Bi-Hamiltonian structures of KdV type

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*To Franco Magri on the occasion of his 70th birthday,
with friendship and admiration.*

Abstract

Combining an old idea of Olver and Rosenau with the classification of second and third order homogeneous Hamiltonian operators we classify compatible trios of two-component homogeneous Hamiltonian operators. The trios yield pairs of compatible bi-Hamiltonian operators whose structure is a direct generalization of the bi-Hamiltonian pair of the KdV equation. The bi-Hamiltonian pairs give rise to multi-parametric families of bi-Hamiltonian systems. We recover known examples and we find apparently new integrable systems whose central invariants are non-zero; this shows that new examples are not Miura-trivial.

Keywords: Infinite-dimensional Hamiltonian systems; completely integrable systems; Bi-Hamiltonian structures.

1 Introduction

Many integrable systems admit a bi-Hamiltonian structure. This means that these systems can be written as Hamiltonian differential equations by means of two compatible Hamiltonian operators P and Q.

It was observed in [32] that in many examples the bi-Hamiltonian structures are, in fact, defined by a compatible trio of Hamiltonian operators. In this paper we consider the special case when P is a first-order Hamiltonian operator and Q is the sum of a first-order Hamiltonian operator and a higher-order Hamiltonian operator, and the three operators are mutually compatible. All these operators are homogeneous in the sense of Dubrovin and Novikov [10, 11].

The first example from [32] is the trio

$$P = P_1 = \partial_x, \qquad Q = Q_1 + R_3, \quad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3. \tag{1}$$

Coupling Q_1 and R_3 one obtains the Poisson pencil of the KdV hierarchy

$$\Pi_\lambda = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3 \tag{2}$$

discovered by Magri in [28], while coupling P_1 and R_3 one obtains the Poisson pencil of the Camassa–Holm hierarchy

$$\tilde{\Pi}_\lambda = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3). \tag{3}$$

Another example (from [16, 25]) is a trio in two components:

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix}, \; R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix} \tag{4}$$

Note that here the operator R_2 is a Dubrovin-Novikov homogeneous operator of order two. The scheme works in the same way: one coupling yields the Poisson pencil of the the so-called AKNS (or two-boson) hierarchy, and the other yields the Poisson pencil of the two component Camassa-Holm hierarchy [16, 25].

Using the language of [32], we say that *the pencils* $\Pi_\lambda = Q_1 + R_3 - \lambda P_1$ *and* $\tilde{\Pi}_\lambda = P_1 + R_3 - \lambda Q_1$ *are related by tri-Hamiltonian duality.* The existence of a reciprocal transformation relating dual hierarchies, generalizing the well-known transformation relating the negative flows of the KdV hierarchy with the positive flows of the Camassa-Holm hierarchy, was recently suggested [22].

Motivated by the above examples, in the present paper we consider the problem of classification of compatible trios of Hamiltonian operators P_1, Q_1, R_n where P_1 and Q_1 are homogeneous first-order Hamiltonian operators (also known as Hamiltonian operator of hydrodynamic type)

$$P_1 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x, \qquad Q_1 = h^{ij}\partial_x + \Gamma^{ij}_k u^k_x, \tag{5}$$

and R_n is a homogeneous Hamiltonian operator

$$R_n = \sum_{l=0}^{n} A^{ij}_{n,l}(u, u_x, \ldots, u_{(l)})\partial_x^{(n-l)} \tag{6}$$

of degree $n > 1$. This means that $A^{ij}_{n,l}$ are homogeneous polynomials of degree l in the variables $u_x, \ldots, u_{(l)}$, where the homogeneous degree is given assigning degree 1 to the derivative w.r.t. x. We recall [10, 11] that the homogeneity requirement implies that the operators P_1, Q_1 and R_n do not change their 'form' under the action of point transformations of the dependent variables

$$\tilde{u}^i = \tilde{u}^i(u^j). \tag{7}$$

The associated Poisson pencils are

$$P_1 + R_n - \lambda Q_1, \qquad P_1 - \lambda(Q_1 + R_n). \tag{8}$$

We call a pencil of one of the above types a *bi-Hamiltonian structure of KdV type*. The above pencils can be thought as a deformation of a Poisson pencil of hydrodynamic type. Due to the general theory of deformations the only interesting cases are $n = 2$ and $n = 3$. In the remaining case the deformations can be always eliminated by Miura type transformations [25]. For this reason we will consider only second and third order Hamiltonian operators R_2 and R_3.

We recall that second-order operators R_2 have been completely described in [9, 33], and third-order operators R_3 have been classified in the m-component case with $m = 1$ (in this case the operator can be reduced to ∂_x^3 by a point transformation (7) [34, 35, 9]) and $m = 2, 3, 4$ [17, 18].

Our strategy uses the normal forms of R_2 and R_3; for each of them we will find all possible compatible first-order Poisson pencils of hydrodynamic type $P_1 - \lambda Q_1$ and, consequently, all possible Poisson pencils of the form (8) with $n = 2$ (or $n = 3$) where the three operators P_1, Q_1, R_2 (or R_3) are mutually compatible.

In the scalar case $m = 1$ there is nothing new: we obtain the KdV and Camassa-Holm hierarchies.

In this paper we focus on the 2-component case, leaving the 3-component case to future investigations. When $m = 2$ there is only one homogeneous second-order Hamiltonian operator:

$$R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2, \tag{9}$$

and there are three homogeneous third-order Hamiltonian operators

$$R_3^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^3, \tag{10}$$

$$R_3^{(2)} = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x, \tag{11}$$

$$R_3^{(3)} = \partial_x \begin{pmatrix} \partial_x & \partial_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1}\partial_x & \frac{(u^2)^2+1}{2(u^1)^2}\partial_x + \partial_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \partial_x. \tag{12}$$

The operators are distinct up to transformations (7).

Our main results are the following Theorems (the coefficients c_i are constants).

Theorem 1. *P_1 is a Hamiltonian operator compatible with R_2 if and only if*

$$g^{11} = c_1 u^1 + c_2, \tag{13a}$$

$$g^{12} = \frac{1}{2} c_3 u^1 + \frac{1}{2} c_1 u^2 + c_5 \tag{13b}$$

$$g^{22} = c_3 u^2 + c_4. \tag{13c}$$

Moreover the above metric is flat for every value of the parameters.

Theorem 2. *P_1 is a Hamiltonian operator compatible with $R_3^{(1)}$ if and only if*

$$g^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{14a}$$

$$g^{12} = c_4 u^1 + c_1 u^2 + c_5 \tag{14b}$$

$$g^{22} = c_6 u^1 + c_4 u^2 + c_7 \tag{14c}$$

together with the algebraic conditions

$$c_1c_4 - c_2c_6 = 0, \quad c_3c_4 - c_7c_2 = 0, \quad c_3c_6 - c_1c_7 = 0. \tag{15}$$

Theorem 3. *P_1 is a Hamiltonian operator compatible with $R_3^{(2)}$ if and only if*

$$g^{11} = c_1u^1 + c_2u^2, \tag{16a}$$

$$g^{12} = c_4u^1 + \frac{c_3}{u^1} + \frac{c_2(u^2)^2}{2u^1}, \tag{16b}$$

$$g^{22} = 2c_4u^2 + \frac{c_6}{u^1} - \frac{c_1(u^2)^2}{u^1} + c_5, \tag{16c}$$

together with the algebraic conditions

$$c_2c_6 + 2c_1c_3 = 0, \quad c_2c_5 = 0, \quad c_1c_5 = 0. \tag{17}$$

Theorem 4. *P_1 is a Hamiltonian operator compatible with $R_3^{(3)}$ if and only if*

$$g^{11} = c_1u^1 + c_2u^2 + c_3, \tag{18a}$$

$$g^{12} = c_4u^1 - \frac{c_2}{2u^1} + \frac{c_3u^2}{u^1} + \frac{c_2(u^2)^2}{2u^1}, \tag{18b}$$

$$g^{22} = 2c_4u^2 + \frac{c_1}{u^1} + \frac{c_5u^2}{u^1} - \frac{c_1(u^2)^2}{u^1} + c_6, \tag{18c}$$

together with the algebraic conditions

$$c_2c_5 + 2c_1c_3 = 0, \quad c_2c_6 - 2c_3c_4 = 0, \quad c_1c_6 + c_4c_5 = 0. \tag{19}$$

The above mentioned algebraic conditions are quadratic in the parameters and define an algebraic variety. The problem of finding Poisson pencils of the form (8) inside the above algebraic variety is mathematically equivalent to finding all the straight lines cointained in this variety. The detailed list of solutions is given (case by case) in Section 3. In the generic case we obtain:

- a 5 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(1)}$ (see Theorem 6 for further details).
- a 4 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(2)}$ (see Theorem 7 for further details).

- a 4 parameter family of mutually commuting pairs P_1, Q_1 that commute with $R_3^{(3)}$ (see Theorem 8 for further details).

The above results can also be read in the framework [15] of Dubrovin and Zhang's perturbative approach. Indeed, all the pencils that we are considering can be regarded as deformations of a Poisson pencil of hydrodynamic type. The classification of deformations with respect to the group of Miura transformations

$$\tilde{u}^i = f^i(u^1, \dots, u^n) + \sum_{k\geq 1} \epsilon^k F^i_k(u, u_x, \dots, u_{(k)}), \tag{20}$$

(where $F^i_k(u, u_x, \dots, u_{(k)})$ are homogeneous differential polynomials of degree k) has been obtained in recent years in the semisimple case (see [25] for the scalar case and [5] for the general case). It turned out that deformations are uniquely determined by their dispersionless limit and by n functions of a single variable called *central invariants.* More precisely, if

$$\begin{aligned}\Pi^{ij}_\lambda = \omega^{ij}_\lambda &+ \sum_{k\geq 1}\epsilon^k \sum_{l=0}^{k+1} A^{ij}_{2;k,l}(u, u_x, \dots, u_{(l)})\partial_x^{(k-l+1)}\\ &-\lambda \sum_{k\geq 1}\epsilon^k \sum_{l=0}^{k+1} A^{ij}_{1;k,l}(u, u_x, \dots, u_{(l)})\partial_x^{(k-l+1)},\end{aligned} \tag{21}$$

($A^{ij}_{1;k,l}$ and $A^{ij}_{2;k,l}$ are homogeneous differential polynomials of degree l) is a deformation of a semisimple Poisson pencil of hydrodynamic type

$$\omega^{ij}_\lambda = (g^{ij}_2 - \lambda g^{ij}_1)\partial_x + (\Gamma^{ij}_{(2)k} - \lambda \Gamma^{ij}_{(1)k})u^k_x,$$

then the central invariants are then defined as [25]:

$$s_i = \frac{1}{(f^i)^2}\left(A^{ii}_{2;2,0} - r^i A^{ii}_{1;2,0} + \sum_{k\neq i}\frac{(A^{ki}_{2;1,0} - r^i A^{ki}_{1;1,0})^2}{f^k(r^k - r^i)}\right), \quad i = 1, \dots, n,$$

where f^i are the diagonal components of the contravariant metric g_1 in canonical coordinates. Here, canonical coordinates are the eigenvalues of the pencil $g^{ij}_2 - \lambda g^{ij}_1$.

The main result of [25] is the following: *Two deformations of the same Poisson pencil of hydrodynamic type are related by a Miura transformation if* *and only if their central invariants coincide. In particular deformations* Π_λ *with vanishing central invariant can be reduced to their dispersionless limit* ω_λ *by a Miura transformation. This means that there exists a transformation of the form* (20) *such that*

$$\Pi_\lambda^{ij} = L_k^{*i} \omega_\lambda^{kl} L_l^j,$$

where

$$L_k^i = \sum_s (-\partial_x)^s \frac{\partial \tilde{u}^i}{\partial u^{(k,s)}}, \qquad L_k^{*i} = \sum_s \frac{\partial \tilde{u}^i}{\partial u^{(k,s)}} \partial_x^s.$$

The vanishing of the central invariants implies the existence of a Miura transformation reducing the pencil to its dispersionless limit. For this reason deformations with vanishing central invariants are said to be trivial.

In Section 4 we will first recover old and recent 2-component examples of bi-Hamiltonian systems of PDEs. In particular we show that the Kaup-Broer system [24] and a more recent multicomponent family of commuting operators [8] are particular cases of hierarchies generated by trios with R_2 and that the coupled Harry-Dym hierarchy [2] and the Dispersive Water Waves system [3] are particular cases of hierarchies generated by trios with $R_3^{(1)}$.

Then, we provide examples of apparently new bi-Hamiltonian systems of PDEs generated by trios with $R_3^{(2)}$ and $R_3^{(3)}$. The systems are expressed via rational functions; this makes them particularly interesting. We also computed their central invariants and proved that none or them is Miura-trivial.

Computations were performed independently with Maple and with the software package CDE [38, 23] of the Reduce computer algebra system. We are ready to supply the computer programs that we used for proving the main results upon requesting them to the authors by email.

2 Homogeneous Hamiltonian and bi-Hamiltonian structures

2.1 First-order operators and flat pencils

First-order Hamiltonian operators of hydrodynamic type

$$P = g^{ij}\partial_x - g^{il}\Gamma_{lk}^j u_x^k = g^{ij}\partial_x + \Gamma_k^{ij} u_x^k$$

have been introduced by Dubrovin and Novikov in [10, 11]. In the non-degenerate case ($\det(g^{ij}) \neq 0$) the operator P is Hamiltonian if and only if g_{ij} (the inverse of g^{ij}) is a flat pseudo-Riemannian metric and Γ^j_{hk} are the Christoffel symbols of the associated Levi-Civita connection.

Poisson pencils of hydrodynamic type have been introduced in the framework of Frobenius manifolds by Boris Dubrovin in [13]; they are defined by a pair of contravariant (pseudo)-metrics g and h satisfying the following conditions:

1. The pencil of metrics $g_\lambda = g - \lambda h$ is flat for any λ.
2. The (contravariant) Christoffel symbols $\Gamma^{ij}_{(\lambda)k}$ of the pencil g_λ coincide with the pencils of Christoffel symbols:
$$\Gamma^{ij}_{(\lambda)k} = \Gamma^{ij}_{(2)k} - \lambda\Gamma^{ij}_{(1)k}, \tag{22}$$
where $\Gamma^{ij}_{(1)k}$ and $\Gamma^{ij}_{(2)k}$ are the Christoffel symbols of the metrics h and g respectively.

A pencil of contravariant metrics g_λ fulfilling the above conditions is called a *flat pencil*. A flat pencil is said to be *semisimple* if the eigenvalues of the affinor gh^{-1} are functionally independent. In this case the eigenvalues define a special set of coordinates, called *canonical coordinates*, where both the metrics of the pencil become diagonal.

2.2 Higher-order operators

General structure theorems for higher-order homogeneous Hamiltonian operators (6) are much weaker. We only consider the case where the coefficient $\ell^{ij} = A^{ij}_{n,0}(u)$ of the leading term is non-degenerate: $\det(\ell^{ij}) \neq 0$. The term $A^{ij}_{n,n}(u, u_x, \ldots, u_{(n)})$ of the above operators contains a summand of the form $d^{ij}_k u^k_{(n)}$. It can be proved that $-\ell_{ih}d^{hj}_k$ transform as the Christoffel symbols of a linear connection; the fact that the operator is Hamiltonian imply that such a connection is symmetric and flat [34, 9]. In flat coordinates we have the following canonical forms of R_2 and R_3, respectively:
$$R_2 = \partial_x \ell^{ij} \partial_x, \tag{23}$$
where $\ell_{ij} = T_{ijk}u^k + T^0_{ij}$ and T_{ijk} are completely skew-symmetric and
$$R_3 = \partial_x \left(\ell^{ij}\partial_x + c^{ij}_k u^k_x\right) \partial_x. \tag{24}$$

Moreover, introducing $c_{ijk} = \ell_{iq}\ell_{jp}c^{pq}_k$, the following conditions must be fulfilled [17]:

$$c_{nkm} = \frac{1}{3}(\ell_{nm,k} - \ell_{nk,m}), \tag{25a}$$

$$\ell_{mn,k} + \ell_{nk,m} + \ell_{km,n} = 0, \tag{25b}$$

$$c_{mnk,l} = -\ell^{pq}c_{pml}c_{qnk}. \tag{25c}$$

Both canonical forms (23) and (24) are defined up to affine transformations. The normal forms of the operators R_2 and R_3 depend on the number of components m. In the case $m = 2$ we have $R_2 = T^{ij}_0\partial^2_x$, where T^{ij}_0 is a constant skew-symmetric matrix. The operator can be reduced to (9) by an affine transformation. There are three canonical forms for the leading term of R_3 when $m = 2$ modulo affine transformations [17], namely (10), (11), (12). One can verify that the metric $\ell^{(2)}$ of $R^{(2)}_3$ is flat, while the metric $\ell^{(3)}$ of $R^{(3)}_3$ is non-flat.

We stress that two homogeneous third-order Hamiltonian operators are equivalent by a point transformation (7) if and only if they have the same normal form (10), (11), or (12). We also remark that the invariance group of R_3 can be enlarged to reciprocal transformations of projective type [17]. When $m = 2$ it can be proved that the same projective transformation reduces the last two cases to constant coefficients. If $m = 3, 4$ there is a classification of normal forms of R_3 up to reciprocal transformations of projective type [17, 18]. However, reciprocal transformations are outside the aims of this paper.

3 Compatible trios P_1, Q_1, R_i in two components

In this Section we classify all trios of two compatible homogeneous first-order Hamiltonian operators P_1, Q_1 and one homogeneous Hamiltonian operator R_i of order i, with $i = 2$ or $i = 3$. Without loss of generality we assume that the operators R_i are in one of the normal forms (9), (10), (11), (12).

First of all, let us prove the main Theorems 1, 2, 3, 4 (see the Introduction).

Proof of Theorems 1, 2, 3, 4. First of all we solved the conditions $[P_1, R_2] = 0$ and $[P_1, R^{(i)}_3] = 0$ using all coefficients g^{ij} and Γ^{ij}_k as unknown functions of the field variables (u^i). Differential operators are identified with variational multivectors, and their Schouten bracket is computed by the formulae that can be found in [15, 20, 21]. The results of the Shouten brackets are variational three-vectors which we require to vanish (up to total divergences). The vanishing of the coefficients of the three-vectors yields an overdetermined system of linear PDEs. It turns out that the solutions linearly depend on a set of parameters c_i, and are given in (13), (14), (16), (18). We checked the solutions using the programs `pdesolve` (Maple) and `crack` [39, 40];

Then we impose that the functions Γ^{ij}_k are the Christoffel symbols of the Levi-Civita connection of g^{ij}:

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s \tag{26}$$

$$\Gamma^{ij}_k + \Gamma^{ji}_k = \partial_k g^{ij} \tag{27}$$

In the case $[P_1, R_2] = 0$ the above conditions are empty, while in the case $[P_1, R_3^{(i)}] = 0$ we obtain quadratic constraints for the coefficients c_i. In particular, we obtain (15) in the case $i = 1$, (17) in the case $i = 2$ and (19) in the case $i = 3$.

In principle we should have further restrictions coming from the flatness of g but in the two-component case this condition does not provide additional constraints (this fact is no longer true already in the three-component case). This completes the proof of main Theorems. □

Now, we find trios of compatible operators between the families of first-order operators that we selected in the main Theorems.

It is easy to realize that the following statement holds.

Theorem 5. *Any pair (g, h) of metrics which are in the family* (13) *yield two first-order compatible Hamiltonian operators P_1, Q_1, and hence a compatible trio (P_1, Q_1, R_2).*

Three Theorems stated below describe all trios of compatible Hamiltonian operators $(P_1, Q_1, R_3^{(i)})$, where $i = 1, 2, 3$, respectively.

Theorem 6. *The solution of the Levi-Civita conditions* (15) *for the metric g^{ij}* (14) *of the operator P_1 that is compatible with $R_3^{(1)}$ are*

1. *if $c_2 \neq 0$ then $c_6 = (c_4c_1)/c_2$, $c_7 = (c_3c_4)/c_2$;*

2. *if* $c_2 = 0$ *and* $c_3 \neq 0$ *then* $c_6 = (c_7 c_1)/c_3$, $c_4 = 0$;

3. *if* $c_3 = 0$, $c_2 = 0$ *then* $c_1 = 0$;

4. *if* $c_3 = 0$, $c_2 = 0$ *and* $c_1 \neq 0$ *then* $c_4 = 0$, $c_7 = 0$.

The compatible pencils $g_{\lambda,kl} = g_k^{ij} - \lambda h_l^{ij}$ *of one metric* g_k^{ij} *from the above case* k *and one metric* h_l^{ij} *from the above case* l *(*l *and* k *run from 1 to 4) are*

- $g_{\lambda,11}$ *if* $c_4 = \frac{d_4 c_2}{d_2}$, *or* $d_3 = \frac{d_2 c_3}{c_2}$, $c_1 = \frac{d_1 c_2}{d_2}$;
- $g_{\lambda,12}$ *if* $d_7 = \frac{d_3 c_4}{c_2}$;
- $g_{\lambda,13}$ *if* $d_6 = \frac{d_4 c_1}{c_2}$, $d_7 = \frac{d_4 c_3}{c_2}$.
- $g_{\lambda,14}$ *if* $d_6 = \frac{c_4 d_1}{c_2}$;
- $g_{\lambda,22}$ *if* $d_7 = \frac{d_3 c_7}{c_3}$, *or if* $d_1 = \frac{d_3 c_1}{c_3}$;
- $g_{\lambda,23}$ *if* $d_4 = 0$, $d_6 = \frac{d_7 c_1}{c_3}$;
- $g_{\lambda,24}$ *if* $d_6 = \frac{c_7 d_1}{c_3}$;
- $g_{\lambda,33}$;
- $g_{\lambda,34}$ *if* $c_4 = c_7 = 0$;
- $g_{\lambda,44}$.

Proof. In order to get a compatible trio (P_1, Q_1, R_3) we have to select among the pairs of flat metrics (g, h) of the above form those defining a flat pencil. To this aim, we first solve the system of Levi-Civita conditions (15). The solutions are given as a numbered list of subcases, from the most generic to the least generic.

Each metric is defined by a point in the space of parameters. We call c_i the values of the parameters that provide the metric g of the operator P_1 and d_i the values of the parameters that provide the metric h of the operator Q_1. They can be interpreted as the coordinates of two points in the algebraic variety defined by the quadratic conditions described above. If the pair (g, h) defines a flat pencil, then the straight line joining these two points is entirely contained in this variety.

We compute the Schouten bracket condition $[P_1, Q_1] = 0$ with (algebraic) unknowns c_i and d_i; here, we make use of Computer Algebra Systems as described in the Main Theorems. We obtain quadratic constraints in both sets of variables. The solutions of these constraints are given as a list of pencils denoted by $g_{\lambda,kl} = g^{ij}_k - \lambda h^{ij}_l$. In this notation the metric g^{ij}_k comes from subcase k and yields the first-order operator P_1, and the metric h comes from subcase l and yields the first-order operator Q_1, and the constants c_i and d_i must fulfill additional conditions to ensure the compatibility of P_1 and Q_1. □

Theorem 7. *The solutions of the Levi-Civita conditions* (17) *for the metric* g^{ij} (16) *of the operator* P_1 *that is compatible with* $R^{(2)}_3$ *are*

1. *if* $c_1 \neq 0$ *then* $c_5 = 0$ *and* $c_3 = -\frac{c_2 c_6}{2c_1}$*;*
2. *if* $c_1 = 0$ *and* $c_2 \neq 0$ *then* $c_5 = c_6 = 0$*;*
3. *otherwise* $c_1 = c_2 = 0$*.*

The compatible pencils $g_{\lambda,kl} = g^{ij}_k - \lambda h^{ij}_l$ *of one metric* g^{ij}_k *from the above case* k *and one metric* h^{ij}_l *from the above case* l *(*l *and* k *run from 1 to 3) are*

- $g_{\lambda,11}$ *if* $d_6 = \frac{d_1 c_6}{c_1}$*, or* $d_2 = \frac{d_1 c_2}{c_1}$*.*
- $g_{\lambda,12}$ *if* $d_3 = -\frac{d_2 c_6}{2c_1}$*.*
- $g_{\lambda,13}$ *if* $d_3 = -\frac{d_6 c_2}{2c_1}$*,* $d_5 = 0$*.*
- $g_{\lambda,22}$*;*
- $g_{\lambda,23}$ *if* $d_5 = d_6 = 0$*.*
- $g_{\lambda,33}$*.*

Proof. Same as in Theorem 6. □

Theorem 8. *The solution of the Levi-Civita conditions* (19) *for the metric* g^{ij} (18) *of the operator* P_1 *that is compatible with* $R^{(3)}_3$ *are*

1. *if* $c_2 \neq 0$ *then* $c_5 = -\frac{2c_1 c_3}{c_2}$ *and* $c_6 = \frac{2c_3 c_4}{c_2}$*;*
2. *if* $c_2 = 0$ *and* $c_3 \neq 0$ *then* $c_1 = c_4 = 0$*;*

3. *if* $c_2 = c_3 = 0$ *and* $c_6 \neq 0$ *then* $c_1 = -\frac{c_4 c_5}{c_6}$*;*

4. *if* $c_2 = c_3 = c_6 = 0$ *and* $c_5 \neq 0$ *then* $c_4 = 0$*;*

5. *otherwise* $c_2 = c_3 = c_5 = c_6 = 0$*.*

The compatible pencils $g_{\lambda,kl} = g_k^{ij} - \lambda h_l^{ij}$ *of one metric* g_k^{ij} *from the above case* k *and one metric* h_l^{ij} *from the above case* l *(*l *and* k *run from 1 to 5) are*

- $g_{\lambda,11}$ *if* $d_3 = \frac{d_2 c_3}{c_2}$*, or* $d_1 = \frac{d_2 c_1}{c_2}$*,* $d_4 = \frac{d_2 c_4}{c_2}$*.*
- $g_{\lambda,12}$ *if* $d_5 = -\frac{2 d_3 c_1}{c_2}$*,* $d_6 = \frac{2 d_3 c_4}{c_2}$*.*
- $g_{\lambda,13}$ *if* $d_6 = \frac{2 d_4 c_3}{2 c_2}$*, with* $d_4 \neq 0$*,* $c_3 \neq 0$*.*
- $g_{\lambda,14}$ *if* $d_5 = -\frac{2 d_4 c_3}{2 c_2}$*, with* $d_4 \neq 0$*,* $c_3 \neq 0$*.*
- $g_{\lambda,15}$ *if* $c_3 = 0$*.*
- $g_{\lambda,22}$*.*
- $g_{\lambda,33}$ *if* $d_5 = d_6 = 0$*, or* $d_5 = \frac{d_6 c_5}{c_6}$*, or* $d_4 = \frac{d_6 c_4}{c_6}$*.*
- $g_{\lambda,34}$ *if* $d_1 = -\frac{d_5 c_4}{c_6}$*.*
- $g_{\lambda,35}$ *if* $d_1 = -\frac{d_4 c_5}{c_6}$*.*
- $g_{\lambda,44}$*.*
- $g_{\lambda,45}$ *if* $d_4 = 0$*.*
- $g_{\lambda,55}$*.*

We stress that $g_{\lambda,23}$*,* $g_{\lambda,24}$ *and* $g_{\lambda,25}$ *do not define flat pencils.*

Proof. Same as in Theorem 6. □

4 Examples

We consider some known and new examples of bi-Hamiltonian structures associated with trios of compatibile operators. Each trio (P_1, Q_1, R_i) $(i = 2, 3)$ defines two pencils $\Pi_\lambda = P_1 + R_i - \lambda Q_1$ and $\tilde{\Pi}_\lambda = Q_1 + R_i - \lambda P_1$. In the case of new examples we compute the first non trivial flows of the associated bi-Hamiltonian hierarchies.

4.1 Case R_2: Cohomology spaces of curves

In [8] the following six-parameter family of pairwise compatible Hamiltonian operators defined by the cohomology spaces of curves is considered:

$$\begin{pmatrix} a(u^1_x + 2u^1\partial_x) + \alpha\partial_x + c\partial_x^3 & au^2\partial_x + \beta\partial_x + \gamma\partial_x^2 \\ a\partial_x u^2 + \beta\partial_x - \gamma\partial_x^2 & \epsilon\partial_x \end{pmatrix} \tag{28}$$

It contains systems by Ito, Kupershmidt, Antonowicz and Fordy, Fokas and Liu, Gümral and Nutku.

For $\gamma = 1$ and $c = 0$ we have a family of commuting operators of our type. It is easy to check that it corresponds to the choice $c_1 = 2a$, $c_2 = \alpha$, $c_4 = \epsilon$ (and all other $c_i = 0$) in the metric g of Theorem 1.

4.2 Case R_2: Kaup-Broer equation

The bi-Hamiltonian property of the Kaup-Broer system was established in [24]. The system is

$$\begin{cases} u^1_t = ((u^1)^2/2 + u^2 + \beta u^1_x)_x, \\ u^2_t = (u^1u^2 + \alpha u^1_{xx} - \beta u^2_x)_x, \end{cases} \tag{29}$$

where α, β are two constants. Indeed, the system is tri-Hamiltonian, two of the operators are of the form

$$B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2\partial_x & \partial_x u^1 - \partial_x^2 \\ u^1\partial_x + \partial_x^2 & u^2\partial_x + \partial_x u^2 \end{pmatrix} \tag{30}$$

and are defined by trio of compatible Hamiltonian operators of our class. Indeed, it is easy to check that the choice $c_2 = 2$, $c_3 = 2$ and all other c_i set to zero in the metric g of Theorem 1 yields the above example (up to the sign of R_2).

According with [3], there exists a Miura transformation that brings the above system into Dispersive Water Waves system.

4.3 Case $R_3^{(1)}$: Dispersive Water Waves

Here we consider the example on page 482 of [3]. The system

$$u^1_t = \frac{1}{4}u^2_{xxx} + \frac{1}{2}u^2u^1_x + u^1u^2_x, \tag{31}$$

$$u^2_t = u^1_x + \frac{3}{2}u^2u^2_x \tag{32}$$

is the DWW equation up to a Miura transformation. It is a tri-Hamiltonian equation with respect to the operators

$$B_0 = \begin{pmatrix} -\frac{1}{2}u^2\partial_x - \frac{1}{2}\partial_x u^2 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{33}$$

$$B_1 = \begin{pmatrix} \frac{1}{4}\partial_x^3 + \frac{1}{2}u^1\partial_x + \frac{1}{2}\partial_x u^1 & 0 \\ 0 & \partial_x \end{pmatrix} \tag{34}$$

$$B_2 = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 + \frac{1}{2}u^1\partial_x + \frac{1}{2}\partial_x u^1 \\ \frac{1}{4}\partial_x^3 + \frac{1}{2}u^1\partial_x + \frac{1}{2}\partial_x u^1 & \frac{1}{2}u^2\partial_x + \frac{1}{2}\partial_x u^2 \end{pmatrix} \tag{35}$$

The pair (B_0, B_2) is defined by a trio of compatible Hamiltonian operators of our class. Indeed, if we choose $c_2 = -1/2$, $c_5 = 1$ and all other values of c_i equal to 0 in g, and $d_4 = 1/2$ with all other values of d_j equal to 0 in h we recover the above example from (14).

4.4 Case $R_3^{(1)}$: coupled Harry-Dym hierarchy

We consider the example on page L273 of [2]. The system

$$u_1^1 = \left(\frac{1}{4(u^2)^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{(u^2)^{1/2}}\right)_x \tag{36}$$

$$u_t^2 = u^1\left(\frac{1}{(u^2)^{1/2}}\right)_x + \frac{u_x^1}{2(u^2)^{1/2}} \tag{37}$$

is tri-Hamiltonian with respect to the following operators

$$B_0 = \begin{pmatrix} -\frac{1}{2}u^1\partial_x - \frac{1}{2}\partial_x u^1 & -\frac{1}{2}u^2\partial_x - \frac{1}{2}\partial_x u^2 \\ -\frac{1}{2}u^2\partial_x - \frac{1}{2}\partial_x u^2 & 0 \end{pmatrix}, \tag{38}$$

$$B_1 = \begin{pmatrix} \frac{1}{4}\partial_x^3 - \alpha\partial_x & 0 \\ 0 & -\frac{1}{2}u^2\partial_x - \frac{1}{2}\partial_x u^2 \end{pmatrix} \tag{39}$$

$$B_2 = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 - \alpha\partial_x \\ \frac{1}{4}\partial_x^3 - \alpha\partial_x & +\frac{1}{2}u^1\partial_x + \frac{1}{2}\partial_x u^1 \end{pmatrix} \tag{40}$$

The pair (B_0, B_2) is defined by a trio of compatible Hamiltonian operators of our class. Indeed, if we choose $c_1 = -1/2$ with all other c_i equal to 0 in g and $d_5 = -\alpha$, $d_6 = 1/2$ with all other d_j equal to 0 in h we recover the above example from (14).

4.5 Case $R_3^{(2)}$: pencil $g_{\lambda,11}$

Choosing

$$c_4 = 0, \quad c_1 = -1, \qquad c_6 = -1, \qquad c_2 = 0, \qquad d_2 = 0, \qquad d_1 = 0$$

we obtain the trio

$$\begin{aligned}
P_1 &= \begin{pmatrix} -u^1 & 0 \\ 0 & \frac{(u^2)^2-1}{u^1} \end{pmatrix} \partial_x + \frac{1}{2} \begin{pmatrix} -u^1_x & u^2_x \\ -u^2_x & \frac{2u^1u^2u^2_x-(u^2)^2u^1_x+u^1_x}{(u^1)^2} \end{pmatrix} \\
Q_1 &= \begin{pmatrix} 0 & -u^1 \\ -u^1 & -2u^2 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & -u^1_x \\ 0 & -u^2_x \end{pmatrix} \\
R_3^{(3)} &= \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x.
\end{aligned}$$

Starting from the Casimirs of Q_1

$$C_1 = \int_{S^1} u^1 \, dx, \qquad C_2 = \int_{S^1} \frac{u^2}{u^1} \, dx,$$

the first flows of the bi-Hamiltonian hierarchy are

$$u_{t_i} = (P_1 + \epsilon^2 R_3)\delta C_i, \qquad i = 1, 2,$$

that is

$$u^1_{t_1} = -\frac{1}{2}u^1_x, \qquad u^2_{t_1} = -\frac{1}{2}u^2_x$$

and

$$\begin{aligned}
u^1_{t_2} &= \frac{3}{2}\frac{u^2_x}{u^1} - \frac{3}{2}\frac{u^2u^1_x}{(u^1)^2} - \frac{u^1_{xxx}}{(u^1)^3} + 9\frac{u^1_xu^1_{xx}}{(u^1)^4} - 12\frac{(u^1_x)^3}{(u^1)^5} \\
u^2_{t_2} &= \frac{3}{2}\frac{(1-(u^2)^2)u^1_x}{(u^1)^3} + \frac{3}{2}\frac{u^2u^2_x}{(u^1)^2} - \frac{30u^2(u^1_x)^3}{(u^1)^6} + 10\frac{u^2_x(u^1_x)^2}{(u^1)^5} + 12\frac{u^2_x(u^1)^2_x}{(u^1)^5} + \\
&\quad - \frac{3u^2_xu^1_{xx}}{(u^1)^4} - 2\frac{u^2u^1_{xxx}}{(u^1)^4} - \frac{u^2_{xx}u^1_x}{(u^1)^4}.
\end{aligned}$$

The canonical coordinates are

$$\lambda^1 = \frac{u^2+1}{u^1}, \qquad \lambda^2 = \frac{u^2-1}{u^1}$$

and the central invariants are

$$s_1 = \frac{1}{2}, \qquad s_2 = -\frac{1}{2}.$$

This shows that the above system is not Miura-trivial.

4.6 Case $R_3^{(2)}$: pencil $g_{\lambda,13}$

Choosing

$$c_3 = 0, \quad d_3 = 1, \quad c_2 = 2, \quad c_4 = 1, \quad d_4 = 0, \quad d_5 = 0$$

we obtain the trio

$$\begin{aligned}
P_1 &= \begin{pmatrix} 2u^2 & \frac{(u^1)^2+(u^2)^2}{u^1} \\ \frac{(u^1)^2+(u^2)^2}{u^1} & 2u^2 \end{pmatrix} \partial_x + \begin{pmatrix} u^2_x & u^1_x \\ \frac{u^2(2u^1u^2_x - u^1_x u^2)}{(u^1)^2} & u^2_x \end{pmatrix} \\
Q_1 &= \begin{pmatrix} 0 & -1/u^1 \\ -1/u^1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 0 \\ \frac{u^1_x}{(u^1)^2} & 0 \end{pmatrix} \\
R_3^{(2)} &= \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x.
\end{aligned}$$

Starting from the Casimirs of Q_1

$$C_1 = \int_{S^1} \frac{1}{2}(u^1)^2\, dx, \qquad C_2 = \int_{S^1} u^2\, dx,$$

the first flows of the bi-Hamiltonian hierarchy are

$$\begin{aligned}
u^1_{t_1} &= u^1_x \\
u^2_{t_1} &= u^2_x
\end{aligned}$$

and

$$\begin{aligned}
u^1_{t_2} &= 2u^2u^1_x + u^1u^2_x \\
u^2_{t_2} &= u^1u^1_x + 2u^2u^2_x - \frac{u^1_x u^1_{xx}}{(u^1)^2} + \frac{u^1_{xxx}}{u^1},
\end{aligned}$$

respectively.

The canonical coordinates are

$$\lambda^1 = (u^1+u^2)^2, \qquad \lambda^2 = (u^1-u^2)^2,$$

and the central invariants are

$$s_1 = -\frac{1}{8\sqrt{\lambda^1}}, \qquad s_2 = \frac{1}{8\sqrt{\lambda^2}}.$$

Again, this example is not Miura-trivial.

4.7 Case $R_3^{(3)}$: pencil $g_{\lambda,12}$

Choosing

$$c_1 = 1, \quad c_2 = -1, \quad d_3 = 1, \quad c_3 = 0, \quad c_4 = 0$$

we obtain the trio

$$P_1 = \begin{pmatrix} u^1 - u^2 & \frac{-(u^2)^2+1}{2u^1} \\ \frac{-(u^2)^2+1}{2u^1} & \frac{-(u^2)^2+1}{u^1} \end{pmatrix} \partial_x +$$

$$\frac{1}{2} \begin{pmatrix} u^1_x - u^2_x & -u^2_x \\ \frac{(u^1)^2 u^2_x - 2u^1 u^2 u^2_x + u^1_x (u^2)^2 - u^1_x}{(u^1)^2} & \frac{-2u^1 u^2 u^2_x + u^1_x (u^2)^2 - u^1_x}{(u^1)^2} \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} -1 & -\frac{u^2}{u^1} \\ -\frac{u^2}{u^1} & -2\frac{u^2}{u^1} \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 0 \\ \frac{-u^1 u^2_x + u^1_x u^2}{(u^1)^2} & \frac{-u^1 u^2_x + u^1_x u^2}{(u^1)^2} \end{pmatrix}$$

$$R_3^{(3)} = \partial_x \begin{pmatrix} 1 & \partial_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1} \partial_x & \frac{(u^2)^2+1}{2(u^1)^2} \partial_x + \partial_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \partial_x.$$

Starting from the Casimirs of Q_1

$$C_1 = \int_{S^1} (u^1 - u^2)\, dx, \qquad C_2 = \int_{S^1} \sqrt{(u^2)^2 - 2u^1 u^2}\, dx,$$

one easily gets the first non trivial flows of the associated bi-Hamiltonian hierarchy.

The canonical coordinates are

$$\lambda^1 = -\frac{1}{2} \frac{(u^2)^2 - 1}{u^2}, \qquad \lambda^2 = \frac{1}{2} \frac{4\,(u^1)^2 - 4\,u^1 u^2 + (u^2)^2 - 1}{2u^1 - u^2},$$

and the central invariants are

$$s_1 = \frac{1}{2} \frac{\lambda^1 \sqrt{(\lambda^1)^2 + 1} - (\lambda^1)^2 - 1}{(\lambda^1)^2 + 1}, \qquad s_2 = -\frac{1}{2} \frac{\lambda^2 \sqrt{(\lambda^2)^2 + 1} + (\lambda^2)^2 + 1}{(\lambda^2)^2 + 1}.$$

This example is also not Miura-trivial.

Remark 9. We observe that the above examples are linear with respect to third-order derivatives, but the matrix of coefficients of the third-order derivatives, which is also known as *separant* is non-constant. Two-component systems of third-order evolution equations with constant separant are classified (see [30]), but at the moment we cannot exclude that our examples are related to known integrable systems by a reciprocal transformation. To include reciprocal transformations in the classification problem would require to consider a more general class of (non local) Poisson pencils of hydrodynamic type since (in general) locality is not preserved by this kind of transformations.

5 Conclusions

The above straightforward generalization of the bi-Hamiltonian structure of the KdV equation yields bi-Hamiltonian systems in great amount even in the 2-component case. Preliminary computations show that a similar situation occurs in the 3 and 4-component case. Unfortunately, in these case there is no affine classification available; the only classification that has been found so far is under the group of reciprocal transformations of projective type [17, 18].

By extending the group of admissible transformations to reciprocal transformations of projective type we are led to consider also Ferapontov–Mokhov non-local Poisson brackets of hydrodynamic type. Interesting projective-geometric issues are likely to appear. We leave this interesting topic to future investigations.

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