Bogus Transformations in Mechanics of Continua

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Abstract

In this note we consider the structure of the symmetry group of some important mechanical theories (nonlinear elasticity and fluids of grade n). We discuss why the invariance with respect to some well-known transformations must be used with care and we explain why some of these universal transformations are useless to obtain invariant solutions of physical significance.

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1. Introduction

Lesson four of Giancarlo Rota's invited address delivered at the meeting of the Mathematical Association of America in 1997, today known by the title *Ten lessons I wish I had learned before I started teaching differential equations*[1], is to **Teach changes of variables**. Rota writes:

> *Worse, no one realizes that changes of variables are not just a trick; they are a coherent theory (it is the differential analogue of classical invariant theory, but let it pass).*

We think that Rota has been taken literally by a large community of applied mathematicians. Today, it is possible to record several books entirely devoted to symmetry groups applied to differential equations (Bluman and Kumei [2], Cantwell [6], Krasil'shchik and Vinogradov [4], Olver [23], Ibragimov [16],

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[1] See http://www.math.toronto.edu/lgoldmak/Rota.pdf

Stephani [31]), a long list of review papers and a tremendous quantity of scientific papers. This means that we have at our disposal a full catalogue of symmetries for a large class of differential equations.

It is well known that the computation of the classical transformations admitted by differential equations is a completely algorithmic procedure and this procedure may be automatized using symbolic software. The reason is that the Lie's method is extremely powerful but, on the other hand, this is indeed also the origin of a major drawback: automatism produces an *abuse* of the methodology with respect to the understanding of the problem under investigation.

Today, many of the published papers about symmetries of differential equations are quite unsatisfactory. Some papers start from an equation without any knowledge of its physical motivations, then they provide a list of symmetries (whose computation sometimes may be considered as just a big exercise) and contain some solutions. These solutions, often, are non sense from the physical standpoint (just mathematical curiosity) and in many cases they are displayed without any discussion on their possible meaning in the context of the mechanical theory investigated.

For people seriously interested in Mechanics the simple knowledge of the symmetries and transformations admitted by a given model is of no interest without an investigation of the physical meaning of this invariance. On the other hand, the standard situation that we observe concerning point symmetries of a given mechanical or physical theory may be very disappointing: the full group of transformations admitted by the differential equations describing the given theory may be guessed by a simple inspection of the basic principles underlying the theory itself. This is the case of uniformity of the material properties, frame indifference and material symmetry, as seen in many textbooks eg [1, 7, 33]. Physical intuition and experience is enough to discover the fundamental transformations but clearly only the general theory may be give us the complete picture.

In (Edelen, 1982 [10]) we read

> *The isovector*[2] *fields of the incompressible Navier-Stokes equations thus generate the already known transformations admitted by those equations. What is new is the fact that the isovector method is exhaustive; there are no other mappings admitted by the incompressible Navier-Stokes equations.*

The above remark by Edelen can be recast in a more essential but communicative language: *nothing new is under the sun but now you know it for sure.*

The knowledge of the symmetry group of a mechanical theory is always of interest if this knowledge is coupled with a clear understanding of these symmetries to uncover the nature of the transformations and the usefulness of their mathematical properties.

[2]The isovector method is a method to compute symmetries of differential equations based on the use of external differential forms.

Some of the symmetries underlying a mechanical theory are fundamental bricks in the construction and characterization of the theory itself, but if such a transformation is used to build exact solutions using reduction methods they are a sort of *bogus* transformation. They are useless because they generate solutions of no mechanical interest. The aim of the present note is to develop this point.

Our arguments are based on general considerations and two basic theories of continuum mechanics: the theory of nonlinear elasticity and the Navier-Stokes equations. General considerations are based on frame-indifference as a basic principle for continuum mechanics that provides the fundamental symmetries of the theory. Frame-indifference has several analogies with the symmetries that are postulated in gauge theories. Then, we specialize our arguments to the above two theories. The fact that the theories are well known will help the reader in understanding our points.

2. Basic Equations

For the sake of simplicity here we are interested in purely mechanical theories of non polar materials. Therefore, let $\boldsymbol{x}$ denote the current position of a particle $\boldsymbol{X}$ in the reference configuration that is assumed to be stress free. The motion of the body is a one-to-one mapping $\boldsymbol{\chi}(\boldsymbol{X}, t)$ that assigns to each point $\boldsymbol{X}$ belonging to the reference configuration the position $\boldsymbol{x}$ at time t, i.e. $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. We make the hypothesis of the existence of a functional $\mathcal{F}$ such that we have the following expression for the stress $\boldsymbol{T}$ at time t

$$\boldsymbol{T} = \mathcal{F}(\boldsymbol{\chi}). \tag{2.1}$$

A basic principle of continuum mechanics, of interest in what follows, is that physical laws be independent of the frame of reference. Usually this principle is denoted as *frame–indifference* (see eg [33]). Given a process $\{\boldsymbol{\chi}, \boldsymbol{T}\}$ and a process $\{\boldsymbol{\chi}^*, \boldsymbol{T}^*\}$ related by

$$\boldsymbol{\chi}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\boldsymbol{\chi}(\boldsymbol{X}, t) + \boldsymbol{c}(t), \quad \boldsymbol{T}^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\boldsymbol{T}(\boldsymbol{X}, t)\boldsymbol{Q}(t)^T,$$

and $t^* = t - a$. Here $\boldsymbol{Q}(t)$ is a rotation and $\boldsymbol{c}(t)$ is an arbitrary point and a arbitrary number. We require the *indifference* (i.e. invariance) of the constitutive equation for $\mathcal{F}$ with respect rigid translations, shifting of the time scale and rigid rotations i.e.

$$\boldsymbol{Q}\mathcal{F}(\boldsymbol{\chi})\boldsymbol{Q}^T = \mathcal{F}(\boldsymbol{Q}\boldsymbol{\chi}). \tag{2.2}$$

We stress that frame indifference is a broader invariance than the usual Galilei group invariance, as the orthogonal transformation $\boldsymbol{Q}$ depends on time and the translation vector $\boldsymbol{c}$ may not depend linearly on time in general. The invariance group of the frame-indifference is thus the infinite-dimensional group of functions

$$(\boldsymbol{Q}, \boldsymbol{c})\colon \mathbb{R} \to SO(3) \times \mathbb{R}^3. \tag{2.3}$$

The class of field theories which are characterized by invariance groups of a similar structure is the class of *gauge theories*. Mathematically, any theory which possess an infinite-dimensional group of symmetries can be regarded as a gauge theory [4]. This definition encompasses not only continuum mechanics but also general relativity and the theory of electro-weak-strong interactions which underlies the Standard Model of particle physics.

The equations of general relativity were derived by Einstein by prescribing a gauge group of symmetries (or, equivalently, by requiring the general covariance of the theory) and by requiring the existence of distinguished conservation laws. In gauge theories this approach does not provide the equations of the theory; however, the existence of gauge symmetries and conservation laws implies that the equations of the theory must be Lagrangian (see [19] and references therein). This means that the invariance group in physical theories plays a foundational role and is not something to be looked for *a posteriori*. This pattern can also be found in many PDEs of mostly mathematical interest which can be shown to be uniquely determined by their symmetry group [18, 20, 21].

Continuum mechanics was derived by some authors in a Newtonian spacetime by a relativistic approach [9], while others found some strong analogies between fluid mechanics and electromagnetism [12]. A derivation of continuum mechanics as a Yang-Mills gauge theory, with a similar structure to electromagnetism or weak and strong interactions, was done in the classical text [11]. In that book gauge theories were proved to be useful at dealing with the continuum mechanics of defects.

The field equations of a gauge theory are by construction invariant with respect to the gauge transformations in strong sense. Indeed, usually there is a Lagrangian density $\mathcal{L}$ whose Euler–Lagrange expression are the balance equations $\mathcal{E} = 0$. The Lagrangian is gauge-invariant, or (like in Chern-Simons theory) it is gauge-invariant up to a total divergence; this implies that the Euler-Lagrange equations are directly gauge-invariant. So, the gauge symmetry is attained off-shell, ie in the whole space and not only if we restrict ourselves on the manifold defined by the solutions of the Euler-Lagrange equations.

When symmetries are looked for with Lie's algorithm the condition $\mathcal{E} = 0$ is always assumed, leading to on-shell symmetries. In the theories of continuum mechanics where frame-indifference is assumed the tensors that yield the field equations are gauge-invariant with respect to the group (2.3), hence there is an off-shell group of symmetries that leave the solutions of $\mathcal{E} = 0$invariant.

It is even more important for us to observe that by a general principle of gauge theories any two solutions of the field equations which are connected by a gauge transformations are physically indistinguishable. In other words, this means that they are just the same up to a change of gauge. As we will see, many authors find apparently new solutions in fluid mechanics or elasticity by applying one of the transformations in the frame-indifference group; this is just another way to express the same solution.

There are further symmetry assumptions on the fundamental structures of the theory. Namely, in what follows we assume the body to be materially uniform so that the only quantity of geometric interest is the deformation gradient

$\boldsymbol{F}$ defined through $\boldsymbol{F} = \partial\boldsymbol{\chi}/\partial\boldsymbol{X}$ and its history $\boldsymbol{F}^{(t)}$ such that

$$\boldsymbol{T}(t) = \mathcal{F}(\boldsymbol{F}^{(t)}). \tag{2.4}$$

A material is defined *isotropic* if there is at least one reference configuration such that

$$\boldsymbol{Q}\mathcal{F}\boldsymbol{Q}^T = \mathcal{F}(\boldsymbol{Q}\boldsymbol{F}^{(t)}\boldsymbol{Q}^T), \tag{2.5}$$

for any $\boldsymbol{Q}$ in the orthogonal group.

If the material is incompressible, and therefore only isochoric motions are admissible ($\det \boldsymbol{F}^{(t)} = 1$), then the constitutive equation for the stress tensor may be splitted as

$$\boldsymbol{T} = -p\boldsymbol{I} + \mathcal{F}^{\text{extra}}(\boldsymbol{F}^{(t)}). \tag{2.6}$$

Here the reactive part of the stress is $-p\boldsymbol{I}$, where p is the unknown Lagrange multiplier associated with the incompressibility constraint, and $\mathcal{F}^{\text{extra}}(\boldsymbol{F}^{(t)})$ is the *extra* stress tensor to which we apply all the previous considerations.

Examples of materials of such kind are:

- Cauchy isotropic elastic materials, such that $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{B})$ where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$;
- isotropic viscoelastic materials of differential type such as $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{B}, \boldsymbol{D})$ where $\boldsymbol{D}$ is the symmetric part of $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$;
- incompressible fluids of the differential type such that

  $$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{T}^{\text{extra}}\left(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n\right);$$

  here $\boldsymbol{A}_{n+1} = d\boldsymbol{A}_n/dt + \boldsymbol{L}^T\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{L}$, where the time derivative is the material derivative and for $n = 0$ we have $\boldsymbol{A}_1 = 2\boldsymbol{D}$. Navier-Stokes equations are a special case of this class of constitutive equations.

In theories of fluids it is convenient to use the Eulerian point of view and in solid mechanics it is usual to consider the Lagrangian point of view.

3. Fluid Mechanics

We cannot determine the first appearance of the theory of transformation groups in fluid mechanics. However, Birkhoff in his book on hydrodynamics [3] was already aware of the relevance of symmetries in this branch of mechanics.

Incompressible Euler, Navier-Stokes and second grade fluids equations are a good concrete example to introduce our point of view in more details. The Cauchy stress tensor for a second grade fluid contains as special case the Navier-Stokes theory which contains Euler theory, ie

$$\boldsymbol{T} = \overbrace{\underbrace{\underbrace{-p\boldsymbol{I}}_{\text{Euler}} + \mu\boldsymbol{A}_1}_{\text{Navier-Stokes}} + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2}^{\text{Second grade}}.$$

Here μ, α_1 and α_2 are the various constitutive parameters.

Let us start with the symmetry group of point transformations of the Euler equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{3.1}$$

Here $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the unknown velocity field. The symmetry group for such systems of equations is well known and is made by:

- Time translation (G_0): $(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to (\boldsymbol{x},\, t + \epsilon,\, \boldsymbol{u}, p)$.
- Transformation in a moving coordinate system ($G_{\boldsymbol{\alpha}}$):

  $$(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to \left(\boldsymbol{x} + \epsilon\boldsymbol{\alpha},\, t,\, \boldsymbol{u} + \epsilon\boldsymbol{\alpha}_t,\, p - \epsilon\boldsymbol{x} \cdot \boldsymbol{\alpha}_{tt} - \frac{1}{2}\epsilon^2\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}_{tt}\right),$$

  where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t)$.
- Scale transformation (G_1): $(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to (\lambda\boldsymbol{x},\, \lambda t,\, \boldsymbol{u}, p)$, where $\lambda = \exp(\epsilon)$.
- Scale transformation (G_2): $(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to (\boldsymbol{x},\, \lambda t,\, \lambda^{-1}\boldsymbol{u}, \lambda^{-2}p)$, where $\lambda = \exp(\epsilon)$.
- Rotations ($SO(3)$): $(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to (\boldsymbol{Q}\boldsymbol{x},\, t,\, \boldsymbol{Q}\boldsymbol{u}, p)$, here $\boldsymbol{Q} \in SO(3)$.
- Pressure changes (G_p): $(\boldsymbol{x},\, t,\, \boldsymbol{u}, p) \to (\boldsymbol{x},\, t,\, \boldsymbol{u}, p + \epsilon\theta(t))$.

All these transformations are connected to basic facts of continuum mechanics. This is well known but let us review in detail this point.

The G_0 is clearly associated to the fact the Euler equations are autonomous.

The G_1 and G_2 are two scale independent groups that are associated with the homogeneity of the space at the actual configuration.

The $SO(3)$ group is usually associated with the constitutive requirements of isotropy and the frame indifference. Isotropy is associated with the reference configuration and frame indifference is associated with the actual configuration. Using the Lagrangian point of view we obtain the invariance with respect two independent group of rotations. Since here we are using the Eulerian point of view and since $\boldsymbol{u} = d\boldsymbol{x}/dt$ we are considering only invariance under frame indifference. The invariance induced by isotropy is hidden (in the next Section, where we will consider elasticity, this point will be discussed in detail). Frame indifference requires invariance under time dependent rotations but here we find only constant in time rotations. This is because clearly for Euler equations, being $\boldsymbol{T} = -p\boldsymbol{I}$, we have for any $\boldsymbol{Q}(t) \in SO(3)$

$$\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T = -p\boldsymbol{Q}\boldsymbol{Q}^T = -p\boldsymbol{I},$$

and frame indifference is satisfied, but here we are considering the invariance of the equations of motion

$$\rho\frac{d}{dt}\boldsymbol{u}^*(\boldsymbol{x}^*, t) = \mathrm{div}_{\boldsymbol{x}^*}\boldsymbol{T}^*(\boldsymbol{x}^*, t),$$

and this means washing-up the time dependence. Indeed, being $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x}$, we have

$$\frac{d}{dt}\boldsymbol{u}^*(\boldsymbol{x}^*, t) = \boldsymbol{Q}(t)\frac{d}{dt}\boldsymbol{u}(\boldsymbol{x}, t) + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{u}(\boldsymbol{x}, t) + \ddot{\boldsymbol{Q}}(t)(\boldsymbol{x} - \boldsymbol{o}),$$

and therefore the balance equation reduces to

$$\rho\left(\boldsymbol{Q}(t)\frac{d}{dt}\boldsymbol{u}(\boldsymbol{x}, t) + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{u}(\boldsymbol{x}, t) + \ddot{\boldsymbol{Q}}(t)(\boldsymbol{x} - \boldsymbol{o})\right) = \boldsymbol{Q}(t)\mathrm{div}_{\boldsymbol{x}}\boldsymbol{T}(\boldsymbol{x}, t),$$

and clearly the invariance of the balance equation is ensured if and only if there is no time dependence in $\boldsymbol{Q}$.

The G_p is just a gauge transformation associated with the fact that the pressure field enters into the equations via a gradient term.

The G_α is once again associated with frame indifference. We are just adding a rigid translation and use the G_p invariance to incorporate in a pressure term the moving coordinate terms. There has been a lot of folklore around this invariance and we shall discuss the details of this folklore in the framework of the Navier-Stokes equations (the model equation originating the folklore).

Since there is a strong connection among the constitutive character of the theory and the various invariance groups it is natural to ask if all these invariances *propagate* to more complex theories. For example, the exercise 2.15 in Olver's book [23] asks[3]:

> *Prove that the symmetry group for the Navier-Stokes equations*
>
> $$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0,$$
>
> ... *is the* **same** *as that of the corresponding system of Euler equations-* ($\nu = 0$).

At first sight the above mentioned exercise seems to be correct: the constitutive theory leading to the Navier-Stokes equations is clearly based on principles that are the same of the Euler equations. This is true up to what happens about the scale transformations.

To make this point clear let us consider a scalar equation in the unknown $u = u(x, t)$ i.e.

$$u_t + uu_x = \nu u_{xx} + \alpha u_{xxt}, \tag{3.2}$$

where ν and α are two constitutive parameters.

Let us consider the scale transformation

$$(t, x, u) \to (\lambda_0 t, \lambda_1 x, \lambda u);$$

[3] The computation of the symmetry group for Navier-Stokes equations has been performed for the plane case in 1960 by Puhnachev [24] and then in the three-dimensional case by Lloyd [17] in 1981. The results are contained in many textbooks and therefore our discussion has to be considered only for its methodological value.

if we set $\nu = \alpha = 0$ in (3.2) we obtain

$$u_t + uu_x = 0 \to u_t + \frac{\lambda\lambda_0}{\lambda_1} uu_x = 0,$$

and the requirement

$$\lambda\lambda_0 = \lambda_1, \tag{3.3}$$

ensures the invariance. In this case we have three scaling parameters and only one equation to be satisfied and therefore we have two scaling groups.

If we consider $\alpha = 0$ under the action of the same group we have

$$u_t + uu_x = \nu u_{xx} \to u_t + \frac{\lambda\lambda_0}{\lambda_1} uu_x = \nu \frac{\lambda_0}{\lambda_1^2} u_{xx},$$

and now to ensure invariance it must be

$$\lambda\lambda_0 = \lambda_1, \quad \lambda_0 = \lambda_1^2. \tag{3.4}$$

Therefore only a scaling group may be admitted.

If we consider the full (3.2) we obtain

$$u_t + uu_x = \nu u_{xx} + \alpha u_{xxt} \to u_t + \frac{\lambda\lambda_0}{\lambda_1} uu_x = \nu \frac{\lambda_0}{\lambda_1^2} u_{xx} + \alpha \frac{1}{\lambda_1^2} u_{xxt},$$

and because here we have to add the requirement $\lambda_1 = 1$ we loose the possibility of the scale invariance.

This is exactly what happens going from Euler equations to Navier-Stokes equations: from two groups of scale transformations we reduce to one. When we move from Navier-Stokes to more complex non-Newtonian models we loose completely the invariance under scale transformation. Only if we also rescale the constitutive parameters we may ensure the possibility of a scale invariance. This is in agreement with the classical theory of dimensional analysis. Indeed if we fulfill (3.3) and we rescale the constitutive parameters such that

$$\nu \frac{\lambda_0}{\lambda_1^2} = \hat{\nu}, \quad \alpha \frac{1}{\lambda_1^2} = \hat{\alpha},$$

we are to maintain all the basic scale transformations. Therefore we need to consider *equivalence* transformations to fully recover dimensional analysis.

The remaining symmetry groups of Euler's equation are not only inherited by Navier-Stokes but by any n-th grade fluid. This is because these invariances are deeply grounded in general constitutive requirements. This fact means that we must be very careful in using these transformations in generating new solutions for our model equations. A good example is obtained considering $G_{\boldsymbol{\alpha}}$.

At first sight the richness of $G_{\boldsymbol{\alpha}}$ seems something of special and very promising. Indeed, in the introduction of (Boisvert et al., 1983 [5]) a paper dedicated to invariant solutions of the Navier-Stokes equations speaking of the previous papers by Puhnachev (1960) and Lloyd (1981) we read:

> *the work of these authors*[4] *is verified and it is shown how the group*

[4]They are speaking of Puhnachev and Lloyd.

permits the association of an infinite number of time-dependent solutions to any steady state solution.

Continuing in (Boisvert et al., 1983) at page 207 we have:

> *Consequently, there is obtained the useful result that any steady-state solution to the two-dimensional equations can be transformed by means of (3.5) and (3.6) into a time-dependent solution involving three arbitrary functions of the time variable. A similar result holds in three dimensions.*

To check the consistency of these affirmations let us consider the following class of homogeneous motions:

$$\boldsymbol{x} = \boldsymbol{A}(t)\boldsymbol{X}, \tag{3.5}$$

where $\boldsymbol{A}(t)$ is a general matrix such that $\det \boldsymbol{A} = 1$ to ensure isochoricity. The Eulerian velocity field and acceleration field associated with (3.1) are

$$\boldsymbol{u} = \dot{\boldsymbol{A}}\boldsymbol{A}^{-1}\boldsymbol{x}, \quad \boldsymbol{a} = \ddot{\boldsymbol{A}}\boldsymbol{A}^{-1}\boldsymbol{x}.$$

If $\boldsymbol{L} = \dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$ is constant we have that the motion defined in (3.5) is steady and the corresponding acceleration field is $\boldsymbol{a} = \boldsymbol{L}^2\boldsymbol{x}$.

We point out that (3.5) is a solution in the steady case of the Euler's equations if and only if

$$\boldsymbol{L}^2 = (\boldsymbol{L}^2)^T, \tag{3.6}$$

because in this case we have that the motion is circulation preserving. Moreover, since $\boldsymbol{L}$ is for the entire class of motions in (3.5) and does not depend on $\boldsymbol{x}$, all the Rivlin-Ericksen tensors $\boldsymbol{A}_i$ do not depend on $\boldsymbol{x}$ and therefore steady isochoric motions (3.5) such that (3.6) is valid are solutions also for all second grade fluids.

Now it is very easy to apply the group $G_{\boldsymbol{\alpha}}$ to $\hat{\boldsymbol{u}} = \dot{\boldsymbol{A}}\boldsymbol{A}^{-1}\hat{\boldsymbol{x}}$ when $\boldsymbol{L} = \dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$ is constant and we obtain

$$\boldsymbol{u} = \boldsymbol{L}\boldsymbol{x} + \boldsymbol{L}\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}'(t). \tag{3.7}$$

The motion (3.7) is indeed an unsteady motion but, the $\boldsymbol{L}$ for $\hat{\boldsymbol{u}}$ and $\boldsymbol{u}$ being the same from a mechanical point of view, we cannot discriminate among the two solutions. Indeed, the stress tensor corresponding to these two motions is identical. The motions are related by a rigid motion and therefore the requirement of frame indifference washes out the effect of the transformation $G_{\boldsymbol{\alpha}}$.

In subsequent papers the fact that the time dependent solutions obtained in such a way are of no interest at all from a mechanical point because they differ for a rigid motion is noticed *en passant* in (Grauel and Steeb, 1985 [14]) where it is remarked that the inertial reaction produced by the acceleration of the frame is balanced at each instant by a spatially constant pressure gradient. A clear discussion of the mechanical *irrelevance* of $G_{\boldsymbol{\alpha}}$ to generate new time dependent solutions is given in the book by Cantwell (2002) [6]:

The arbitrary functions translating the coordinates imply that the Navier-Stokes equations are invariant for all moving observers as long as the observer moves irrotationally. An observer translating and accelerating arbitrarily in three dimensions will sense the same equations of motion as an observer at rest. This invariance implies a great degree of flexibility in the choice of the observer used to view a flow. ... The term added to the pressure in (11.14) represents a spatially uniform effective body force induced by the acceleration of the observer. This force is purely hydrostatic in nature in that it is exactly balanced everywhere by the rate of change of the velocity field (the derivative of the translation term in the transformation of the velocity) and has no dynamical significance; it produces no net force on the flow field.

Cantwell does not connect this property in a direct way to frame indifference but it is clear that he is speaking of the same idea contained in such a concept.

Let us summarize the discussion of the present Section. First of all we have pointed out that there is a strong correlation among the constitutive invariance and the invariance of the balance equation. It is not the same invariance because balance equations involve inertial terms. The basic invariant transformations are in common within all the theories in the same constitutive class. This interesting fact may introduce bogus transformation (with respect the idea to obtain new solution from old ones) in continuum mechanics. Scale transformation have a special status in this framework.

4. Cauchy Elasticity

It is known that the traditional theory of Cauchy Elasticity can be regarded as a particular case of a broader theory of Elasticity [25, 26]. Also this theory admits examples of solutions with remarkable symmetry properties.

In this paper we will focus on the Cauchy theory of Elasticity, where we found interesting examples to be compared with what we have previously determined in fluid mechanics. We observe that this theory is usually casted in the Lagrangian point of view. In general the theory of Cauchy Elasticity has been less studied from the point of view of group analysis. However we mention [8, 28] where group analysis was used in different ways and [30] for an application of *generalized* symmetries to Cauchy Elasticity.

We start by considering hyperelastic isotropic materials. We define the strain-energy density function W as a function $W = W(I_1, I_2, I_3)$, of the principal invariants $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2$ and $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$. Here λ_i^2 are the principal stretches of $\boldsymbol{B}$.

Let us restrict to plane strains[5], i.e. using Cartesian coordinates in the

[5] We point out that this is different from plane stresses hypotheses.

reference and actual configurations

$$x_1 = \phi_1(X_1, X_2, t), \quad x_2 = \phi_2(X_1, X_2, t), \quad x_3 = X_3.$$

Now the components of the Piola-Kirchhoff stress tensor in the plane are given by

$$T_{R\alpha\beta} = (\alpha_0 - j^2\alpha_2)F^{-T}_{\alpha\beta} + (\alpha_1 + h\alpha_2)F_{\alpha\beta}, \tag{4.1}$$

where

$$\alpha_0 = 2I_3W_3, \quad \alpha_1 = 2\left(W_1 + I_1W_2\right), \quad \alpha_2 = -2W_2, \tag{4.2}$$

and

$$h = \lambda_1^2 + \lambda_2^2, \qquad j = \lambda_1\lambda_2. \tag{4.3}$$

In the above expression Greek indexes run over $1, 2$ and $W_i = \partial W/\partial I_i$, moreover

$$I_1 = h + 1, \quad I_2 = j^2 + h, \quad I_3 = j^2.$$

The balance equations read

$$\gamma_1\Delta\phi_1 + \left(\frac{\gamma_0}{j}\right)_1 \phi_{2,2} - \left(\frac{\gamma_0}{j}\right)_2 \phi_{2,1} + \gamma_{1,1}\phi_{1,1} + \gamma_{1,2}\phi_{1,2} = \rho_0\phi_{1,tt} \tag{4.4}$$

and

$$\gamma_1\Delta\phi_2 + \left(\frac{\gamma_0}{j}\right)_1 \phi_{1,1} - \left(\frac{\gamma_0}{j}\right)_2 \phi_{1,2} + \gamma_{1,1}\phi_{2,1} + \gamma_{1,2}\phi_{2,2} = \rho_0\phi_{2,tt}, \tag{4.5}$$

where

$$\gamma_0 = \alpha_0 - j^2\alpha_2, \quad \gamma_1 = \alpha_0 + h\alpha_2,$$

Δ is the two dimensional Laplace operator and ρ_0 is the density of the material in the reference configuration.

If the material is incompressible $j = 1$ and the equation looks the same but now γ_0 may be interpreted as the Lagrange multiplier necessary to impose the constraint of incompressibility. Therefore γ_0 is an unknown and we have the kinematical equation

$$\phi_{1,1}\phi_{2,2} - \phi_{1,2}\phi_{2,1} = 1. \tag{4.6}$$

In the literature, the symmetries of the equations for the Neo-hookean materials have been computed in (Lei and Blume, 1996 [15]). Neo-hookean materials are hyperelastic incompressible materials such that $W = \frac{1}{2}\mu(I_1 - 3)$ where μ is the infinitesimal shear modulus.

In this case the equations (4.4) and (4.5) are simplified to

$$\Delta\phi_1 + \frac{\gamma_{0,1}}{\mu}\phi_{2,2} - \frac{\gamma_{0,2}}{\mu}\phi_{2,1} = \frac{\rho_0}{\mu}\phi_{1,tt} \tag{4.7}$$

and

$$\Delta\phi_2 + \frac{\gamma_{0,1}}{\mu}\phi_{1,1} - \frac{\gamma_{0,2}}{\mu}\phi_{1,2} = \frac{\rho_0}{\mu}\phi_{2,tt}, \tag{4.8}$$

to which we add the (4.6).

The infinitesimal generator of a Lie group is denoted

$$\mathbf{V} = \mathcal{X}_0 \frac{\partial}{\partial t} + \mathcal{X}_1 \frac{\partial}{\partial X_1} + \mathcal{X}_2 \frac{\partial}{\partial X_2} + \mathcal{Y}_1 \frac{\partial}{\partial \phi_1} + \mathcal{Y}_2 \frac{\partial}{\partial \phi_2} + \mathcal{Y}_3 \frac{\partial}{\partial \gamma_0},$$

and the results of Lei and Blume (1996) are obvious. We have exactly the same situation that we have recorded for the Navier-Stokes equations. It is only necessary to take care that we are considering equations in Lagrangian variables. Therefore, we find the group of scalings, the groups of rotations in the (x_1, x_2, t)–space (the reference configuration) and the $(\phi_1, \phi_2, \gamma_0)$–space (the actual reference), the groups of time-independent translations in the reference and actual configurations. All these groups are enforced in the structure of the field equations because we are considering uniform and isotropic materials.

Let us discuss the three groups which depend on arbitrary functions. The first group is just a gauge time dependent function that we may add to the pressure. Then we have (we are quoting Lei and Blume (1996)) *the groups of time dependent translations in the* (ϕ_1, ϕ_2)*–space and redistributions of the pressure field,* γ_0*, which balance the inertial forces induced from the time-dependent translations.* These groups are defined by

$$\hat{\boldsymbol{x}} = \boldsymbol{x},\ \hat{\phi}_1 = \phi_1 + f(t)s,\ \hat{\phi}_2 = \phi_2,\ \hat{\gamma}_0 = -\frac{\rho_0}{\mu} f''(t) \left[f(t)\frac{s^2}{2} + \phi_1 s \right] + \frac{\gamma_0}{\mu},$$

and

$$\hat{\boldsymbol{x}} = \boldsymbol{x},\ \hat{\phi}_1 = \phi_1,\ \hat{\phi}_2 = \phi_2 + g(t)s,\ \hat{\gamma}_0 = -\frac{\rho_0}{\mu} g''(t) \left[g(t)\frac{s^2}{2} + \phi_2 s \right] + \frac{\gamma_0}{\mu}.$$

Here s is the infinitesimal parameter of the group, $f(t)$ and $g(t)$ are arbitrary functions.

It is clear that the authors do not realize that these symmetries are only a manifestation of the frame indifference *mediated* with the Galilean invariance (as in the Navier-Stokes equations). The presence of the arbitrary pressure field allows to retain any rigid motion whose associated acceleration field is self-equilibrated.

On the other hand, we have to notice that Lei and Blume (1996) were able to point out that invariant solutions cannot be always associated with the symmetries of the given differential system. This is another argument that can give some bogus transformation with respect the possibility to use symmetries groups to obtain exact solutions of the partial differential equations of Mechanics.

Indeed, if we consider rotations in the material space, i.e. $G = SO(2)$ in $X_1 \times X_2$, and we reduce considering, as usual,

$$\phi_1 = \phi_1(\eta),\ \phi_2 = \phi_2(\eta), \quad \eta = X_1^2 + X_2^2,$$

the equation (4.6) (the isochoricity condition) we obtain an absurd i.e. $0 = 1$.

In polar coordinates

$$R = \sqrt{X_1^2 + X_2^2}, \quad \Theta = \text{atan}(X_2/X_1),$$

the determinant j and the trace h are rewritten as

$$j = \frac{1}{R}\left(\phi_{1,R}\phi_{2,\Theta} - \phi_{1,\Theta}\phi_{2,R}\right), \; h = \phi_{1,R}^2 + \phi_{2,R}^2 + \frac{1}{R^2}\left(\phi_{1,\Theta}^2 + \phi_{2,\Theta}^2\right).$$

Using polar coordinates it is clear that both j and h are invariant under rotations, i.e. these quantities do not change under the transformation $\hat{\Theta} = \Theta + \alpha$.

The *invariants* j and h involve first order derivatives. The first order *differential invariants* of G in the framework under investigation are: $\phi_{1,\Theta} = 0$ and $\phi_{2,\Theta} = 0$. Therefore h and j are both functions of the differential invariants, and $j = 0$ when $\phi_{1,\Theta} = \phi_{2,\Theta} = 0$. All the motions ϕ_1 and ϕ_2 invariant under the action of G satisfy $j = 0$.

In nonlinear elasticity one needs to satisfy $j > 0$. Therefore although the equations are clearly invariant under the group of material rotations, no $G-$invariant solutions are possible.

The computations in Lei and Blume (1996) are quite general but the authors are not able to realize this fact. The results in (Lei and Blume, 1996) can be straightforwardly extended to all incompressible materials and with a minor modification to all compressible elastic materials. The true problem (which in any case seems to be an academic problem) in the framework of such computations is: does there exist special constitutive equations for which the symmetry group is enlarged with respect the universal group?

We point out that (Suhubi and Bakkaloglu, 1997 [32]) is devoted to the computation of symmetry groups associated with field equations which govern finite motions of a wholly arbitrary, anisotropic and heterogeneous hyperelastic solida. After long computations they are able to provide some of the above results for compressible materials.

It is also possible to extend these results to more complex theories of solid mechanics as for example Kelvin–Voigt nonlinear materials. The structure of the symmetry group is always the same modulo the scale invariance that usually will be lost.

The fact that a great family of mechanical models share the same group of symmetries is fundamental to compare these theories. This is the true interesting point of this history: it is a remarkable property that allows us to understand in a deeper way the role of the various terms of these theories.

5. Concluding Remarks

Rota's point of view was right: the theory of symmetries of differential equations is a beautiful and elegant chapter of mathematics which allows synthesis and completeness. Moreover, this theory is powerful because it is simple and algorithmic. The crucial point is to connect mathematics and physics.

In some literature about symmetry groups applied to differential equations, the invariance under a transformation group seems to be just a mathematical happenstance. A beautiful property of the differential equations that seems to born of sea-foam like Venus. Clearly this is not the case; invariance in the framework of a mechanical theory comes always from deep physical reasons.

The role of transformation groups in continuum mechanics is not to be underestimated. First of all symmetry groups have a major role in the constitutive characterization of material behaviour. Then there is a clear connection among symmetries and solutions of the partial differential equations governing the mechanical theories.

In continuum mechanics nearly all the exact solutions we know have been obtained by using the semi–inverse method (see [22]). With this term we denote a set of ad hoc methods that are able to work in the framework of a certain variety of problems. In his 1977 review paper on elastostatics [13] Ericksen remembers that:

> *Generally, a semi–inverse method is one that reduces the basic equations to equations involving fewer independent or dependent variables, or both, for a limited set of solutions. Commonly, this involves exploiting some invariance of the equations. It seems probable that, by better developing the underlying group theory, one could make the search for such methods more routine. Here, we take a small step toward this goal by exploring what can be done with a simple but popular kind of group.*

Still today we trust that transformation groups must be the unifying paradigm for the semi–inverse method but we have not been able to unravel the problem. Indeed, the classical theory of symmetries for differential equations it is clearly insufficient to explain the semi-inverse method (examples are given in Saccomandi (2004) [29]). In spite of this situation, many invariant solutions under classical symmetries provide interesting classes of solutions that have a major role in our understanding of mechanical theories. An example are the Carroll wave solutions: a beautiful set of general exact solutions (see Destrade and Saccomandi 2005 [8], Rogers et al. 2014 [28]) whose mechanical and mathematical reasons are indeed clarified using symmetries (Saccomandi and Vitolo 2014 [30])

Despite this situation most of the current literature about symmetries of partial differential equations contains only marginal results and some paper contains several bogus results. We hope that the results and examples contained in our note will help to stop the misapplication of the beautiful theory of symmetries.

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