# On the Mathematical and Geometrical Structure of the Determining Equations for Shear Waves in Nonlinear Isotropic Incompressible Elastodynamics

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**Abstract**

Using the theory of 1+1 hyperbolic systems we put in perspective the mathematical and geometrical structure of the celebrated circularly polarized waves solutions for isotropic hyperelastic materials determined by Carroll in Acta Mechanica 3 (1967) 167–181. We show that a natural generalization of this class of solutions yields an infinite family of *linear* solutions for the equations of isotropic elastodynamics. Moreover, we determine a huge class of hyperbolic partial differential equations having the same property of the shear wave system. Restricting the attention to the usual first order asymptotic approximation of the equations determining transverse waves we provide the complete integration of this system using generalized symmetries.

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# 1 Introduction and Basic Equations

The propagation of transverse or shear waves in incompressible isotropic nonlinear hyperlasticity is governed by a system of non-linear equations in $1+1$ independent variables $(x,t)$

$$\begin{aligned} &\varrho u_{tt} - \left[Q(u_x^2+v_x^2)u_x\right]_x = 0, \\ &\varrho v_{tt} - \left[Q(u_x^2+v_x^2)v_x\right]_x = 0, \end{aligned} \tag{1.1}$$

where $u=u(x,t)$ and $v=v(x,t)$ are the unknown functions (the transverse motions), $\varrho$ is the constant density and $Q=Q(u_x^2+v_x^2)$ is the generalized shear modulus[15].

It is usual to derive (1.1) with respect to $x$, and to introduce as new unknowns the strains $U=u_x$ and $V=v_x$, to rewrite (1.1) as a first-order homogeneous quasilinear system

$$\boldsymbol{U}_t + \mathcal{A}(\boldsymbol{U})\boldsymbol{U}_x = 0, \tag{1.2}$$

see, for example, [29] or system (7.1.14) page 176 in [13]. Indeed, introducing $\tilde{Q}=Q/\varrho$ in terms of the strains

$$U_t = M_x, \quad M_t = [\tilde{Q}U]_x, \quad V_t = N_x, \quad N_t = [\tilde{Q}V]_x. \tag{1.3}$$

On the other hand, the highly symmetric structure of (1.1) suggest an alternative compact form for this system. Using the complex function $W$ with modulus $\Omega$ and argument $\theta$, defined by

$$W(x,t) = \Omega(x,t)\exp[i\theta(x,t)] = U(x,t) + iV(x,t), \tag{1.4}$$

so that

$$U = \mathrm{Re}\{W\} = \Omega\cos(\theta), \quad V = \mathrm{Im}\{W\} = \Omega\sin(\theta),$$

we rewrite (1.1) as a single complex equation

$$\varrho W_{tt} - [Q(\Omega)W]_{xx} = 0. \tag{1.5}$$

This complex format has been used in the study of the nonlinear string by Rubin, Rosenau and co-workers[28, 37, 38] and by Destrade and Saccomandi in nonlinear elasticity and some related theories[14, 15, 17], to obtain some similarity reductions for the system (1.1) and several exact solutions. This format has been fundamental to unify all of the Carroll's results[9, 10, 11] and to generalize them. Moreover, this format has been used by Rogers[36] to show a relationship between (1.1) and the Ermakov–Ray–Reid system.

For example, let us consider a special class of such similarity solutions: the Carroll's finite amplitude circularly-polarized harmonic progressive waves[9]. These solutions are obtained considering $\Omega = A$ and $\theta = kx - \omega t$ and are defined as

$$U(x,t) = A\cos(kx - \omega t), \quad V(x,t) = \pm A\sin(kx - \omega t), \tag{1.6}$$

where the amplitude $A$, the wave number $k$, and the frequency $\omega$ are real positive constants, and the plus (minus) sign for $V$ corresponds to a left (right) circularly-polarized wave. The waves (1.6) are solutions for the system (1.1) if the dispersion relation

$$\varrho\omega^2 = k^2 Q(A^2). \tag{1.7}$$

is satisfied. Under very mild conditions on the generalized shear modulus (1.7) is clearly satisfied and therefore (1.6) are an example of smooth global solutions of the Cauchy problem in the whole space composed by (1.1) and the initial conditions

$$\begin{aligned} u(x,0) &= A\cos(kx), & v(x,0) &= \pm A\sin(kx), \\ u_t(x,0) &= A\omega\sin(kx), & v_t(x,0) &= \mp A\omega\cos(kx). \end{aligned}$$

The (1.6) are interesting not only because they are beautiful exact closed form solutions for a large class of materials but, to the well educated reader, it is clear that these solutions are *exceptional* in their mathematical character: global solutions of permanent form of a nonlinear hyperbolic system.

The possibility of smooth global solution for equations (1.1) has been first noticed by John in his celebrated 1974 paper[30]:

> *On the other hand, there are also known special non-singular wave solutions (e.g., the transverse waves in Hadamard materials (see* [4]*), or the Caroll waves in arbitrary materials (see* [6]*). This raises the question whether the results of the present paper that depend on* genuine non-linearity *of the system ever apply to plane elastic waves. It will be proved here that there is indeed a large class of plane waves where we have no genuine non-linearity, namely all those in which the wave front contains a principal direction of strain.*

John is pointing out that there is a special class of materials (Mooney-Rivlin materials in the incompressible setting) for which (1.1) reduces to a linear system of equations. (For Mooney-Rivlin materials we have $Q \equiv \mu$ where $\mu$ is the constant infinitesimal shear modulus).

On the other hand, for general class of materials, the special structure of the matrix $\mathcal{A}$ in (1.2) reduces the problem to one involving the ordinary simple waves of a homogeneous system comprising only two equations. Each simple wave is associated with one of the four eigenvalues[1] $\lambda_{\pm}^{(1)}$ and $\lambda_{\pm}^{(2)}$ of $\mathcal{A}$. Being $\lambda^{(1)} > \lambda^{(2)}$, using a terminology introduced in [12], it is usual to refer to such waves as *fast* and *slow* waves.

Slow waves are *exceptional* or *linearly degenerate* in the hyperbolic systems language, i.e. the gradient of the corresponding eigenvalue is orthogonal to the corresponding eigenvector. It is well known (see corollary 8.2.6 page 211 in [13]) that when a characteristic family of an hyperbolic system is linearly degenerate travelling waves solutions are possible. This corollary contains the *mathematical reason* of the existence of Carroll waves.

We record several papers concerning linearly degenerate or exceptional systems in elastodynamics. Some papers are dedicated to existence theorems, for example [33], others, mainly by the Boillat's school [5, 7, 18], are more focused on the interplay between this peculiar mathematical structure and the constitutive nature of some material laws. The aim of such papers is to determine special classes of elastic strain-energy densities for which the mathematical resolution is simpler than usual because the equations are completely exceptional. Moreover, there is a huge literature about plane transverse acceleration waves in elastic solids (see for example [39]), where it is noticed that in incompressible materials there is a direction where waves may propagate without a change in amplitude because of the exceptional character of one of their eigenvalues.

On the other hand, these exact solutions must play a role in the framework of the well-posedness of the Cauchy problem for incompressible dynamic elasticity. This subject is denoted as problem number 12 in John Ball's review about open problems in nonlinear elasticity[3][2]. In reviewing this problem Ball cites the works by Ebin and coworkers[21, 22, 23] and the work by Hrusa and Renardy[25]. The paper [22] (dated 1996) opens in a very significant, communicative and direct way:

> **Theorem.** *The initial value problem for the equations of motion of an incompressible hyperelastic homogeneous isotropic material has classical solutions for all time, if the initial displacement and velocity are small.*

It is clear that in [21, 22, 23, 25] the investigations are not restricted to transverse waves, but clearly Carroll's solutions are an example of global existence for large amplitude initial data in unbounded domains.

The more recent paper by Sideris and Thomases[40] points out that since in isotropic nonlinear elastic systems shear waves are linearly degenerate the so called *null condition* is automatically satisfied. This fact is used to confirm the *intuitive idea* that suitable weighted local decay estimates for the perturbative equations can be expected via the generalized

[1]The plus and minus sign are associated to forwards and backwards waves respectively.

[2]The problem 12 in [3] is: *Prove the global existence and uniqueness of solutions to initial-boundary value problems for properly formulated dynamic theories of nonlinear elasticity.*

energy method and therefore the existence of global-in-time classical solutions to the Cauchy problem for incompressible elastic materials is proved for *small* initial displacements. This fact is confirmed in [32] where the possibility to obtain such kind of results is connected to the exceptionality of the equations. In [32] we read:

> *The equations of incompressible elastodynamics display a linear degeneracy in the isotropic case; i.e., the equation inherently satisfies a null condition. By taking the advantage of this structure, we prove that the 2-D incompressible isotropic nonlinear elastic system is almost globally well-posed for small initial data.*

Once again the connection between the linearly degenerate structure of the equations and the possibility to find global solutions for the Cauchy problem (and this also for large data) seems to have been not noticed.

This situation is strange because the paper [32] focuses on the neo-Hookean material in the two dimensional case. (We point out that in the two dimensional case the neo-Hookean material cannot be distinguished from the Mooney-Rivlin material.) The fact that many solutions of the neo-Hookean model may be found solving linear equations is well known to the practitioners of nonlinear elasticity, see for example [31].

The aim of the present note is to push forward the connection between the exceptionality of the system (1.1) and the existence of exact solutions as (1.6). In so doing we determine explicitly a huge class of smooth solutions. Moreover, we are able to point out that this kind of exact solutions are peculiar of an entire class of linearly degenerate second order differential equations.

The plan of the paper is the following. In Section 2 we generalize the (1.6). In so doing, we provide a huge class of new exact solutions for the equations of non-linear elastodynamics. In Section 3 with derive the usual first order asymptotic model associated with (1.1). This allows to introduce a simpler format for our investigations. We realize that the asymptotic system corresponding to (1.1) is a Temple system[41, 1]. Using symmetry transformations we are able to provide not only the full class of *linear* solutions for such system, but to derive also the general integral. The last Section is devoted to concluding remarks.

Symbolic computations were performed in CDIFF[44], a freely available REDUCE[35] package for computations in the geometry of differential equations.

# 2 A Generalization of Carroll Solutions

Let us consider $\Omega = \text{const.}$ in the system (1.4) . In this case starting from (1.5) we obtain an overdetermined system in the unknown $\theta = \theta(t, x)$. The general solution of this overdetermined system, when $Q > 0$, is simple and given by the solutions of the first order wave equations

$$\theta_t \pm \sqrt{Q/\varrho}\theta_x = 0.$$

This means that the Carroll's solutions (1.6) are only one possible choice among infinite possibilities.

In general it is possible to have solutions in the form

$$U(x,t) = A\cos(\theta), \quad V(x,t) = \pm A\sin(\theta), \tag{2.1}$$

where $\theta = F(x \pm \sqrt{Q/\varrho}t)$ and $F$ is an arbitrary function. Of course, new solutions cannot be obtained as sums of solutions $\theta$ with different signs since they are solutions of nonlinear equations. This is the *large class of plane waves where we have no genuine non-linearity* identified in [30]. To the best of our knowledge, the explicit determination of such exact solution have been unnoticed.

If in (1.4) we set $\theta =$ const., i.e. we consider a plane polarized wave, from (1.5) we obtain the single real second order partial differential equation $\varrho\Omega_{tt} - [Q(\Omega)\Omega]_{xx} = 0$ for which, if $Q \neq$ const., any solution blows up[27].

We point out that to ensure that the generalized shear modulus $Q$ is positive it is sufficient to impose that the strain-energy density satisfies the usual *empirical inequalities*[2].

The possibility to find explicitly a similar huge class of exact solution for a nonlinear system of partial differential equations is not restricted to the system (1.1). Let us consider the abstract mathematical system

$$U_{tt} - [P(U,V)U]_{xx} = 0, \tag{2.2a}$$

$$V_{tt} - [P(U,V)V]_{xx} = 0, \tag{2.2b}$$

where $P$ is a suitable *constitutive* function. We suppose that $P(U,V) > 0$ in the domain of interest. We can set $P(U,V) = A$ where $A > 0$ is a constant and therefore, under suitable assumptions on $P$, it is possible to write $V = \Psi(U;A)$. Then the system (2.2) is transformed in the overdetermined system

$$U_{tt} - AU_{xx} = 0, \tag{2.3a}$$

$$[\Psi(U;A)]_{tt} - A\,[\Psi(U;A)]_{xx} = 0. \tag{2.3b}$$

It is easy to check, by a direct computation, that $U = F(x \pm \sqrt{A}t)$ solves this overdetermined system. On the other hand, the system (2.2) is compatible if $V = kU$ where $k$ is an arbitrary constant and in this case we reduce the system to a single genuinely non-linear partial differential equation of the second order.

Therefore, we have pointed out a very general result peculiar to the system (2.2) of second order hyperbolic differential equations in $1+1$ dimensions in two unknowns. To the best of our knowledge the explicit characterization of the solutions we have provided has never been noticed.

**Remark 1.** *If $P = P(U/V)$ the substitution $V = kU$ reduces the system* (2.2) *to a set of two uncoupled linear differential equation. This choice of $P$ in the family* (2.2) *is special as we can check when $P = U/V$. This system is completely exceptional[6].*

# 3 An Asymptotic Model

In nonlinear acoustics it is usual to derive an asymptotic model for the system (1.1). This Section is devoted to a detailed discussion of such a system. The system has a mechanical interest, and its mathematical structure has been deeply studied (see for example [26]).

Let us introduce the Taylor expansion of the generalized shear modulus

$$Q(U^2 + V^2) = \mu_0 + \mu_1(U^2 + V^2) + \dots,$$

and let us assume $U = \epsilon\hat{U}$, $V = \epsilon\hat{V}$ introducing the new independent variables $X = \epsilon^2 x$ and $\tau = t - x/\mathfrak{c}_0$, where $\mathfrak{c}_0 = \mu_0/\varrho$. Here $\epsilon$ is a small parameter. Considering only terms up to $\mathcal{O}(\epsilon^3)$ and introducing the notation $U := \partial\hat{U}/\partial\tau$, $V := \partial\hat{V}/\partial\tau$ we obtain the first order hyperbolic system[3]

$$U_X - \beta\left[(U^2 + V^2)U\right]_\tau = 0, \tag{3.1a}$$
$$V_X - \beta\left[(U^2 + V^2)V\right]_\tau = 0, \tag{3.1b}$$

where $\beta = \mathfrak{c}_1/(2\mathfrak{c}_0^2)$ (here $\mathfrak{c}_1 = \mu_1/\varrho$).

Introducing polar coordinates[4]

$$U = \rho(X,\tau)\cos\vartheta(X,\tau), \quad V = \rho(X,\tau)\sin\vartheta(X,\tau), \tag{3.2}$$

the system (3.1) is rewritten as

$$\vartheta_X - \beta\rho^2\vartheta_\tau = 0, \tag{3.3a}$$
$$\rho_X - 3\beta\rho^2\rho_\tau = 0 \rightarrow \rho_X - \beta(\rho^3)_\tau = 0. \tag{3.3b}$$

If $\vartheta$ = const. we have plane polarized waves and the system (3.3) reduces to a single partial differential equation whose exact solution is $\rho = \Phi(\tau - 3\beta X\rho^2)$.

[3]We point out that the system we derive is exactly the *toy* hyperbolic system introduced in [13] page 182 formula (7.2.11).

[4]here $\rho$ is not to be confused with the density $\varrho$.

On the other hand, a class of remarkable solutions for system (3.3) is obtained when $\rho = A$, where $A$ as in the circularly polarized wave solutions (1.6) is an arbitrary constant. This class of solutions is obtained solving a linear equation, indeed, in this case we have the solutions

$$U = A\cos\left(\Theta(\xi)\right), \quad V = A\sin\left(\Theta(\xi)\right), \tag{3.4}$$

where $\vartheta = \Theta(\xi)$ and $\xi = \beta A^2 X + \tau$. This is an infinite family of smooth solutions of the nonlinear system (3.1).

The system (3.3) maybe easily rewritten in conservative form as

$$(\rho\vartheta)_X - \beta(\rho^3\vartheta)_\tau = 0, \tag{3.5a}$$

$$\rho_X - \beta(\rho^3)_\tau = 0. \tag{3.5b}$$

The general solution of the system (3.5) may be represented introducing the potential variable $\phi = \phi(X, \tau)$ such that

$$\phi_\tau = \rho, \quad \phi_X = \beta\rho^3. \tag{3.6}$$

Indeed, the general integral (in implicit form) of (3.5b) is well known and for any given solution $\rho(X, \tau)$ it is possible to define a corresponding $\phi$ from (3.6). If we consider $\vartheta = F(\phi)$, where $F$ is an arbitrary function, we obtain the general solution of (3.3). By a direct check denoting $dF/d\phi = F'$ we compute

$$(\rho F)_X - \beta(\rho^3 F)_\tau \equiv \rho F'\phi_X - \beta\rho^3 F'\phi_\tau \equiv \rho F'\left(\phi_X - \beta\rho^2\phi_\tau\right) \equiv 0.$$

Another representation of the exact solution of this system (when $\rho \neq$ const.) is obtained by the so-called *generalized hodograph method*[42]. The method makes use of families of commuting generalized (or higher) symmetries (see, for example, [4, 34]) of the system (3.3). Such symmetries are generalized (or higher) vector fields $\varphi^\vartheta\partial/\partial\vartheta + \varphi^\rho\partial/\partial\rho$. Here the word "generalized" means that the coefficients are functions of derivatives of an arbitrarily high order, *i.e.* $\varphi^\vartheta = \varphi^\vartheta(X, \tau, \vartheta, \rho, \vartheta_X, \vartheta_\tau, \rho_X, \rho_\tau, \ldots)$ and analogously for $\varphi^\rho$. Generalized vector fields are generalized symmetries if they are solutions of the linearized system

$$D_X(\varphi^\vartheta) - 2\beta\rho\vartheta_\tau\varphi^\rho - \beta\rho^2 D_\tau(\varphi^\vartheta) = 0 \tag{3.7a}$$

$$D_X(\varphi^\rho) - 6\beta\rho\vartheta_\tau\varphi^\rho - 3\beta\rho^2 D_\tau(\varphi^\rho) = 0 \tag{3.7b}$$

*over* the system (3.3). *Hydrodynamic type symmetries* are generalized symmetries that have the simplest structure with respect to derivatives: for our equation they are of the form $\varphi^\vartheta = \varphi^\vartheta(\vartheta, \rho)\vartheta_\tau$ and analogously for $\varphi^\rho$. Note that a vector field on the space of dependent and independent variables $\xi^\tau\partial/\partial\tau+\xi^X\partial/\partial X+\eta^\vartheta\partial/\partial\vartheta+\eta^\rho\partial/\partial\rho$ is a classical (point) symmetry if and only if its vertical part $(\eta^\vartheta-\vartheta_X\xi^X-\vartheta_\tau\xi^\tau)\partial/\partial\vartheta+(\eta^\rho-\rho_X\xi^X-\rho_\tau\xi^\tau)\partial/\partial\rho$ is a solution of the above linearized system; in this sense the hydrodynamic type symmetries are the simplest generalized symmetries.

It is known[42] that diagonal hydrodynamic-type systems in 2 dependent variables admit a space of hydrodynamic-type symmetries which are parametrized by two arbitrary functions and commute, as vector fields, with the vector field defined by the right-hand side of the differential equation.

In our case, the system (3.3) admits the hydrodynamic symmetries

$$\phi=-\left(\frac{s_3(\theta)}{\rho}+s_4(\rho)\right)\theta_\tau\frac{\partial}{\partial\theta}-\left(\frac{d(s_4(\rho)\rho)}{d\rho}\right)\rho_\tau\frac{\partial}{\partial\rho} \tag{3.8}$$

The above symmetries commute with the vector field $\varphi=\beta\rho^2\vartheta_\tau\partial/\partial\vartheta+3\beta\rho^2\rho_\tau\partial/\partial\rho$ which is given by the right-hand side of the equation (3.3) (see the Appendix for more details). Then every solution of the algebraic system

$$-\left(\frac{s_3}{\rho}+s_4\right)=\beta\rho^2X+\tau \tag{3.9a}$$

$$-\left(\frac{ds_4}{d\rho}\rho+s_4\right)=3\beta\rho^2X+\tau \tag{3.9b}$$

(where $s_3=s_3(\theta)$ and $s_4=s_4(\rho)$ are two arbitrary functions) is a solution of the system (3.3) with the property that $u^i_\tau\neq0$, and conversely any solution of (3.3) with the property $u^i_\tau\neq0$ can be locally represented as a solution of (3.9). Indeed, the system (3.3) can be solved through the hodograph transformation $(X,\tau)\mapsto(\theta,\rho)$. If one performs that transformation the system (3.3) becomes

$$\tau_\rho+\beta\rho^2X_\rho=0$$
$$\tau_\theta+3\beta\rho^2X_\theta=0$$

which is equivalent to

$$(\beta\rho^2\tau+X)_\rho=(\beta\rho^2)_\rho\tau$$
$$(3\beta\rho^2\tau+X)_\theta=(3\beta\rho^2)_\theta\tau=0$$

On solutions of (3.3) the above system is fulfilled since it is equivalent to the commutativity condition $[\phi,\psi]=0$ (see the Appendix). Algebraic solving system (3.9) we obtain the implicit solutions:

$$X=\frac{s_3}{2\beta\rho^3}-\frac{s_4'}{2\beta\rho},\quad \tau=-\frac{3s_3}{2\rho}-s_4+\frac{\rho s_4'}{2}. \tag{3.12}$$

Due to their implicit character, the solutions in (3.12) will develop a shock in finite time.

The Cauchy problem for (3.3) have been considered into details in[24], but the possibility to find in a simple a direct way a large class of exact smooth solutions seems to have been always skipped. Indeed, the main theorem contained in [24] speaks of *piecewise smooth solutions with locally finitely many shocks*. This kind of solutions are exactly the ones represented by (3.12). On the other hand, we point out that Theorem 2[24] speaks of *soliton-like solutions*, i.e. a rotational wave emerging as *a travelling wave of unchanged shape*. This kind of solutions seems to be exactly the ones we have explicitly determined in (3.4).

# 4 Temple systems

Let us put in the right framework the geometrical structure of system (3.1). To this end let us consider a general $2 \times 2$ autonomous and uniform hyperbolic system

$$u_t = [A(u, v)]_x, \quad u_t = [B(u, v)]_x, \tag{4.1}$$

where $A$ and $B$ are smooth solutions. This system of differential equations (for a given choice of $A$ and $B$) is defined in the base space $(t, x) \times (u, v)$ prolonged to the jet space over this base space containing the first order derivatives. Our goal is to understand if in the subvariety of the solutions of (4.1), say $\mathcal{S}$, it is possible a subset $L\mathcal{S} \subset \mathcal{S}$, which may be determined solving a linear differential equations.

This may be done using several methods, but in solving (3.1) we have used the following one. We have considered a subset of $(t, x) \times (u, v)$ defined by a relation of the kind

$$\varphi(u, v) = k, \tag{4.2}$$

where $k$ is a constant. Then we have shown that when we restrict (4.1) to (4.2) in the base space we obtain an overdetermined but compatible system of equations which is linear. Indeed since in the base space (4.2) holds then whe should have in the jet space

$$\varphi_u u_x + \varphi_v v_x = 0, \quad \varphi_u u_t + \varphi_v v_t = 0,$$

and therefore from (4.1) (we impose $\varphi_v \neq 0$) we obtain the overdetermined system

$$u_t = \left( A_u - A_v \frac{\varphi_u}{\varphi_v} \right) u_x, \quad \frac{\varphi_u}{\varphi_v} u_t = \left( B_u - B_v \frac{\varphi_u}{\varphi_v} \right) u_x. \tag{4.3}$$

By using the standard Lagrange-Charpit method we have that (4.3) is fully compatible if and only if

$$B_u \varphi_v^2 + (A_u - B_u)\varphi_u \varphi_v - A_v \varphi_u^2 = 0, \tag{4.4}$$

and the single differential equation to which the overdetermined system is reduced is linear, for example, if and only if

$$A_u - A_v \frac{\varphi_u}{\varphi_v} = k, \tag{4.5}$$

where we point out again that $k$ is constant.

The general solution of (4.4) and (4.5) is given by

$$A(u, v) = H(\varphi)u + \Phi(\varphi), \tag{4.6a}$$

$$B(u, v) = H(\varphi)v + \Psi(\varphi), \tag{4.6b}$$

where $H, \Phi$ and $\Psi$ are arbitrary functions of $\varphi(u, v)$ and we have considered that we are solving the (4.4) and (4.5) in the conical subset of the basic space defined by (4.2).

We have therefore determined a class of hyperbolic systems containing the Temple system[41, 1]

$$u_t - [P(u, v)u]_x = 0, \tag{4.7a}$$

$$v_t - [P(u, v)v]_x = 0, \tag{4.7b}$$

as special case.

For this system the corresponding Cauchy problem have been extensively studied[41] but once again we notice that the possibility to deduce an infinity of solutions for (4.7) solving a linear equation seems to have been skipped. Here we perform some general consideration about system (4.7). Let us rewrite (4.7) as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} P_u u + P & P_v u \\ P_u v & P_v v + P \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} \tag{4.8}$$

The right-eigenvalues of $A$ in (4.8) are

$$\lambda_1 = P + P_u u + P_v v, \quad \lambda_2 = P, \tag{4.9}$$

and the corresponding right-eigenvectors are

$$\boldsymbol{d}_1 = (1, v/u), \quad \boldsymbol{d}_2 = (1, -P_u/P_v). \tag{4.10}$$

About (4.8) we remark:

- The eigenvector $\boldsymbol{d}_2$ is exceptional: indeed $\nabla\lambda_2 \cdot \boldsymbol{d}_2 = 0$.

- We have $P(u,v) = P(u/v)$ if and only if $\lambda_1 = \lambda_2$.
- The system (4.8) is *completely* exceptional if and only if $uP_u + vP_v = u^{-1}H(u/v)$, with $H$ arbitrary function.
- The system (4.8) may be derived by a classical Hamiltonian density $\mathcal{H}[u] = \int H dx$ and rewritten as $u_t = [H_v]_x$, $v_t = [H_u]_x$ if and only if $vP_v = uP_u$, i.e. $P = P(uv)$.

Here we are interested in the case where only one eigenvalue is exceptional. In this case we use only the Riemann invariant corresponding to this eigenvalue to transform the system in a form where it is clear how to detect the class of *linear* equations. We introduce the point transformation of the dependent variables $\alpha = \alpha(u,v)$ and $\beta = \beta(u,v)$ such that $P(u,v) = R(\alpha)$ and $\beta = u/v$. In so doing (4.7) is rewritten in the form

$$\alpha_t - (R'(\alpha_u u + \alpha_v v) + R(\alpha))\alpha_x = 0, \tag{4.11a}$$

$$\beta_t - R(\alpha)\beta_x = 0. \tag{4.11b}$$

This is a diagonal two-component hydrodynamic type system and, as such, it can always be solved with the generalized hodograph method of the previous section. However, for concrete cases of $R$ and $\alpha$ the system could be still difficult to solve.

When $\alpha$ is constant (i.e. $P(u,v)$ is constant) the system (4.11) collapses to a single *linear* differential equation. Therefore it always possible to find an infinity of *linear* solutions for the nonlinear system (4.7).

In general the first equation (4.11) is of the form $\alpha_t - f(\alpha,\beta)\alpha_x = 0$. However, it may happen that for special functional forms of $R$ and $\alpha$ this equation decouples from the second one. The condition for the decoupling is

$$\frac{\partial(\alpha_u u + \alpha_v v)}{\partial\beta} = 0 \quad \Rightarrow \tag{4.12}$$

$$\frac{\partial u}{\partial\beta}(\alpha_{uu}u + \alpha_{uv}v + \alpha_u) + \frac{\partial v}{\partial\beta}(\alpha_{uv}u + \alpha_{vv}v + \alpha_v) = 0.$$

For instance, the above condition holds if $\alpha_u u + \alpha_v v$ is constant; a function $\alpha$ fulfilling this last property $\alpha = e^{a(u-v)+c}$ with $a$, $b$, $c$ three constants. Another possibility is that $\alpha(u,v) = uv$.

The decoupled system is the general category to which the asymptotic system associated with the elastic system (1.1) belongs. In this case

$$\alpha_t - f(\alpha)\alpha_x = 0, \tag{4.13a}$$

$$\beta_t - R(\alpha)\beta_x = 0. \tag{4.13b}$$

The same method as the asymptotic system applies to the above system. Indeed, the hydrodynamic symmetries are

$$\phi = s_1\alpha_x\frac{\partial}{\partial\alpha} + s_2\beta_x\frac{\partial}{\partial\beta} \tag{4.14}$$

where $s_1 = s_1(\alpha)$ is arbitrary and $s_2 = s_2(\alpha,\beta)$ is subject to the following linear ODE:

$$R_\alpha(s_2 - s_1) + s_{2\,\alpha}(f - R) = 0$$

which can be solved in a standard way. The resulting symmetries commute, hence the generalized hodograph method provides the generic solution. For example, in the particular case $\alpha(u, v) = uv$ we have $f(\alpha) = R_\alpha 2\alpha + R$ and the above equation becomes $s_2 - s_1 + 2\alpha s_{2\,\alpha} = 0$ whose explicit solution does not depend on $R$ and can be given in quadratures after the form of $s_1$ is specified.

In the general situation when $f = f(\alpha,\beta)$ the constraint on $s_2$ in the hydrodynamic symmetries is a partial differential equation whose solutions must be investigated for each explicit form of $f$.

**Remark 2.** *The case when* (4.7) *is completely exceptional has been extensively studied[6]. In this case using Riemann invariants and then an hodograph transformation the system may be linearized. A general theory for the linearization of completely exceptional second order hyperbolic conservative equations is provided in [19]. For example, let us consider the completely exceptional system obtained by setting $P(u, v) = u/v$. Now* (4.11) *is*

$$u_t = \left(\frac{u^2}{v}\right)_x, \quad v_t = u_x, \tag{4.15}$$

*equivalent to*

$$\psi_x = u, \quad \psi_t = \frac{u^2}{v}, \quad \phi_x = v, \quad \phi_t = u. \tag{4.16}$$

*Being $\psi_x = \phi_t$ we have a* stream *function $\chi$ such that $\psi = \chi_t, \quad \phi = \chi_x$ and the equation $\psi_t = u^2/v$ is therefore equivalent to the classical homogeneous Monge-Ampere equation $\chi_{tt}\chi_{xx} - \chi_{xt}^2 = 0$, which, it is well known, is a linearizable equation.*

# 5 Concluding Remarks

We have considered the celebrated polarized circularly waves found by Carroll in [9]: this is a class of beautiful and simple general smooth exact solutions for the nonlinear theory of isotropic elasticity. We have provided evidence that the mathematical reason for the existence of such solutions is the exceptional character of the hyperbolic system of determining equations for such waves. Indeed, we have been able to generalize such solutions in a straightforward way. To our knowledge, the huge class of exact solutions (clearly obtainable by similarity methods) that we have obtained was not noticed before, despite the fact that several papers have been devoted to the group analysis of systems of wave equations in $1+1$ dimensions (see for example [43]).

To simplify the algebra we have restricted our attention to the asymptotic first order system. For the system (2.2) the computation of the generalized symmetries is a possible but cumbersome procedure. For this reason, to determine similarity reductions the approach proposed by Carroll[10, 11] used by Destrade and Saccomandi[14, 15, 16, 17] via the complex-coordinates formalism in (1.5) may be relevant.

For example, going back to the system of second order differential equations (2.2) it is easy to understand how the Carroll's method may be extended. Clearly this extension it is not trivial, elegant and beautiful as for system (1.1), but it is still an effective method to find by reduction to ordinary differential equation some exact solutions of (2.2).

A non trivial example of how it is possible to extend the ideas of Carroll is given considering $P = P(uv)$ in (2.2). In this case we start considering

$$u = \phi(t)\psi(x), \quad v = \frac{\phi(t)}{\psi(x)}.$$

In so doing we get $P = P(\phi^2)$ and we obtain the class of solution

$$u = \phi(t)\exp(kx), \quad v = \phi(t)\exp(-kx),$$

where $\phi_{tt} = k^2 P(\phi^2)\phi$.

On the other hand, our discussion opens some interesting mathematical questions. First of all if there is the possibility of the existence of smooth solutions (in the whole space) and this also for large initial data for the equations of non-linear elasticity. Then if the global existence theorems by Temple[41] may be for certain conditions on the initial data reformulated in this more strong setting. A first step in this direction seems given in [8].

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# Appendix: symmetries and conservation laws

The system (3.3) is a hydrodynamic-type quasilinear system in diagonal form and the coefficients of $\theta_\tau$ and $\rho_\tau$ are its Riemann invariants []. However, the system is not completely exceptional [] (or linearly degenerate, according with another terminology []): the gradients of its eigenvalues are not orthogonal to its eigenvectors. An entropy pair (or a conservation law) for (3.3) is given by any relation

$$D_\tau T(\theta,\rho) - D_x X(\theta,\rho) = 0,$$

satisfied by all solutions of (4.11). (Here $D_\tau$ and $D_x$ are the usual total derivatives). The system admits the following hydrodynamic conservation law densities:

$$X = -\frac{dc_1}{d\rho}\rho^2 - 3c_1\rho - c_2\rho, \quad T = \beta\left(-3\frac{dc_1}{d\rho}r^4 - 3c_1\rho^3 - c_2\rho^3\right), \tag{5.1}$$

where $c_1$ is an arbitrary function of $\rho$ and $c_2$ is an arbitrary function of $\theta$. The corresponding characteristic vector is the pair of functions

$$\begin{aligned}\frac{\delta X}{\delta\theta} &= -\frac{dc_2}{dv}\rho,\\ \frac{\delta X}{\delta\rho} &= \left(-\frac{d^2c_1}{d\rho^2}\rho^2 - 5\frac{dc_1}{d\rho}\rho - 3c_1 - c_2\right)\end{aligned}$$

where $\delta/\delta\theta$, $\delta/\delta\rho$ are variational derivatives. The characteristic vector vanishes if and only if the above hydrodynamic conservation laws are trivial, *i.e.* they are the total divergence of a quantity defined on the whole coordinate space. This happens if and only if $c_2$ is a constant and $c_1$ fulfills the above Cauchy–Euler ODE. So, the space of nontrivial hydrodynamic conserved quantities is still parametrized by two functions which are 'almost' arbitrary.

The system (3.3) also admits the following symmetries whose characteristic function depends on first-order derivatives:

$$\phi = -s_1\frac{\partial}{\partial\theta} + s_2\rho_\tau\frac{\partial}{\partial\rho} \tag{5.2}$$

where $s_1 = s_1(\theta,\rho,\theta_\tau)$ and $s_2 = s_2(r)$ fulfill the additional PDE

$$\frac{ds_1}{d\rho}\rho + \frac{ds_1}{d\theta_\tau}\theta_\tau + s_2\theta_\tau = 0.$$

If we require that $\phi$ be of hydrodynamic type [20], *i.e.* $s_1 = s_{10}(\rho,\theta)\theta_\tau$ then we conclude that $s_{10} = s_3(\theta)(1/\rho) + s_4(\rho)$ and $s_2 = -(ds_4/d\rho\cdot\rho + s_4)$. The hydrodynamic symmetries $\phi$ commute with the vector field whose characteristic function is given by the right-hand side of the equation. More precisely, if

$$\psi = \beta\rho^2\theta_\tau\frac{\partial}{\partial\theta} + 3\beta\rho^2\rho_\tau\frac{\partial}{\partial\rho}, \quad \phi = -\left(\frac{s_3}{\rho} + s_4\right)\theta_\tau\frac{\partial}{\partial\theta} - \left(\frac{ds_4}{d\rho}\rho + s_4\right)\rho_\tau\frac{\partial}{\partial\rho}$$

then we have

$$\{\phi,\psi\} = \Big(\phi^\theta\frac{\partial\psi^\theta}{\partial\theta} + \phi^\rho\frac{\partial\psi^\theta}{\partial\rho} + D_\tau(\phi^\theta)\frac{\partial\psi^\theta}{\partial\theta_\tau} + D_\tau\phi^\rho\frac{\partial\psi^\theta}{\partial\rho_\tau}$$
$$-\psi^\theta\frac{\partial\phi^\theta}{\partial\theta} - \psi^\rho\frac{\partial\phi^\theta}{\partial\rho} - D_\tau(\psi^\theta)\frac{\partial\phi^\theta}{\partial\theta_\tau} - D_\tau\psi^\rho\frac{\partial\phi^\theta}{\partial\rho_\tau}\Big)\frac{\partial}{\partial\theta}+$$
$$\Big(\phi^\theta\frac{\partial\psi^\rho}{\partial\theta} + \phi^\rho\frac{\partial\psi^\rho}{\partial\rho} + D_\tau(\phi^\theta)\frac{\partial\psi^\rho}{\partial\theta_\tau} + D_\tau\phi^\rho\frac{\partial\psi^\rho}{\partial\rho_\tau}$$
$$-\psi^\theta\frac{\partial\phi^\rho}{\partial\theta} - \psi^\rho\frac{\partial\phi^\rho}{\partial\rho} - D_\tau(\psi^\theta)\frac{\partial\phi^\rho}{\partial\theta_\tau} - D_\tau\psi^\rho\frac{\partial\phi^\rho}{\partial\rho_\tau}\Big)\frac{\partial}{\partial\rho} = 0,$$

where $\{\phi,\psi\}$ is the Jacobi bracket (see, for example, [4]), or the commutator of the two flows $\psi$ and $\phi$.

We stress that the same computation can be repeated for the system 4.13.

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