra formed by the 2 translations, the rotation, the dilation and the 2 special conformal vector fields. Let us introduce the notation $y^0 = x$ and $y^i = u^i$. Then, in any dimension, special conformal vector field are given by

$$
\Xi_j = (2(y^j)^2 - \|y\|^2)\partial_{y^j} + 2\sum_{i\neq j} y^j y^i \partial_{y^i},
$$

and their local flow is

$$
{y_0^i}' = \frac{y_0^i - \delta_j^i t\|y_0\|^2}{1 - 2ty_0^j + t^2\|y_0\|^2}.
$$

Special conformal transformations preserve circles of $\mathbb{R}^n$. By adopting the method described in the beginning of Section 4, we proved by a straightforward computation that the only system of ODEs admitting $\mathcal{C}(\mathbb{R}^3)$ as Lie symmetry subalgebra is system of item 3 of Theorem 5.

We observe that in a recent paper [11] a system of ODEs is associated with the conformal symmetry algebra of an anti-self-dual conformal structure defined by a pseudo-Riemannian metric which is a solution of the Plebanski equation. If the dimension of the conformal symmetry algebra is high enough, the associated system of ODEs is Lie remarkable (like in Example 1 of [11]), and can be determined by our methods, as an alternative to the method proposed in [11].

4.2.4 Systems od ODEs associated with Lie algebra $\mathcal{P}(\mathbb{R}^3)$

The generators of Lie algebra $\mathcal{P}(\mathbb{R}^3)$ are

$$
\frac{\partial}{\partial a}, \quad a\frac{\partial}{\partial b}, \quad a\left(x\frac{\partial}{\partial x} + \sum_{i=1}^{m} u^i\frac{\partial}{\partial u^i}\right),
$$

for all $a, b \in \{x, u^i\}$. We can prove, by using the reasonings contained in the beginning of Section 4, that when $r = 6$ there exists the following Lie remarkable equation (u and v denote the dependent variables):

$$
\begin{aligned}
u_{xxxxxx} =&(-144u_{xx}^5 v_{xxx}^2 v_{xxxxx}^2 + 360u_{xx}^5 v_{xxx} v_{xxxx}^2 v_{xxxxx} - 225u_{xx}^5 v_{xxxx}^4 +\\
&288u_{xx}^4 u_{xxx} v_{xx} v_{xxx} v_{xxxxx}^2 - 360u_{xx}^4 u_{xxx} v_{xx} v_{xxxx}^2 v_{xxxxx}\\
&- 480u_{xx}^4 u_{xxx} v_{xxx}^2 v_{xxxx} v_{xxxxx} + 600u_{xx}^4 u_{xxx} v_{xxx} v_{xxxx}^3\\
&- 720u_{xx}^4 u_{xxxx} v_{xx} v_{xxx} v_{xxxx} v_{xxxxx} + 900u_{xx}^4 u_{xxxx} v_{xx} v_{xxxx}^3\\
&+ 960u_{xx}^4 u_{xxxx} v_{xxx}^3 v_{xxxxx} - 1200u_{xx}^4 u_{xxxx} v_{xxx}^2 v_{xxxx}^2\\
&+ 288u_{xx}^4 u_{xxxxx} v_{xx} v_{xxx}^2 v_{xxxxx} - 360u_{xx}^4 u_{xxxxx} v_{xx} v_{xxx} v_{xxxx}^2\\
&+ 240u_{xx}^4 u_{xxxxx} v_{xxx}^3 v_{xxxx} - 144u_{xx}^3 u_{xxx}^2 v_{xx}^2 v_{xxxxx}^2\\
&+ 960u_{xx}^3 u_{xxx}^2 v_{xx} v_{xxx} v_{xxxx} v_{xxxxx} - 600u_{xx}^3 u_{xxx}^2 v_{xx} v_{xxxx}^3\\
&- 640u_{xx}^3 u_{xxx}^2 v_{xxx}^3 v_{xxxxx} + 400u_{xx}^3 u_{xxx}^2 v_{xxx}^2 v_{xxxx}^2\\
&+ 720u_{xx}^3 u_{xxx} u_{xxxx} v_{xx}^2 v_{xxxx} v_{xxxxx} - 2400u_{xx}^3 u_{xxx} u_{xxxx} v_{xx} v_{xxx}^2 v_{xxxxx}\\
&+ 600u_{xx}^3 u_{xxx} u_{xxxx} v_{xx} v_{xxx} v_{xxxx}^2 + 400u_{xx}^3 u_{xxx} u_{xxxx} v_{xxx}^3 v_{xxxx}\\
&- 576u_{xx}^3 u_{xxx} u_{xxxxx} v_{xx}^2 v_{xxx} v_{xxxxx} + 360u_{xx}^3 u_{xxx} u_{xxxxx} v_{xx}^2 v_{xxxx}^2\\
&- 240u_{xx}^3 u_{xxx} u_{xxxxx} v_{xx} v_{xxx}^2 v_{xxxx} + 640u_{xx}^3 u_{xxx} u_{xxxxx} v_{xxx}^4\\
&+ 360u_{xx}^3 u_{xxxx}^2 v_{xx}^2 v_{xxx} v_{xxxxx} - 1350u_{xx}^3 u_{xxxx}^2 v_{xx}^2 v_{xxxx}^2\\
&+ 2400u_{xx}^3 u_{xxxx}^2 v_{xx} v_{xxx}^2 v_{xxxx} - 800u_{xx}^3 u_{xxxx}^2 v_{xxx}^4\\
&+ 720u_{xx}^3 u_{xxxx} u_{xxxxx} v_{xx}^2 v_{xxx} v_{xxxx} - 1200u_{xx}^3 u_{xxxx} u_{xxxxx} v_{xx} v_{xxx}^3\\
&- 144u_{xx}^3 u_{xxxxx}^2 v_{xx}^2 v_{xxx}^2 - 480u_{xx}^2 u_{xxx}^3 v_{xx}^2 v_{xxxx} v_{xxxxx}\\
&+ 1920u_{xx}^2 u_{xxx}^3 v_{xx} v_{xxx}^2 v_{xxxxx} - 800u_{xx}^2 u_{xxx}^3 v_{xx} v_{xxx} v_{xxxx}^2\\
&+ 1920u_{xx}^2 u_{xxx}^2 u_{xxxx} v_{xx}^2 v_{xxx} v_{xxxxx} + 600u_{xx}^2 u_{xxx}^2 u_{xxxx} v_{xx}^2 v_{xxxx}^2\\
&- 2000u_{xx}^2 u_{xxx}^2 u_{xxxx} v_{xx} v_{xxx}^2 v_{xxxx} + 288u_{xx}^2 u_{xxx}^2 u_{xxxxx} v_{xx}^3 v_{xxxxx}\\
&- 240u_{xx}^2 u_{xxx}^2 u_{xxxxx} v_{xx}^2 v_{xxx} v_{xxxx} - 1920u_{xx}^2 u_{xxx}^2 u_{xxxxx} v_{xx} v_{xxx}^3\\
&- 360u_{xx}^2 u_{xxx} u_{xxxx}^2 v_{xx}^3 v_{xxxxx} - 3000u_{xx}^2 u_{xxx} u_{xxxx}^2 v_{xx}^2 v_{xxx} v_{xxxx}\\
&+ 2800u_{xx}^2 u_{xxx} u_{xxxx}^2 v_{xx} v_{xxx}^3 - 720u_{xx}^2 u_{xxx} u_{xxxx} u_{xxxxx} v_{xx}^3 v_{xxxx}\\
&+ 3120u_{xx}^2 u_{xxx} u_{xxxx} u_{xxxxx} v_{xx}^2 v_{xxx}^2 + 288u_{xx}^2 u_{xxx} u_{xxxxx}^2 v_{xx}^3 v_{xxx}\\
&+ 900u_{xx}^2 u_{xxxx}^3 v_{xx}^3 v_{xxxx} - 1200u_{xx}^2 u_{xxxx}^3 v_{xx}^2 v_{xxx}^2\\
&- 360u_{xx}^2 u_{xxxx}^2 u_{xxxxx} v_{xx}^3 v_{xxx} - 1920u_{xx} u_{xxx}^4 v_{xx}^2 v_{xxx} v_{xxxxx}\\
&+ 400u_{xx} u_{xxx}^4 v_{xx}^2 v_{xxxx}^2 - 480u_{xx} u_{xxx}^3 u_{xxxx} v_{xx}^3 v_{xxxxx}\\
&+ 2800u_{xx} u_{xxx}^3 u_{xxxx} v_{xx}^2 v_{xxx} v_{xxxx} + 240u_{xx} u_{xxx}^3 u_{xxxxx} v_{xx}^3 v_{xxxx}\\
&+ 1920u_{xx} u_{xxx}^3 u_{xxxxx} v_{xx}^2 v_{xxx}^2 + 600u_{xx} u_{xxx}^2 u_{xxxx}^2 v_{xx}^3 v_{xxxx}\\
&- 3200u_{xx} u_{xxx}^2 u_{xxxx}^2 v_{xx}^2 v_{xxx}^2 - 2640u_{xx} u_{xxx}^2 u_{xxxx} u_{xxxxx} v_{xx}^3 v_{xxx}\\
&- 144u_{xx} u_{xxx}^2 u_{xxxxx}^2 v_{xx}^4 + 1800u_{xx} u_{xxx} u_{xxxx}^3 v_{xx}^3 v_{xxx}\\
&+ 360u_{xx} u_{xxx} u_{xxxx}^2 u_{xxxxx} v_{xx}^4 - 225u_{xx} u_{xxxx}^4 v_{xx}^4 + 640u_{xxx}^5 v_{xx}^3 v_{xxxxx}\\
&- 1200u_{xxx}^4 u_{xxxx} v_{xx}^3 v_{xxxx} - 640u_{xxx}^4 u_{xxxxx} v_{xx}^3 v_{xxx} + 1200u_{xxx}^3 u_{xxxx}^2 v_{xx}^3 v_{xxx}\\
&+ 720u_{xxx}^3 u_{xxxx} u_{xxxxx} v_{xx}^4 - 600u_{xxx}^2 u_{xxxx}^3 v_{xx}^4)/(160u_{xx}^4 v_{xxx}^4 -\\
&640u_{xx}^3 u_{xxx} v_{xx} v_{xxx}^3 + 960u_{xx}^2 u_{xxx}^2 v_{xx}^2 v_{xxx}^2 - 640u_{xx} u_{xxx}^3 v_{xx}^3 v_{xxx} + 160u_{xxx}^4 v_{xx}^4),\\
v_{xxxxxx} =&(-144u_{xx}^4 v_{xx} v_{xxx}^2 v_{xxxxx}^2 + 360u_{xx}^4 v_{xx} v_{xxx} v_{xxxx}^2 v_{xxxxx}\\
&- 225u_{xx}^4 v_{xx} v_{xxxx}^4 + 720u_{xx}^4 v_{xxx}^3 v_{xxxx} v_{xxxxx}
\end{aligned}
$$

$$
\begin{aligned}
&- 600u_{xx}^4 v_{xxx}^2 v_{xxxx}^3 + 288u_{xx}^3 u_{xxx} v_{xx}^2 v_{xxx} v_{xxxxx}^2\\
&- 360u_{xx}^3 u_{xxx} v_{xx}^2 v_{xxxx}^2 v_{xxxxx} - 2640u_{xx}^3 u_{xxx} v_{xx} v_{xxx}^2 v_{xxxx} v_{xxxxx}\\
&+ 1800u_{xx}^3 u_{xxx} v_{xx} v_{xxx} v_{xxxx}^3 - 640u_{xx}^3 u_{xxx} v_{xxx}^4 v_{xxxxx}\\
&+ 1200u_{xx}^3 u_{xxx} v_{xxx}^3 v_{xxxx}^2 - 720u_{xx}^3 u_{xxxx} v_{xx}^2 v_{xxx} v_{xxxx} v_{xxxxx}\\
&+ 900u_{xx}^3 u_{xxxx} v_{xx}^2 v_{xxxx}^3 + 240u_{xx}^3 u_{xxxx} v_{xx} v_{xxx}^3 v_{xxxxx}\\
&+ 600u_{xx}^3 u_{xxxx} v_{xx} v_{xxx}^2 v_{xxxx}^2 - 1200u_{xx}^3 u_{xxxx} v_{xxx}^4 v_{xxxx}\\
&+ 288u_{xx}^3 u_{xxxxx} v_{xx}^2 v_{xxx}^2 v_{xxxxx} - 360u_{xx}^3 u_{xxxxx} v_{xx}^2 v_{xxx} v_{xxxx}^2\\
&- 480u_{xx}^3 u_{xxxxx} v_{xx} v_{xxx}^3 v_{xxxx} + 640u_{xx}^3 u_{xxxxx} v_{xxx}^5\\
&- 144u_{xx}^2 u_{xxx}^2 v_{xx}^3 v_{xxxxx}^2 + 3120u_{xx}^2 u_{xxx}^2 v_{xx}^2 v_{xxx} v_{xxxx} v_{xxxxx}\\
&- 1200u_{xx}^2 u_{xxx}^2 v_{xx}^2 v_{xxxx}^3 + 1920u_{xx}^2 u_{xxx}^2 v_{xx} v_{xxx}^3 v_{xxxxx}\\
&- 3200u_{xx}^2 u_{xxx}^2 v_{xx} v_{xxx}^2 v_{xxxx}^2 + 720u_{xx}^2 u_{xxx} u_{xxxx} v_{xx}^3 v_{xxxx} v_{xxxxx}\\
&- 240u_{xx}^2 u_{xxx} u_{xxxx} v_{xx}^2 v_{xxx}^2 v_{xxxxx} - 3000u_{xx}^2 u_{xxx} u_{xxxx} v_{xx}^2 v_{xxx} v_{xxxx}^2\\
&+ 2800u_{xx}^2 u_{xxx} u_{xxxx} v_{xx} v_{xxx}^3 v_{xxxx} - 576u_{xx}^2 u_{xxx} u_{xxxxx} v_{xx}^3 v_{xxx} v_{xxxxx}\\
&+ 360u_{xx}^2 u_{xxx} u_{xxxxx} v_{xx}^3 v_{xxxx}^2 + 1920u_{xx}^2 u_{xxx} u_{xxxxx} v_{xx}^2 v_{xxx}^2 v_{xxxx}\\
&- 1920u_{xx}^2 u_{xxx} u_{xxxxx} v_{xx} v_{xxx}^4 + 360u_{xx}^2 u_{xxxx}^2 v_{xx}^3 v_{xxx} v_{xxxxx}\\
&- 1350u_{xx}^2 u_{xxxx}^2 v_{xx}^3 v_{xxxx}^2 + 600u_{xx}^2 u_{xxxx}^2 v_{xx}^2 v_{xxx}^2 v_{xxxx}\\
&+ 400u_{xx}^2 u_{xxxx}^2 v_{xx} v_{xxx}^4 + 720u_{xx}^2 u_{xxxx} u_{xxxxx} v_{xx}^3 v_{xxx} v_{xxxx}\\
&- 480u_{xx}^2 u_{xxxx} u_{xxxxx} v_{xx}^2 v_{xxx}^3 - 144u_{xx}^2 u_{xxxxx}^2 v_{xx}^3 v_{xxx}^2\\
&- 1200u_{xx} u_{xxx}^3 v_{xx}^3 v_{xxxx} v_{xxxxx} - 1920u_{xx} u_{xxx}^3 v_{xx}^2 v_{xxx}^2 v_{xxxxx}\\
&+ 2800u_{xx} u_{xxx}^3 v_{xx}^2 v_{xxx} v_{xxxx}^2 - 240u_{xx} u_{xxx}^2 u_{xxxx} v_{xx}^3 v_{xxx} v_{xxxxx}\\
&+ 2400u_{xx} u_{xxx}^2 u_{xxxx} v_{xx}^3 v_{xxxx}^2 - 2000u_{xx} u_{xxx}^2 u_{xxxx} v_{xx}^2 v_{xxx}^2 v_{xxxx}\\
&+ 288u_{xx} u_{xxx}^2 u_{xxxxx} v_{xx}^4 v_{xxxxx} - 2400u_{xx} u_{xxx}^2 u_{xxxxx} v_{xx}^3 v_{xxx} v_{xxxx}\\
&+ 1920u_{xx} u_{xxx}^2 u_{xxxxx} v_{xx}^2 v_{xxx}^3 - 360u_{xx} u_{xxx} u_{xxxx}^2 v_{xx}^4 v_{xxxxx}\\
&+ 600u_{xx} u_{xxx} u_{xxxx}^2 v_{xx}^3 v_{xxx} v_{xxxx} - 800u_{xx} u_{xxx} u_{xxxx}^2 v_{xx}^2 v_{xxx}^3\\
&- 720u_{xx} u_{xxx} u_{xxxx} u_{xxxxx} v_{xx}^4 v_{xxxx} + 960u_{xx} u_{xxx} u_{xxxx} u_{xxxxx} v_{xx}^3 v_{xxx}^2\\
&+ 288u_{xx} u_{xxx} u_{xxxxx}^2 v_{xx}^4 v_{xxx} + 900u_{xx} u_{xxxx}^3 v_{xx}^4 v_{xxxx}\\
&- 600u_{xx} u_{xxxx}^3 v_{xx}^3 v_{xxx}^2 - 360u_{xx} u_{xxxx}^2 u_{xxxxx} v_{xx}^4 v_{xxx}\\
&+ 640u_{xxx}^4 v_{xx}^3 v_{xxx} v_{xxxxx} - 800u_{xxx}^4 v_{xx}^3 v_{xxxx}^2 + 240u_{xxx}^3 u_{xxxx} v_{xx}^4 v_{xxxxx}\\
&+ 400u_{xxx}^3 u_{xxxx} v_{xx}^3 v_{xxx} v_{xxxx} + 960u_{xxx}^3 u_{xxxxx} v_{xx}^4 v_{xxxx} - 640u_{xxx}^3 u_{xxxxx} v_{xx}^3 v_{xxx}^2\\
&- 1200u_{xxx}^2 u_{xxxx}^2 v_{xx}^4 v_{xxxx} + 400u_{xxx}^2 u_{xxxx}^2 v_{xx}^3 v_{xxx}^2\\
&- 480u_{xxx}^2 u_{xxxx} u_{xxxxx} v_{xx}^4 v_{xxx} - 144u_{xxx}^2 u_{xxxxx}^2 v_{xx}^5\\
&+ 600u_{xxx} u_{xxxx}^3 v_{xx}^4 v_{xxx} + 360u_{xxx} u_{xxxx}^2 u_{xxxxx} v_{xx}^5 - 225u_{xxxx}^4 v_{xx}^5)\big/(160u_{xx}^4 v_{xxx}^4\\
&- 640u_{xx}^3 u_{xxx} v_{xx} v_{xxx}^3 + 960u_{xx}^2 u_{xxx}^2 v_{xx}^2 v_{xxx}^2 - 640u_{xx} u_{xxx}^3 v_{xx}^3 v_{xxx} + 160u_{xxx}^4 v_{xx}^4).
\end{aligned}
$$

Acknowledgments. Work supported by G.N.F.M. and by G.N.S.A.G.A of I.N.d.A.M. We thank Paola Morando of the University of Milan for stimulating discussions.

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