

Symmetries and conservation laws for the Karczewska–Rozmej–Rutkowski–Infeld equation

A. Sergyeyev^a and R. Vitolo^b

^a Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 746 01 Opava, Czech Republic
`Artur.Sergyeyev@math.slu.cz`

^b Department of Mathematics and Physics “E. De Giorgi”,
University of Salento, Lecce, Italy
`raffaele.vitolo@unisalento.it`

Nonlinear Analysis: Real World Applications 32 (2016) 1–9
--

We give a complete description of generalized symmetries and local conservation laws for the fifth-order Karczewska–Rozmej–Rutkowski–Infeld equation describing shallow water waves in a channel with variable depth. In particular, we show that this equation has no genuinely generalized symmetries and thus is not symmetry integrable.

Introduction

Investigation of dynamics of shallow water waves has a long and distinguished history and remains a subject of intense research nowadays, see e.g. [2, 4, 17] and references therein. Recently the authors of [2] have developed a systematic procedure for deriving an equation for surface elevation of shallow water waves for a prescribed relation between the orders of the two expansion parameters. This procedure was *mutatis mutandis* applied in [6, 8] for deriving a fifth-order equation describing unidirectional shallow-water waves in channels with variable bottom geometry.

Consider the equation in question [6, 8], to which we shall refer to as to the Karczewska–Rozmej–Rutkowski–Infeld (KRRI) equation,

$$\begin{aligned} u_t = & -u_x - \frac{3}{2}auu_x - \frac{1}{6}bu_{xxx} + \frac{3}{8}a^2u^2u_x - \frac{23}{24}abu_xu_{xx} - \frac{5}{12}abuu_{xxx} \\ & + \frac{d}{2}(h'u + hu_x) + \frac{1}{4}bd(-h'''u - h''u_x + h'u_{xx} + hu_{xxx}) - \frac{19}{360}b^2u_{xxxxx}, \end{aligned} \quad (1)$$

where a , b , and d are constants (denoted in [6, 8] by α , β and δ), and $h = h(x)$ is a smooth function of x giving the dimensionless channel depth, and the primes indicate x -

derivatives of h ; the dependent variable u , i.e., the dimensionless wave elevation, denoted in [6, 8] by η , is a function of x and t .

For $a = 0$ equation (1) becomes linear, while if $b = 0$ we obtain a first-order equation

$$u_t = -u_x - \frac{3}{2}auu_x + \frac{3}{8}a^2u^2u_x + \frac{d}{2}(h'u + hu_x),$$

so both of these special cases of (1) are obviously integrable.

On the other hand, the authors of [6] have conjectured that in general the KRRI equation (1) is not integrable by the inverse scattering transform. We provide strong evidence to support this conjecture by rigorously proving that (1) is not symmetry integrable (see e.g. [13] and references therein for details on symmetry integrability), i.e., it has only finitely many generalized symmetries, all of which are equivalent to the Lie point ones, and just one local conservation law (2). This also lends substantial support to the conjectured absence of (non-dissipating) multisoliton solutions for the equation under study, as well as of the zero-curvature representation involving a nonremovable parameter (as for the latter, cf. e.g. [3, 12] and references therein). The proofs make substantial use of the formal series technique, see e.g. [13] and [14] and references therein; cf. also [20].

1 Main results

Theorem 1. *The KRRI equation (1) with $a \neq 0$, $b \neq 0$ and $d \neq 0$ has no generalized symmetries of order greater than 9 and no local cosymmetries of order greater than 10.*

Using this result we can readily compute *all* generalized symmetries, cosymmetries and conservation laws for (1), as it suffices to find all symmetries up to order 10 and all cosymmetries up to order 9.

Proposition 1. *If $a \neq 0$, $b \neq 0$, and $d \neq 0$, then the KRRI equation (1) has just one local conservation law*

$$u_t = \left(u - \frac{3}{4}au^2 - \frac{1}{6}bu_{xx} + \frac{1}{8}a^2u^3 - \frac{13}{48}abu_x^2 - \frac{5}{12}abuu_{xx} + \frac{d}{2}hu \right. \\ \left. + \frac{1}{4}bd(-h''u + hu_{xx}) - \frac{19}{360}b^2u_{xxxx} \right)_x \quad (2)$$

with the density u associated to the only cosymmetry $\gamma = 1$ of (1).

The result of Proposition 1 can be seen as an ultimate amplification of the results of [7] on conservation laws of (1) for constant h .

As an aside note that since the density of the conservation law (2) is just u , equation (1) is in normal form with respect to low-order conservation laws in the sense of [15].

Denote by F the right-hand side of (1).

Proposition 2. *If $a \neq 0$, $b \neq 0$, and $d \neq 0$ then all generalized symmetries of the KRRI equation (1) are equivalent to the Lie point ones.*

If $a \neq 0$, $b \neq 0$, $d \neq 0$, and $h' \neq 0$, then the only generalized symmetry of (1) is the one with the characteristics equal to F ; this corresponds to the Lie point symmetry $\partial/\partial t$, i.e., the time translation.

If $a \neq 0$, $b \neq 0$, $d \neq 0$ and $h = \text{const}$, then, in addition to the time translation, we have a symmetry with the characteristics u_x , which corresponds to the Lie point symmetry $\partial/\partial x$, i.e., the space translation.

Moreover, if $a \neq 0$, $b \neq 0$, $d \neq 0$, $h = \text{const}$, and $hd = 4$, then in addition to the space and time translations equation (1) admits a symmetry with the characteristics $5tF + (x + 2t)u_x + 2u - 4/a$, which corresponds to a Lie point symmetry $5t\partial/\partial t + (x + 2t)\partial/\partial x + (4/a - 2u)\partial/\partial u$.

One of the immediate consequences of the above result is that (1) is not Lie remarkable in the sense of [11], i.e., it is not uniquely determined by its Lie point symmetries. Note also that it could be of interest to explore the point equivalence transformations for (1), cf. e.g. [18] and references therein.

2 Preliminaries

In this section we recall a number of known definitions and results from the so-called formal symmetry approach to integrability mostly following [13, 14]; cf. also [5, 9].

As usual, denote by F the right-hand side of (1), by u_i the i th x -derivative of u (so $u_0 \equiv u$, $u_1 \equiv u_x$, etc.), and by D_x and D_t total x - and t -derivatives restricted to the differential equation (1) and its differential consequences, i.e.,

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} u_{j+1} \frac{\partial}{\partial u_j}, \quad D_t = \frac{\partial}{\partial t} + \sum_{j=0}^{\infty} D_x^j(F) \frac{\partial}{\partial u_j}.$$

For any *local* function, i.e., a smooth function $f = f(t, x, u_0, u_1, \dots)$ which may depend on x, t and at most finitely many u_j , define its order $\text{ord } f$ as the greatest integer k such that $\partial f / \partial u_k \neq 0$, and if $f = f(x, t)$ we set $\text{ord } f = -\infty$ by definition. We denote by \mathcal{A} the algebra of local functions with respect to the usual multiplication.

Consider a local *conserved vector* for (1), cf. e.g. [15] and references therein, i.e., a pair of local functions (ρ, σ) , that is, the density ρ and the flux σ , which satisfy the equation

$$D_t(\rho) = D_x(\sigma).$$

Two local conserved vectors (ρ, σ) and $(\tilde{\rho}, \tilde{\sigma})$ are said to be equivalent if there exists a function $\zeta \in \mathcal{A}$ such that $\rho = \tilde{\rho} - D_x(\zeta)$ and $\sigma = \tilde{\sigma} - D_t(\zeta)$. An equivalence class of conserved vectors is called a local *conservation law* for (1).

Next, consider an algebra \mathcal{B} of formal series of the form

$$L = \sum_{i=-\infty}^k a_i D^i \tag{3}$$

where the coefficients a_i belong to \mathcal{A} and the indeterminate D can be informally seen as just another avatar of D_x .

The multiplication defined on monomials by the generalized Leibniz rule

$$aD^i \circ bD^j = a \sum_{k=0}^{\infty} \frac{i(i-1)\cdots(i-k+1)}{k!} D_x^k(b) D^{i+j-k}$$

extends by linearity to the whole \mathcal{B} . It is readily seen that this multiplication is associative and hence the commutator $[A, B] = A \circ B - B \circ A$ turns \mathcal{B} into a Lie algebra. We shall omit \circ whenever this does not lead to confusion.

The action of total derivatives extends from \mathcal{A} to \mathcal{B} in an obvious manner,

$$D_t \left(\sum_{i=-\infty}^k a_i D^i \right) = \sum_{i=-\infty}^k D_t(a_i) D^i,$$

and likewise for D_x .

As usual, the greatest k such that $a_k \neq 0$ in (3) is called the degree of L and denoted by $\deg L$, with the convention that $\deg 0 = -\infty$.

An $L \in \mathcal{B}$ with $\deg L \neq 0$ is called a *formal recursion operator* (or a *formal symmetry*) of (1) of rank k if we have

$$\deg(D_t(L) - [F_*, L]) \leq \deg L + \deg F_* - k,$$

where for any $f \in \mathcal{A}$ we define

$$f_* = \sum_{i=0}^{\text{ord } f} \frac{\partial f}{\partial u_i} D^i.$$

Likewise, an $L \in \mathcal{B}$ is called a *formal symplectic operator* (or a *formal conservation law*) for (1) of rank k if we have

$$\deg(D_t(L) + F_*^\dagger \circ L + L \circ F_*) \leq \deg L + \deg F_* - k. \quad (4)$$

Here for any

$$Q = \sum_{j=-\infty}^q b_j D^j$$

from \mathcal{B} the formal adjoint Q^\dagger is defined as

$$Q^\dagger = \sum_{j=-\infty}^q (-D)^j \circ b_j.$$

In this connection recall (see e.g. [13, 14, 5] and references therein) that the characteristics G of generalized symmetries of (1) are local solutions of the equation

$$D_t(G) = \ell_F(G), \quad (5)$$

where $\ell_F = F_*|_{D=D_x}$, while local cosymmetries are identified with local solutions γ of the equation

$$D_t(\gamma) = -\ell_F^\dagger(\gamma), \quad (6)$$

where $\ell_F^\dagger = F_*^\dagger|_{D=D_x}$.

It is well known that if (ρ, σ) is a local conserved vector for (1) then $\delta\rho/\delta u$ is a cosymmetry for (1). Here the Euler operator $\delta/\delta u$ is defined as

$$\frac{\delta}{\delta u} = \sum_{i=0}^{\infty} (-D_x)^i \frac{\partial}{\partial u_i}.$$

As the image of \mathcal{A} with respect to D_x lies in the kernel of $\delta/\delta u$, the cosymmetry $\delta\rho/\delta u$ actually corresponds to a conservation law rather than to a particular conserved vector. On the other hand, it is in general *not* true that to any given cosymmetry there corresponds a conservation law, cf. e.g. [9, 14, 15] and references therein for details.

We identify the order of symmetry with that of its characteristics, and the order of cosymmetry is handled in a similar manner.

In closing note that (see e.g. [10, 14]) a generalized symmetry of (1) is equivalent to a Lie point one if and only if the characteristics G of the symmetry in question can be written in the form

$$G = c(t)F + g_1(x, t, u)u_x + g_0(x, t, u)$$

for suitable smooth functions c, g_1, g_0 . The associated Lie point symmetry then reads, up to the sign,

$$c(t)\frac{\partial}{\partial t} + g_1(x, t, u)\frac{\partial}{\partial x} - g_0(x, t, u)\frac{\partial}{\partial u}.$$

3 Formal recursion and symplectic operators for the KRRI equation

Theorem 1 is actually an immediate corollary of two stronger results:

Theorem 2. *If $a \neq 0$, $b \neq 0$, and $d \neq 0$, then the KRRI equation (1) has no formal recursion operator of rank greater than 9.*

Theorem 3. *If $a \neq 0$, $b \neq 0$, and $d \neq 0$, then the KRRI equation (1) has no formal symplectic operator of rank greater than 10.*

Indeed, it is well known, see e.g. [13, 14] and references therein, that if a (1+1)-dimensional scalar evolution equation admits no formal recursion operator of nonzero degree of rank k or greater then it cannot have generalized symmetries of order k or greater. Likewise, if a (1+1)-dimensional scalar evolution equation admits no formal symplectic operator of rank k or greater then it cannot have local cosymmetries of order k or greater.

On the other hand, finding generalized symmetries (resp. local cosymmetries) up to a given order k is just a matter of straightforward albeit somewhat tedious computation.

Finally, as to any local conservation law of (1) there corresponds a local cosymmetry of (1), finding all local conservation laws for (1) when all local cosymmetries for (1) are known becomes a straightforward matter too.

Let us also note that the absence of formal symplectic operator of rank 10 or higher (and hence *a fortiori* of infinite rank) implies that the KRRI equation (1) admits no Hamiltonian or symplectic structure that can be written as a formal power series in the total x -derivative with local coefficients, cf. Section 2. While in principle it could happen that the KRRI equation admits some very exotic Hamiltonian or symplectic structure involving complicated nonlocalities, cf. e.g. [16] and references therein, this is extremely unlikely.

3.1 Proof of Theorem 2

Seeking a contradiction, suppose that there exists an $L \in \mathcal{B}$ with $\deg L \neq 0$ such that

$$\deg(D_t(L) - [F_*, L]) \leq \deg L + \deg F_* - 10. \quad (7)$$

Without loss of generality we can assume (cf. e.g. [13, 14]) that our formal recursion operator L has $\deg L = 1$, and let

$$L = fD + \sum_{j=0}^{\infty} s_j D^{-j},$$

where $f \in \mathcal{A}$ and $s_j \in \mathcal{A}$, so (7) boils down to

$$\deg(D_t(L) - [F_*, L]) \leq -4. \quad (8)$$

Thus, we need to equate to zero the coefficients at D^j for $j = 5, 4, \dots, -3$ (the coefficients at the higher powers of D vanish automatically) in

$$M = D_t(L) - [F_*, L].$$

Equating to zero the coefficient at D^5 in M yields

$$b^2 D_x(f) = 0,$$

so, as $b \neq 0$ by assumption, we have $D_x(f) = 0$ and hence

$$f = f(t), \quad (9)$$

i.e., f is an arbitrary smooth function of t alone.

Next, equating to zero the coefficient at D^4 in M while using (9) yields

$$b^2 D_x(s_0) = 0,$$

whence

$$s_0 = f_0(t),$$

where f_0 again is an arbitrary smooth function of t . As $f_0(t)$ commutes with all elements of \mathcal{B} , without loss of generality put $s_0 = 0$.

In a similar fashion as above, equating to zero the coefficients at D^i , $i = 3, 2, \dots, -3$ in M yields equations of the form

$$D_x(s_j) = K_j, \quad (10)$$

where $j = 1, \dots, 7$ and K_j are some local functions.

As the image of \mathcal{A} with respect to D_x lies in the kernel of the Euler operator $\delta/\delta u$, see e.g. [13, 14] for details, and s_j are local functions by assumption, we have the necessary conditions for (10) to hold of the form

$$\delta K_j / \delta u = 0, \quad j = 1, \dots, 7. \quad (11)$$

Using our assumptions that $a, b, d \neq 0$, f depends on t alone, and $s_0 = 0$, straightforward but somewhat tedious computations show that the only nontrivial conditions among (11) are those for $j = 5$ and $j = 7$.

Recursively solving equations (10) for $j = 1, \dots, 4$ we find that the condition (11) for $j = 5$ reads

$$\frac{\partial f}{\partial t} \frac{a}{b} = 0.$$

As $a \neq 0$ by assumption, we see that f is actually a constant rather than a function of t .

With this in mind we can readily solve (10) for $j = 5, 6$, and then we find that differentiating the condition

$$\delta K_7 / \delta u = 0,$$

with respect to u_{xx} and then to u_x yields

$$a^3 f / b = 0.$$

As $a \neq 0$ and $b \neq 0$ by assumption, we see that $f = 0$, i.e., $\deg L < 1$, which contradicts our initial assumption. Hence, a formal recursion operator for (1) of rank greater than nine and of nonzero degree does not exist, and the result follows.

3.2 Proof of Theorem 3

Now, seeking a contradiction, just as in the proof of Theorem 2, assume that

$$L = \sum_{j=-\infty}^r s_{j-r} D^j,$$

where $s_j \in \mathcal{A}$, is a formal symplectic operator of rank 10 for (1).

Let

$$N = D_t(L) + F_*^\dagger \circ L + L \circ F_*.$$

The condition (4) for $k = 10$ is satisfied if and only if the coefficients of N at the powers D^j , $j = r + 4, \dots, r - 4$ vanish (vanishing of the coefficients at D^{r+5} and higher powers of D occurs automatically).

In particular, vanishing of the coefficient at D^{r+4} yields

$$b^2 D_x(s_0) = 0.$$

Hence

$$s_0 = f_0(t),$$

where again $f_0(t)$ is an arbitrary smooth function of t .

It is readily checked that vanishing of the coefficients at D^j for $j = r + 3, \dots, r - 4$ yields equations of familiar structure (cf. the preceding subsection)

$$D_x(s_j) = H_j, \quad (12)$$

where $j = 1, \dots, 8$ and H_j are some local functions.

In complete analogy with the proof of Theorem 2 we have the necessary conditions for solvability of (12) of the form

$$\delta H_j / \delta u = 0, \quad j = 1, \dots, 8. \quad (13)$$

Recursively solving (12) with respect to s_j we readily find that the first nontrivial condition among (13) is the one for $j = 4$, which reads

$$as_0 d \frac{\partial h}{\partial x} = 0.$$

We see that if $h \not\equiv \text{const}$ then, as $a \neq 0$ and $d \neq 0$ by assumption, $s_0 = 0$ and hence a formal symplectic operator of rank 10 or higher for (1) does not exist.

Moreover, as the breakdown occurs at $j = 4$, for $h \not\equiv \text{const}$ we actually have a much stronger result, namely, a formal symplectic operator of rank 6 or higher for (1) in this case does not exist too.

Now turn to the case of $h \equiv \text{const}$ when s_0 does not have to vanish. Then we find that the next nontrivial condition among (13) is the one for $j = 6$, and this condition reads

$$ab \frac{\partial f_0}{\partial t} = 0,$$

so $s_0 = f_0 = \text{const}$.

With this in mind we can readily solve recursively further equations from (12), and we see that the next nontrivial condition from (13) occurs for $j = 8$.

In particular, we have

$$\frac{\partial^2(\delta H_8 / \delta u)}{\partial u_{xx} \partial u_x} = \frac{27}{1444b} a^3 s_0 (157r + 837).$$

The expression on the right-hand side of this equation must vanish because of (13) for $j = 8$. As r is an integer and $a \neq 0$ by assumption, this is only possible if $s_0 = 0$, and hence $\deg L < r$, which contradicts our initial assumption. Thus, a formal symplectic operator of rank 10 or higher for (1) does not exist even if $h \equiv \text{const}$.

Acknowledgments

This research was supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under RVO funding for IČ47813059, and by the Grant Agency of the Czech Republic (GA ČR) under grant P201/12/G028. The computations in the paper were mostly performed using the computer algebra packages *Jets* [1], *CDE* [19] and *CRACK* [21].

RV thanks Thomas Wolf for stimulating discussions and gratefully acknowledges warm hospitality extended to him in the course of his visit to Silesian University in Opava where the work on the present paper was initiated.

References

- [1] H. Baran, M. Marvan, Jets. A software for differential calculus on jet spaces and diffieties, available online at <http://jets.math.slu.cz>
- [2] G.I. Burde, A. Sergyeyev, Ordering of two small parameters in the shallow water wave problem, *J. Phys. A: Math. Theor.* 46 (2013), no.7, 075501, arXiv:1301.6672
- [3] D. Catalano Ferraioli, L.A. de Oliveira Silva, Nontrivial 1-parameter families of zero-curvature representations obtained via symmetry actions, *J. Geom. Phys.* 94 (2015), 185–198.
- [4] A. Constantin, K. Kalimeris, O. Scherzer, Approximations of steady periodic water waves in flows with constant vorticity, *Nonlinear Analysis: Real World Applications* 25 (2015), 276–306, arXiv:1503.04032
- [5] G. Gaeta, *Nonlinear symmetries and nonlinear equations*, Kluwer, Dordrecht, 1994.
- [6] A. Karczewska, P. Rozmej, E. Infeld, Shallow-water soliton dynamics beyond the Korteweg–de Vries equation, *Phys. Rev. E* 90 (2014) 012907, arXiv:1401.4859
- [7] A. Karczewska, P. Rozmej, E. Infeld, Energy invariant for shallow water waves and the Korteweg – de Vries equation. Is energy always an invariant?, arXiv:1503.09089
- [8] A. Karczewska, P. Rozmej, Ł. Rutkowski, A new nonlinear equation in the shallow water wave problem, *Phys. Scr.* 89 (2014), 054026, arXiv:1401.4261
- [9] J. Krasil’shchik, A. Verbovetsky, Geometry of jet spaces and integrable systems, *J. Geom. Phys.* 61 (2011), no. 9, 1633–1674, arXiv:1002.0077
- [10] B.A. Magadeev, Group classification of nonlinear evolution equations, *Algebra i Analiz* 5 (1993), no. 2, 141–156 (in Russian); English translation in *St. Petersburg Math. J.* 5 (1994), no. 2, 345–359.
- [11] G. Manno, F. Oliveri, R. Vitolo, Differential equations uniquely determined by algebras of point symmetries, *Teoret. Mat. Fiz.* 151 (2007), no. 3, 486–494 (in Russian); English translation in *Theoret. and Math. Phys.* 151 (2007), no. 3, 843–850.
- [12] M. Marvan, On the spectral parameter problem, *Acta Appl. Math.* 109 (2010), no. 1, 239–255, arXiv:0804.2031.
- [13] A.V. Mikhailov, V.V. Sokolov, Symmetries of differential equations and the problem of integrability, in: *Integrability*, ed. by A.V. Mikhailov, *Lecture Notes in Phys.*, 767, Springer, Berlin, 2009, 19–88.
- [14] P.J. Olver, *Applications of Lie groups to differential equations*, 2nd ed., Springer-Verlag, New York, 1993.
- [15] R.O. Popovych, A. Sergyeyev, Conservation laws and normal forms of evolution equations, *Phys. Lett. A* 374 (2010), no. 22, 2210–2217, arXiv:1003.1648.
- [16] A. Sergyeyev, A strange recursion operator demystified, *J. Phys. A: Math. Gen.* 38 (2005), no. 15, L257–L262, arXiv:nlin/0406032

- [17] S. Szatmari, A. Bihlo, Symmetry analysis of a system of modified shallow-water equations, *Commun. Nonlinear Sci. Numer. Simul.* 19 (2014), 530–537, arXiv:1212.5823.
- [18] O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Equivalence transformations in the study of integrability, *Physica Scripta* 89 (2014), 038003, arXiv:1308.5126
- [19] R. Vitolo, CDE: a REDUCE package for integrability of PDEs, freely available at <http://gdeq.org/Category:Software>, 2014.
- [20] J. Vodová-Jahnová, On symmetries and conservation laws of the Majda–Biello system, *Nonlinear Analysis: Real World Applications* 22 (2015), 148–154, arXiv:1405.7858
- [21] T. Wolf, A. Brand, Investigating DEs with CRACK and Related Programs, *SIGSAM Bulletin* 29 (1995), Special Issue, 1–8.