The geometry of real reducible polarizations in quantum mechanics

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Abstract

The formulation of Geometric Quantization contains several axioms and assumptions. We show that for real polarizations we can generalize the standard geometric quantization procedure by introducing an arbitrary connection on the polarization bundle. The existence of reducible quantum structures leads to considering the class of Liouville symplectic manifolds. Our main application of this modified geometric quantization scheme is to Quantum Mechanics on Riemannian manifolds. With this method we obtain an energy operator without the scalar curvature term that appears in the standard formulation, thus agreeing with the usual expression found in the Physics literature.

Keywords: Geometric quantization, real polarization, Schrödinger equation in curved spaces.

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1 Introduction

Quantization of Classical Mechanics is a procedure introduced by Dirac that starting from a classical observable yields a quantum operator. It is well known that it is not possible to consistently quantize all observables $f(q,p)$ on a classical phase space, as this would lead to physical and mathematical contradictions (see *e.g.* [24, 31]). A *polarization* is a mathematical structure that allows a coordinate-free definition of a subset of quantizable observables and a Hilbert space of wave functions on which the quantum operators act.

Polarizations can be real or complex. The latter have attracted the interest of more researchers mainly due to two reasons. In first place complex polarizations exist even when real polarizations cannot exist (consider, for instance, the symplectic manifold S^2) and in second place they have applications to representation theory [11]. During our research we have realized that a class of real polarizations, the reducible polarizations, have a geometry that is richer than what appears in the standard expositions of quantization. In this paper we will study the geometry of real reducible polarizations, and discuss some interesting consequences for the quantization process.

Our framework is that of Geometric Quantization (GQ, [24, 31]), where we have a classical Hamiltonian system on a symplectic manifold (M,ω), $\dim M = 2n$. A real polarization is an integrable Lagrangian distribution $P \subset TM$. The choice of a polarization is the infinitesimal analogue of the choice of a set of (locally defined) Poisson–commuting observables. There are well-known existence theorems concerning polarizations and other geometric structures in GQ; the theorems impose restrictions on the topology of the underlying symplectic manifold.

Quantum states are represented by sections $\psi\colon M \to \mathcal{L}$ of a Hermitian complex line bundle $\mathcal{L} \to M$ endowed with a Hermitian connection $\nabla^{\mathcal{L}}$ whose curvature is proportional to the symplectic form ω.

Quantizable observables $\mathcal{O}_P$ are then just those classical observables $f \in C^\infty(M)$ whose Hamiltonian vector field X_f preserves the polarization. Such vector fields act on *polarized* sections ψ, that is sections which are (covariantly) constant along the polarization, thus providing a candidate for quantum operators. However the scalar product $\int \psi_1\overline{\psi_2}$ of two wave functions ψ_1, ψ_2 is ill-defined since the space of leaves M/P, even if it exists as a Hausdorff manifold, does not have in general a natural choice of volume element. A solution is to redefine wave functions to have values in $\mathcal{L} \otimes \sqrt{\wedge^n_{\mathbb{C}} P}$. The L^2-completion of the subspace of polarized wave functions is a Hilbert space $\mathcal{H}_P$ that represents quantum states.

The polarization also allows us to define a *representation* $\mathcal{O}_P \to \mathrm{End}(\mathcal{H}_P)$ of quantizable classical observables into quantum operators.

The best known example of a real polarization is the vertical polarization $P = VT^*Q = \ker \tau_Q^\vee$ of the phase space of any *natural mechanical system* described by the symplectic manifold $M = T^*Q$ endowed with the standard symplectic form ω_0 or a charged symplectic form $\omega_0 + B$ and with a Riemannian metric g on the configuration space Q. If we consider the natural Darboux coordinates (q^i, p_i) for ω_0, then the vertical polarization is spanned by the vector fields $\partial/\partial p_i$, the family of commuting observables is (q^i) and quantizable observables $f \in \mathcal{O}_P$ are linear functions of momenta $f = X^i(q^j)p_i + f^0(q^j)$. This is the *Schrödinger representation.* Note that P is naturally isomorphic to $T^*Q \times_Q T^*Q$.

One of the key problems that appear in geometric quantization is the need to quantize observables whose Hamiltonian vector fields do not respect the chosen polarization. The most important and basic example is the kinetic energy $K = \frac{1}{2m}\sum_i p_i^2$ of a natural mechanical system with respect to the Schrödinger representation. The standard method for quantizing such observables is the Blattner-Kostant-Sternberg (BKS) method [24, 31]. The quantum operator corresponding to "bad" observables is defined as a kind of Lie derivative of the wave functions which are first dragged along the flow of the corresponding Hamiltonian vector field and then projected to the initial Hilbert space through the BKS kernel. Recent research [9] has shown that the BKS procedure can be interpreted as parallel transport with respect to a connection on an infinite–dimensional space, and is related to the Coherent State Transform [12]. BKS also plays an important role in studying the Maslov correction to semiclassical states [8]. BKS can be used for observables which are different from the Hamiltonian, in principle. For example, Stäckel systems exhibit many conserved quantities that are quadratic in momenta [3]. However, proving that each of them is quantizable using the BKS method is a nontrivial task, and we postpone this problem to future research.

The remark by which we started our investigation is that *real reducible polarizations may have an additional geometric structure.* Our main example is the vertical polarization $P = T^*Q \times_Q T^*Q$ of a natural mechanical system (T^*Q, ω_0), where (Q, g) is a Riemannian manifold. Clearly, P can be endowed with the connection obtained by pulling-back the Levi-Civita connection ∇^g of the Riemannian manifold (Q, g). Strangely enough, this connection has never been considered in GQ.

Therefore, we decided to explore the consequences implied by using such a connection on the polarization bundle. With this aim in mind we had to understand the mathematical hypotheses under which the introduction of a new connection would be most natural. A derivation along P is needed in GQ in order to define the Hilbert space of states, *i.e.* the polarized wave functions. When the space of leaves M/P is a manifold it may happen that both the line bundle $\mathcal{L}$ and the polarization P descend to M/P, as in the case of natural mechanical systems, and the action of quantum operators on polarized sections is given through the Lie derivative with respect to fundamental horizontal vector fields on the factor $\mathcal{L}$ and through the ordinary Lie derivative on the factor $\sqrt{\wedge^n_{\mathbb{C}} P}$.

It turns out that he most natural framework for considering alternative connections on the polarization bundle P is that of *real reducible polarizations* (Section 3). A reducible polarization P on a symplectic manifold M is by definition isomorphic to the vertical bundle of a surjective submersion $\pi\colon M \to Q$. One also needs reducibility hypotheses that guarantee the descent of $\mathcal{L}$ and, to some extent, of the connection $\nabla^{\mathcal{L}}$. The reducibility of the prequantum structure $(\mathcal{L}, \langle\ ,\ \rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ is defined as the existence of a complex line bundle $L \to Q$ with Hermitian product $\langle\ ,\ \rangle$ such that π^*L is isomorphic to $\mathcal{L}$ and $\pi^*\langle\ ,\ \rangle$ is isomorphic to $\langle\ ,\ \rangle^{\mathcal{L}}$. If the reduced line bundle $L \to Q$ admits a Hermitian connection ∇, then the compatibility condition that we require between $\nabla^{\mathcal{L}}$ and ∇ is that $\nabla^{\mathcal{L}}$ and $\pi^*\nabla$ must define the same space of polarized sections.

Using the above definition of reducibility we are able to prove one of the main results of our paper.

Theorem 3.9 *A quantum structure $(\mathcal{L}, \langle\ ,\ \rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ is reducible over a submersion $\pi\colon M \to Q$ if and only if $\mathcal{L}$ admits a relative connection $\widehat{\nabla}$ along the fibres of π that has trivial holonomy groups. In this case P turns out to be isomorphic to π^*T^*Q. If there exists a reduction $(L, \langle\ ,\ \rangle, \nabla)$ of the quantum structure such that ∇ is compatible with $\nabla^{\mathcal{L}}$ then M is endowed with the structure of a Liouville manifold [30].*

The above Theorem leads to interesting consequences at the quantum level. In this paper we develop the Geometric Quantization of Liouville manifolds and we give a complete description of quantizable observables (Theorem 3.10): they are functions which are linear with respect to the fibre coordinates of π. Such a definition is a generalization to Liouville manifolds of what was already known for natural mechanical systems.

We prove that polarized sections have the form $\psi\colon Q \to L \otimes K_Q^{1/2}$, where $K_Q^{1/2}$ is the square root of the determinant bundle of $T^*Q \to Q$. Then, we define quantum operators acting on polarized sections using a more general idea than the usual GQ approach. Namely, instead of operating on $K_Q^{1/2}$ using Lie derivatives, we allow for an arbitrary linear connection ∇^Q. The consequences of this definition are quite interesting in our opinion.

First of all, we have to check that the modified quantum operator satisfy the usual properties dictated by Physics. In this direction we prove the following result:

Theorem 3.14

- *the commutation identity* $\widehat{[f_1, f_2]} = i\hbar[\hat{f}_1, \hat{f}_2]$ *involving quantizable observables* f_i *and the corresponding quantum operators* $\hat{f}_i$ *holds if and only if* ∇^Q *is a flat connection;*

- *the quantum operator* $\hat{f}$ *is symmetric if and only if the gradient of* f *with respect to the fibres of* π *is a divergence-free vector field.*

The flatness of ∇^Q is really a very mild restriction since the determinant bundle $K_Q^{1/2}$ is trivial when Q is orientable. In particular, when quantizing natural mechanical systems using the vertical polarization it is possible to act on half-forms via ∇^g instead of the Lie derivative. Indeed, ∇^g is flat on the determinant bundle. The divergence-free condition is not really a significant restriction since in the main physical examples, like the quantization of positions, momenta or the components of the angular momentum, it always holds. The only change that we will notice at this stage with respect to GQ is the fact that the divergence terms in the quantum momentum operators will disappear.

When quantizing observables that do not preserve the polarization via the BKS method, one might use the parallel transport with respect to the arbitrary connection ∇^Q instead of the flow of the Hamiltonian vector field. In natural mechanical systems with the vertical polarization, the usual quantum energy operator obtained by applying the standard rules of GQ includes a multiplicative term of the form $\hbar^2 k r_g$, where r_g is the scalar curvature of g, and k is a constant. If we act on half-forms via ∇^g instead of the Lie derivative, we have an interesting consequence.

Theorem 4.5. *The quantization of a particle moving in a Riemannian manifold using the above modified GQ procedure acting on half-forms by parallel transport with respect to the Levi-Civita connection* ∇^g *yields the quantum energy operator*

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + V.$$

Therefore, the constant term in front of the scalar curvature vanishes, $k = 0$.

Let us summarize in a table the main differences between the standard GQ and the modified GQ for the case of natural mechanical systems. We use canonical coordinates (q^i, p_i) on T^*Q. Quantum operators are computed on half-form wavefunctions $\psi = s \otimes \sqrt{\nu_g}$, where $s\colon Q \to L$ is a section, $s = \psi_0 b^0$, b^0 is a local basis of sections of Q, and $\sqrt{\nu_g}$ is the square root of the metric volume element on Q.

	Standard GQ	Modified GQ
Quantizable observables	$f = f^0 + f^i p_i$ where $f^0, f^i \in \mathcal{C}^\infty(Q)$	$f = f^0 + f^i p_i$ where $f^0, f^i \in \mathcal{C}^\infty(Q)$ and $f^i\partial/\partial q^i$ is divergence-free
Quantization of position	$\widehat{q^i}(\psi) = i\hbar q^i s \otimes \sqrt{\nu_g}$	$\widehat{q^i}(\psi) = i\hbar q^i s \otimes \sqrt{\nu_g}$
Quantization of momentum	$\widehat{f^i p_i}(\psi) = -i\hbar\big(f^i \frac{\partial}{\partial q^i}\psi^0 + \frac{1}{2}\frac{\partial_i(f^i\sqrt{\lvert g\rvert})}{\sqrt{\lvert g\rvert}}\psi^0\big) b_0 \otimes \sqrt{\nu_g}$	$\widehat{f^i p_i}(\psi) = -i\hbar f^i \frac{\partial}{\partial q^i}\psi^0 b_0 \otimes \sqrt{\nu_g}$
Quantization of energy via BKS	$\widehat{H}(\psi) = -\frac{\hbar^2}{2}\Big(\Delta(s) - \frac{r_g}{6}s\Big)\otimes\sqrt{\nu_g} + V s \otimes \sqrt{\nu_g}$.	$\widehat{H}(\psi) = -\frac{\hbar^2}{2}\Delta(s) \otimes \sqrt{\nu_g} + V s \otimes \sqrt{\nu_g}$.

The scalar curvature term appeared for the first time in [7] with value $k = 1/12$ in the context of canonical quantization, which is GQ in Darboux coordinates. The author also stated that he got the same result with Feynman path integral; the proof was published later [5] (with $k = 1/6$) (see also [1], with $k = -1/12$ and [23] with $k = 1/8$, although in a slightly different context). Weyl quantization also shows a degree of ambiguity with $k = 1/12$, $k = 1/8$ or $k = 1/4$ depending on ordering problems in [18] (see also [17]). In a recent paper [2] the Darboux III oscillator is considered, and it is found that maximal superintegrability is achieved if the Laplacian is supplemented with the conformal term $(\hbar^2(n-2)/(8(n-1)))r_g$, where $n = \dim Q$. In GQ the factor is $k = 1/12$ [24, 31, 32].

The ambiguity in the factor k in front of r_g was already noted by De Witt:

> The choice of a numerical factor to stand in front of $\hbar^2 r_g$ is undetermined; all choices lead to the same classical theory in the limit $\hbar \to 0$. This, however, is the only ambiguity in the quantum Hamiltonian. [7, p. 395].

Another geometric approach to quantum mechanics, namely Covariant Quantum Mechanics [4, 13], provides further insight into the problem. The approach aims at a covariant formulation of classical and quantum time-dependent particle mechanics by means of a Lagrangian formulation, with the vertical polarization as a 'fixed' representation. In that approach one uses half-form valued wave functions and quantum operators act on half-forms through the Levi-Civita covariant derivative. The Schrödinger operator is derived through a Lagrangian approach. The scalar curvature appears in one of the summands of the Lagrangian. It was proved in [14] that there is a unique covariant Schrödinger operator up to the multiplicative term $\hbar^2 k r_g$

in which k cannot be determined by covariance arguments. This is obviously compatible with our results.

After the above considerations it is natural to ask if there is any evidence for the scalar curvature term in the energy spectrum of a physical system. The answer is that to our best knowledge the only examples with a nonzero scalar curvature have *constant* scalar curvature. Here we mention the rigid body, either free or subject to a rotationally invariant magnetic field [21, 22, 25], and the Landau problem leading to the Hall effect [19, 20, 26, 27, 15]. Needless to say, the effect of constant scalar curvature on the energy spectrum is a non-measurable overall constant shift. Indeed, it is quite a difficult task to find a classical system which would pass all the requirements and topological obstructions for GQ and still has a nonzero and nonconstant scalar curvature.

It is our opinion that the only way to decide weather the scalar curvature is present or not in the energy operator, and eventually its numerical factor, is by means of an experiment.

It is at the moment unclear if such an experiment can be performed on a microscopic scale or if any evidence can be inferred from *eg* cosmic observations. However, we stress that a major argument in favour of $k = 0$ (or no scalar curvature term in the quantum energy operator) and eventually our modified geometric quantization is the following one.

As the dynamics of a free particle is only ruled by the metric and the topology of the Riemannian manifold (Q, g) the classical trajectories are geodesics. The natural way to lift trajectories to the tangent and cotangent bundle is parallel transport. *So, it is more likely, in our opinion, to expect that the quantization procedure should be developed using parallel transport of volumes rather than Lie derivative with respect to Hamiltonian vector fields.*

The plan of the paper is the following. In section 2, after some mathematical preliminaries on the theory of connections, we give a summary of those aspects of GQ which will be analyzed later. Section 3 is devoted to a detailed description of the modified GQ. In section 4, we consider systems whose configuration space is a Riemannian manifold and apply to them the modified GQ method in the Schrödinger representation. A final discussion with future research perspectives will close the paper.

2 Preliminaries

All differentiable manifolds considered in this paper are assumed to be Hausdorff and paracompact and maps between them are C^∞. Submersions are always assumed to be surjective.

2.1 Connections and half-forms

We start by recalling some mathematical results that will be used throughout the paper. Given a manifold M we denote by $\tau_M\colon TM \to M$, the tangent projection and by $\tau_M^\vee\colon T^*M \to M$ the cotangent projection.

If $p\colon P \to M$ is a vector bundle and ∇ is a linear connection on it, then there is a natural splitting $TP = H^\nabla(P) \oplus VP$, where $VP = \ker Tp$ is the *vertical bundle*, or the space of vectors tangent to the fibres of p and $H^\nabla(P)$ is the *horizontal bundle*, whose fibres are isomorphic to the fibres of TM. Given a vector field $Y\colon P \to TP$, the connection allows us to compute its components with respect to the above direct sum splitting. If (x^α) are coordinates on M, (b_k) is a local basis of P with dual coordinates (y^k), $\partial/\partial x^\alpha = \partial_\alpha$ is a local basis of TM such that $\nabla_{\partial_\alpha} b_j = \Gamma^k_{\alpha j} b_k$ and $Y = Y^\alpha \partial_\alpha + Y^k b_k$, then the coordinate expression of the horizontal part $h^\nabla(Y)$, or *horizontalization of* Y is

$$h^\nabla(Y) = Y^\alpha(\partial_\alpha - \Gamma^k_{\alpha j} y^j b_k), \tag{1}$$

where $Y^\alpha \in \mathcal{C}^\infty(P)$. One can also lift a vector field $X\colon M \to TM$ to a horizontal vector field $h^\nabla(Y)$ that projects onto X. If $X = X^\alpha \partial_\alpha$, then we have the coordinate expression $h^\nabla(X) = X^\alpha(\partial_\alpha - \Gamma^k_{\alpha j} y^j b_k)$, with $X^\alpha \in \mathcal{C}^\infty(M)$. This theory is well-established in differential geometry and can be found, for instance, in [16].

It is well known that we can define the Lie derivative of a section ξ of the vector bundle $p\colon P \to M$ with respect to any vector field Y on P projectable on M; this gives a new section $L_Y\xi$ of P. If X is a vector field on M that has a lift X^P to a vector field on P, then we denote simply by $L_X\xi$ the Lie derivative of ξ with respect to X^P. There is a natural relationship between covariant and Lie derivatives induced by horizontalization. More precisely, if $Y\colon P \to TP$ is a vector field that projects onto a vector field $X\colon M \to TM$, then one has [16, p. 376]

$$L_{h^\nabla(Y)}\xi = \nabla_X \xi. \tag{2}$$

In particular, if we start from the horizontalization of a vector field on the base space $X\colon M \to TM$, then we obtain $L_{h^\nabla(X)}\xi = \nabla_X\xi$.

It is well known that a complex line bundle $K \to M$ whose first Chern class reduced modulo 2 vanishes, $0 = [c_1(K)] \in H^2(M, \mathbb{Z}/2)$, admits a square root bundle $\sqrt{K} \to M$ which is not necessarily unique in general. The Lie derivative of a section of this bundle with respect to a vector field X on M that has a lift X^K to K can be easily defined by requiring that Leibniz's rule holds. That is, if $\sqrt{\nu}\colon M \to K$ is a section such that $\sqrt{\nu} \otimes \sqrt{\nu} = \nu\colon M \to K$

then

$$L_X\sqrt{\nu}\otimes\sqrt{\nu}+\sqrt{\nu}\otimes L_X\sqrt{\nu}=L_X\nu. \tag{3}$$

In coordinates if $\sqrt{\nu}=f\sqrt{b}$, where $f\in\mathcal{C}^\infty(M)$ and $\sqrt{b}$ is a local basis of $\sqrt{K}$, we have

$$L_X\sqrt{\nu}=\left(L_Xf+\frac{f}{2}(L_Xb)_0\right)\sqrt{b}, \tag{4}$$

where $(L_Xb)_0\in\mathcal{C}^\infty(M)$ is defined by $L_Xb=(L_Xb)_0b$. For instance, if M is oriented, endowed with a Riemannian metric g and if $K=\wedge^n_{\mathbb{C}}T^*M$ is the complexification of the bundle of volume forms, then there exists a unique trivial square root bundle $\sqrt{\wedge^n_{\mathbb{C}}T^*M}\to M$ whose sections are called *half-forms*. There exists a distinguished nowhere-vanishing half-form $\sqrt{\nu_g}$ whose square is the oriented Riemannian volume element ν_g. Since every vector field X on M has a natural lift to the bundle of volume elements, one can define the Lie derivative of any half-form with respect to it and one gets $L_X\sqrt{\nu_g}=\frac{1}{2}\operatorname{div}_g(X)\sqrt{\nu_g}$, where $\operatorname{div}_g(X)$ is the divergence of X with respect to the Riemannian metric g. If (x^α) are oriented coordinates on M then $\sqrt{\nu_g}=\sqrt[4]{|g|}\sqrt{dx^1\wedge\cdots\wedge dx^n}$ and

$$L_X\sqrt{\nu_g}=\frac{1}{2}\frac{\partial_\alpha(X^\alpha\sqrt{|g|})}{\sqrt{|g|}}\sqrt{\nu_g}. \tag{5}$$

The same procedure can be used to define a linear connection in $\sqrt{K}\to M$ from a linear connection ∇ in $K\to M$. Indeed,

$$\nabla_X\sqrt{\nu}\otimes\sqrt{\nu}+\sqrt{\nu}\otimes\nabla_X\sqrt{\nu}=\nabla_X\nu. \tag{6}$$

This implies $\nabla_X\sqrt{\nu}=\left(L_Xf+\frac{f}{2}(\nabla_Xb)_0\right)\sqrt{b}$ with an obvious meaning of symbols. In particular the Levi-Civita connection ∇ on the bundle $\wedge^n_{\mathbb{C}}T^*M$ of volume elements induces a connection on the bundle of half-forms. If we take $b=\sqrt{dx^1\wedge\cdots\wedge dx^n}$ then we have

$$\nabla_X\sqrt{\nu}=\left(L_Xf-\frac{f}{2}X^\alpha\Gamma^\beta_{\alpha\beta}\right)\sqrt{dx^1\wedge\cdots\wedge dx^n}. \tag{7}$$

Remark 2.1. Since the oriented Riemannian volume element ν_g is parallel for the Levi-Civita connection, $\nabla\nu_g=0$, it follows that we also have $\nabla\sqrt{\nu_g}=0$. This means that both $\wedge^n_{\mathbb{C}}T^*M$, endowed with the Levi-Civita connection, and its square root $\sqrt{\wedge^n_{\mathbb{C}}T^*M}$, with the induced connection defined above, are flat line bundles.

We recall that if $F \to Q$ is a vector bundle and $\pi\colon M \to Q$ is a submersion then, given a connection $\overline{\nabla}$ on $F \to Q$, the pull-back connection $\pi^*\overline{\nabla}$ is defined by

$$\pi^*\overline{\nabla}_X \pi^* s = \pi^*(\overline{\nabla}_{\bar{X}} s) \tag{8}$$

where $s\colon Q \to F$ is a section of F and $X\colon M \to TM$ is a vector field that projects onto the vector field $\bar{X}\colon Q \to TQ$. The Christoffel symbols of $\pi^*\overline{\nabla}$ in the direction of $VM = \ker T\pi$ are zero.

On the other hand the pull-back bundle $\pi^*F \to M$ is endowed with a natural *partial connection*, or *π-relative connection* $\widehat{\nabla}^\pi$ in the direction of $VM \subset TM$, which is defined as $\widehat{\nabla}^\pi_V(f\,\pi^*s) = Vf \cdot \pi^*s$ for any $f \in C^\infty(M)$, $s \in \Gamma(Q,F)$ and any vertical vector field V. Obviously this partial connection is flat.

Any connection ∇ on a vector bundle $p\colon E \to M$ induces by restriction to the vertical bundle $V\pi$ a π-relative connection that we denote $\pi_{M/Q}(\nabla)$. A straightforward computation proves the following result.

Lemma 2.2. *Let $\pi\colon M \to Q$ be a submersion and let $q\colon F \to Q$ be a vector bundle. For any connection $\overline{\nabla}$ on F, the pullback connection $\pi^*(\overline{\nabla})$ on the pullback vector bundle $\pi^*F \to M$ induces the natural π-relative connection $\widehat{\nabla}^\pi$ on π^*F. That is*

$$\pi_{M/Q}(\pi^*(\overline{\nabla})) = \widehat{\nabla}^\pi.$$

2.2 Geometric quantization

In this section we recall the basic ingredients of GQ. Our main sources are [24, 31]. We will focus on the aspects that will play a key role for the results of the present paper. A *prequantum structure* on a symplectic manifold (M,ω) is a triple $\mathcal{Q} = (\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$, where $\mathcal{L} \to M$ is a complex line bundle endowed with a Hermitian metric $\langle\,,\rangle^{\mathcal{L}}$ and a Hermitian connection $\nabla^{\mathcal{L}}$ such that

$$R[\nabla^{\mathcal{L}}] = -i\frac{\omega}{\hbar}\,\mathrm{Id}_{\mathcal{L}},$$

where $R[\nabla^{\mathcal{L}}]$ is the curvature of $\nabla^{\mathcal{L}}$ and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant. It can be proved that the symplectic manifold (M,ω) admits a prequantum structure if and only if the cohomology class $[\frac{\omega}{h}]$ is integral, that is

$$[\frac{\omega}{h}] \in i(H^2(M,\mathbb{Z})) \subset H^2(M,\mathbb{R}),$$

where i is the map induced in cohomology by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. If the above condition is fulfilled, then (M,ω) is said to be *quantizable*.

One says that two Hermitian line bundles $(L, \langle\,,\rangle)$, $(L', \langle\,,\rangle')$ (Hermitian line bundles with connection $(L, \langle\,,\rangle, \nabla)$, $(L', \langle\,,\rangle', \nabla'))$ are equivalent Hermitian bundles (with connection) if there exists an isomorphism of line bundles $\phi\colon L \to L'$ such that $\phi^*\langle\,,\rangle' = \langle\,,\rangle$ (and $\phi^*\nabla' = \nabla$). The set of equivalence classes of prequantum structures is parametrized by $H^1(M, U(1)) \simeq \mathrm{Hom}(\pi_1(M), U(1))$. Therefore, a symplectic manifold M can admit non-equivalent quantizations only if it is not simply connected.

Definition 2.3. We say that a prequantization structure $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ on (M, ω) is *reducible* over a submersion $\pi\colon M \to Q$ if there exists a Hermitian line bundle (L, ∇) on Q such that $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}})$ is equivalent to $(\pi^*L, \pi^*\langle\,,\rangle)$. In this case we say that $(L, \langle\,,\rangle, \nabla)$ is *a reduction* of the prequantization structure.

Let us stress that $\nabla^{\mathcal{L}}$ cannot be equivalent to $\pi^*\nabla$, for a Hermitian connection ∇ on $L \to Q$, since $i\hbar\pi^*R[\nabla]$ is not a symplectic form because the vertical vector fields of the submersion $\pi\colon M \to Q$ belong to its radical. We have the following result.

Proposition 2.4. *If a prequantization structure $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ on (M, ω) is reducible over a submersion $\pi\colon M \to Q$ with reduction $(L, \langle\,,\rangle, \nabla)$, then there exists a 1-form $\alpha \in \Omega^1(M)$ such that*

$$\omega = \pi^*\bar{\omega} + d\alpha, \tag{9}$$

where $\bar{\omega} = i\hbar R[\nabla] \in \Omega^2(Q)$. Moreover, $d\alpha$ is a symplectic structure on M.

Proof. Since $\mathcal{L} \simeq \pi^*L$ one has $\nabla^{\mathcal{L}} - \pi^*\nabla = -\frac{i}{\hbar}\alpha$ for a certain 1-form $\alpha \in \Omega(M)$. The first claim follows by computing the curvatures of these connections bearing in mind the quantization condition. Since $d\alpha = \omega - \pi^*\bar{\omega}$, and ω is symplectic, it follows immediately that $d\alpha$ is a symplectic form. □

Note that when a symplectic manifold (M, ω) admits a prequantization structure that is reducible over a submersion $\pi\colon M \to Q$, then M is not compact. Indeed, by Proposition 2.4, there exists a 1-form $\alpha \in \Omega^1(M)$ such that $(M, d\alpha)$ is a symplectic manifold, and it is well known that an exact symplectic manifold is always not compact.

One has the following descent result, whose proof can be found in [29].

Theorem 2.5. *Let $\pi\colon M \to Q$ be a surjective submersion and consider a vector bundle $p\colon E \to M$ that admits an absolute parallelism $\widehat{\nabla}$ relative to π. Then, there exists a vector bundle $q\colon F \to Q$, unique up to isomorphism over Q, and a vector bundle isomorphism $\varphi\colon E \xrightarrow{\sim} \pi^*F$ that transforms $\widehat{\nabla}$ into the natural flat relative connection $\widehat{\nabla}^{\pi}$ defined on the pullback vector bundle $\pi^*F \to M$.*

This together with a straightforward computation, see [29], gives the following result

Corollary 2.6. *A prequantization structure* $(\mathcal{L}, \langle\, , \rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ *on a symplectic manifold* (M, ω) *is reducible over a submersion* $\pi\colon M \to Q$ *if and only if* $\mathcal{L}$ *admits a* π*-relative connection* $\widehat{\nabla}$ *which is an absolute parallelism relative to* π*; that is, if and only if the holonomy groups of* $\widehat{\nabla}$ *along the fibers of* π *are trivial.*

An example is provided by the cotangent bundle $\tau_Q^\vee\colon T^*Q \to Q$ of a manifold Q. Taking into account Corollary 2.6, the topological triviality of the fibers of $\tau_Q^\vee$ implies that every prequantum structure on the symplectic manifold $(T^*Q, \omega_0 = d\theta)$, where θ is the Liouville form, is reducible over $\tau_Q^\vee$.

A *real polarization* on a symplectic manifold (M, ω) is an involutive Lagrangian distribution $P \subset TM$. A *complex polarization* on a symplectic manifold (M, ω) is an involutive Lagrangian distribution P of the complexified tangent bundle of M, $P \subset T_{\mathbb{C}}M$ that fulfills additional hypothesis [24, 31].

A polarization is needed in order to define the Hilbert space of quantum states and also to select the class of observables to be quantized.

A *quantization* of a symplectic mechanical system consists of a prequantum structure $(\mathcal{L}, \langle\, , \rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ and a polarization P. After such a choice, a Hilbert space $\mathcal{H}_P$ can be defined as follows. Sections which are covariantly constant in the direction of P represent quantum states. In the particular case $M = T^*Q$, the simplest real polarization is the vertical one, $P = VT^*Q \simeq T^*Q \times_Q T^*Q$, and the corresponding quantization is called the *Schrödinger representation.* Notice however that the scalar product $(\psi_1, \psi_2) = \int_M \langle \psi_1, \psi_2 \rangle \operatorname{vol}_\omega$ of two parallel sections ψ_1, ψ_2 might diverge.

The bundle of *half-forms* $N_P^{1/2} = \sqrt{\wedge_{\mathbb{C}}^n P} \to M$ was introduced in order to avoid the above problem. Such bundle does only exist provided that the square of the first Stiefel-Whitney class of P vanishes, $w_1(P)^2 = 0$, [24, 31]. This amounts to the possibility of making a consistent choice for the square root of the transition functions of the line bundle $\wedge_{\mathbb{C}}^n P \to M$ to allow for consistent coordinate changes in $N_P^{1/2}$. This is equivalent to the datum of a *metaplectic structure* for the symplectic manifold (M, ω) [24, 31].

In order to be able to achieve our goals, it is of great importance the fact that $N_P^{1/2}$ admits a partial flat connection $\widehat{\nabla}^B$ in the direction of P; $\widehat{\nabla}^B$ is called the *Bott connection* (see Subsection 3.1). Indeed, P admits a local basis $\xi_1, \ldots, \xi_n$ formed by Hamiltonian vector fields. Such fields commute as they take their values in a Lagrangian subspace. Moreover, it can be proved [24] that the existence of a metaplectic structure implies that any two such local bases are connected by a change of coordinates which is constant along the polarization. This means that if $\nu = \nu_0\sqrt{\xi_1 \wedge \cdots \wedge \xi_n}$, where $\nu_0 \in \mathcal{C}^\infty(M)$, and $Y \colon M \to P$ then

$$\widehat{\nabla}^B_Y \nu = (L_Y \nu_0)\sqrt{\xi_1 \wedge \cdots \wedge \xi_n}, \tag{10}$$

is well defined. It can be verified that $\widehat{\nabla}^B$ is flat [24].

One can then consider *polarized sections*, that is sections of the prequantization line bundle twisted by half-forms, $\mathcal{L} \otimes N_P^{1/2} \to M$, which are covariantly constant in the direction of P with respect to the partial connection on the tensor product obtained from $\nabla^{\mathcal{L}}$ and $\widehat{\nabla}^B$. Such sections can be integrated over the space of leaves of the polarization M/P. The *Hilbert space* $\mathcal{H}_P$ *of quantum states* is the L^2-completion of the space of square-integrable sections ψ of $\mathcal{L} \otimes N_P^{1/2}$ which are covariantly constant along P.

The space of (straightforwardly) *quantizable observables* $\mathcal{O}_P \subset \mathcal{C}^\infty(M)$ consists of functions $f \in \mathcal{C}^\infty(M)$ whose Hamiltonian vector field X_f preserves the polarization P; in other words, $[X_f, P] \subset P$. A choice of polarization corresponds to what in the Physics literature is called a choice of *representation*.

At this point we need to define Dirac's correspondence between the space of quantizable observables $\mathcal{O}_P$ and quantum operators acting on the Hilbert space $\mathcal{H}$. Every classical observable $f \in \mathcal{C}^\infty(M)$ yields a Hamiltonian vector field X_f on M. Such a vector field can always be lifted to a vector field on the prequantum line bundle $\mathcal{L} \to M$ that operates on its sections $\psi \colon M \to \mathcal{L}$. However, we also need to operate on the factor $N_P^{1/2}$ and the Lie derivative can be defined only if we use vector fields that preserve P.

Every quantizable observable $f \in \mathcal{O}_P$ corresponds to a symmetric operator $\hat{f} \colon \mathcal{H}_P \to \mathcal{H}_P$ and in many important cases one can prove that it is self-adjoint. The operator $\hat{f}$ is defined as follows. The Hamiltonian vector field corresponding to f can be lifted to a complex Hermitian vector field $X_f^{\mathcal{L}}$ on $\mathcal{L}$ that preserves the connection $\nabla^{\mathcal{L}}$, in the sense that $L_{X_f^{\mathcal{L}}}\nabla^{\mathcal{L}} = 0$, regarding the connection as a tensor field, see [16]. The expression of $X_f^{\mathcal{L}}$ is

$$X_f^{\mathcal{L}} = h^{\nabla^{\mathcal{L}}}(X_f) + i\frac{f}{\hbar}E^{\mathcal{L}}, \tag{11}$$

where $E^{\mathcal{L}}$ is the Euler vector field of $\mathcal{L}$, which is the vertical vector field such that for every $z \in \mathcal{L}$ one has $E_z^{\mathcal{L}} = (z, z)$ via the natural isomorphism $V\mathcal{L} = \mathcal{L} \times_M \mathcal{L}$. The vector field $X_f^{\mathcal{L}}$ acts on sections $s \colon M \to \mathcal{L}$ through its flow $\phi_t^{f,\mathcal{L}}$ as $\phi_{t,*}^{f,\mathcal{L}}(s) = \phi_{-t}^{f,\mathcal{L}} \circ s \circ \phi_t^f$, where ϕ_t^f is the flow of X_f. The generalized Lie derivative (see *eg* [16]) of the section s is then defined via a limit $t \to 0$; one has [16, p. 378]

$$i\hbar L_{X_f^{\mathcal{L}}} s = i\hbar \nabla^{\mathcal{L}}_{X_f} s - fs. \tag{12}$$

On decomposable sections $\psi = s \otimes \sqrt{\nu} \colon M \to \mathcal{L} \otimes N_P^{1/2}$ which are covariantly constant along the direction of P (*ie, wave functions*) we can define the action of *quantum operators*:

$$\hat{f}(\psi) = i\hbar L_{\tilde{X}_f}\psi = i\hbar L_{X_f^{\mathcal{L}}}\, s \otimes \sqrt{\nu} + i\hbar s \otimes L_{X_f}\sqrt{\nu}. \tag{13}$$

Here $\tilde{X}_f$ is the vector field induced on $\mathcal{L} \otimes N_P^{1/2}$ by $X_f^{\mathcal{L}}$ and X_f. Note that the action of X_f on sections of $N_P^{1/2} \to M$ is well defined since X_f preserves P.

One can prove that the well-known relation between classical and quantum commutators holds:

$$[\hat{f}_1, \hat{f}_2] = i\hbar\widehat{\{f_1, f_2\}}. \tag{14}$$

We stress that not all physically or mathematically interesting classical observables can be quantized. For instance, in the case of the Schrödinger representation for a natural mechanical system $f \in \mathcal{O}_P$ if and only if f is linear in the momenta, [24, 31]. More intrinsically, if $f \in \mathcal{O}_P$ then one has $f = (\tau_Q^{\vee})^* h + \theta(X^{\vee})$, where $h \in C^{\infty}(Q)$ and $X^{\vee}$ is the cotangent lift of a vector field X on Q. This implies that the Hamiltonian, which is the total energy function $H = K_g + V$, where $K_g(q^i, p_i) = \frac{1}{2} g^{ij} p_i p_j$ and $V \in \mathcal{C}^{\infty}(Q)$, is not quantizable.

The quantization of energy is usually achieved in the framework of the BKS theory of quantization of observables that do not preserve the polarization. If H is such an observable, we denote by ϕ_t^H the flow of X_H; we have $\phi_{t,*}^H(P) \neq P$. We assume that for $0 < t < \epsilon$ the two polarizations $\phi_{t,*}^H(P)$ and P are *strongly admissible*: this means that they must be transverse and that some technical hypothesis have to be verified [24]. Let us denote by $\mathcal{H}_t = \mathcal{H}_{\phi_{t,*}^H(P)}$ the Hilbert space of quantum observables corresponding to the polarization $\phi_{t,*}^H(P)$, in particular one has $\mathcal{H}_0 = \mathcal{H}_P$. Then a map $\mathcal{H}_{0t} \colon \mathcal{H}_t \to \mathcal{H}_0$, the BKS kernel, is defined for every $\psi_2 \in \mathcal{H}_t$ by

$$(\psi_1, \mathcal{H}_{0t}(\psi_2)) = \int_{M/P} \langle \psi_1, \psi_2 \rangle \quad \forall \psi_1 \in \mathcal{H}_0. \tag{15}$$

The integral on the right-hand side is the BKS pairing, see [24] for details. The quantum operator associated to H is defined, for $\psi = s \otimes \sqrt{\nu} \in \mathcal{H}_0$, as

$$\hat{H}(\psi) = i\hbar \frac{d}{dt} \mathcal{H}_{0t} \circ \phi_{t,*}^{H,\mathcal{L}}(s) \otimes \phi_{t,*}^H(\sqrt{\nu})\big|_{t=0}, \tag{16}$$

In the Schrödinger representation of a natural mechanical system, if (q^i, p_i) are coordinates on T^*Q we have the following quantum operators:

$$\widehat{q^i}(\psi) = i\hbar q^i s \otimes \sqrt{\nu_g}, \tag{17}$$

$$\widehat{f^i p_i}(\psi) = -i\hbar \left(f^i \frac{\partial}{\partial q^i}\psi^0 + \frac{1}{2}\frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}}\psi^0 \right) b_0 \otimes \sqrt{\nu_g}, \tag{18}$$

$$\widehat{H} = \left(-\frac{\hbar^2}{2}\Big(\Delta(s) - \frac{r_g}{6}s\Big) + V s \right) \otimes \sqrt{\nu_g}. \tag{19}$$

In the above formulae, f^i is a function on Q and $\psi = s \otimes \sqrt{\nu_g}$ denotes an element of the Hilbert space, where $s = \psi^0 b_0$ is a section of $L \to Q$ which is expressed through a local basis b_0. Moreover, Δ is the Bochner Laplacian of ∇ and r_g is the scalar curvature of the Riemannian metric g. The above formulae can be found, for example, in [24].

3 The geometry of polarizations and the modified quantization method

In this section we want to show how the existence of a partial flat connection on a vector bundle is related to infinitesimal and global descent properties of the vector bundle. This is relevant for geometric quantization since this is exactly the situation that one has when a prequantum line bundle and a polarization on a symplectic manifold are given.

3.1 The geometry of polarizations

Let (M, ω) be a symplectic manifold. For any real polarization P of (M, ω) we denote by $N(P) = TM/P$ the normal bundle of the polarization and $\pi^N : TM \to N(P)$ is the natural projection. It is well known that $N(P)$ has a natural partial connection relative to P, we simply say a P-relative connection, defined by

$$\widehat{\nabla}^B_V(\pi^N(D)) = \pi^N([V, D]), \quad \text{for any } V \in \Gamma(M, P), D \in \Gamma(M, TM).$$

The P-relative connection $\widehat{\nabla}^B$ is called the Bott connection of (M, P) and one easily checks that it is flat. This connection induces a connection on $N(P)^*$; the symplectic form gives an isomorphism $P \xrightarrow{\sim} N(P)^*$ by which we define a flat P-relative connection on the vector bundle $P \to M$ that we also denote by $\widehat{\nabla}^B$.

We recall a definition which is of key importance to our aims.

Definition 3.1. A real polarization $P \hookrightarrow TM$ is called *reducible* if there exists a submersion $\pi \colon M \to Q$ such that P coincides with the vertical bundle $V(M/Q) \to M$ of the submersion π.

Hence $\pi \colon M \to Q$ is a Lagrangian submersion and there is an equivalence between reducible real polarizations on M and Lagrangian submersions whose total space is M.

If P is reducible over a submersion π, then P-relative connections are π-relative connections (see Subsection 2.1).

Proposition 3.2. *Let (M, ω) be a symplectic manifold and let $P \to M$ be a reducible real polarization with associated submersion $\pi \colon M \to Q$. Then*

1. *the symplectic form yields an isomorphism $P \xrightarrow{\sim} \pi^* T^* Q$ that maps $\widehat{\nabla}^B$ into $\widehat{\nabla}^\pi$;*
2. *The relative connection induced on $P \to M$ by the pullback of any connection ∇^Q on the cotangent bundle $T^*Q \to Q$ coincides with the flat π-relative connection $\widehat{\nabla}^\pi$ induced by the Bott connection, that is*
$$\pi_{M/Q}(\pi^*(\nabla^Q)) = \widehat{\nabla}^\pi.$$

For any real polarization $P \hookrightarrow TM$ on a symplectic manifold of dimension $2n$ the bundle of P-transversal volume forms is the complex line bundle $\Lambda^n_{\mathbb{C}} N(P)^* \to M$. The symplectic form yields an isomorphism of complex line bundles $N_P = \Lambda^n_{\mathbb{C}} P \xrightarrow{\sim} \Lambda^n_{\mathbb{C}} N(P)^*$, so that we have a flat P-relative connection on the line bundle $N_P \to M$. One says that the polarization P admits a metalinear structure if the complex line bundle $N_P \to M$ admits a square root $N_P^{1/2} \to M$, which is called the *bundle of half-forms* transverse to P; it is endowed by a flat P-relative connection $\widehat{\nabla}^B$.

If we consider the Hermitian connection $\nabla^{\mathcal{L}}$ of a quantum structure for the symplectic manifold (M, ω), then the associated P-relative connection $\widehat{\nabla}^{\mathcal{L}} = \pi_{M/P}(\nabla^{\mathcal{L}})$ is flat. Indeed one has
$$R[\widehat{\nabla}] = \pi_{M/P}(R[\nabla^{\mathcal{L}}]) = \pi_{M/P}(-i\frac{\omega}{\hbar} \otimes \mathrm{Id}_{\mathcal{L}}) = -i\frac{\pi_{M/P}(\omega)}{\hbar} \otimes \mathrm{Id}_{\mathcal{L}} = 0,$$
since P is Lagrangian.

Therefore, the line bundle $\mathcal{L} \otimes N_P^{1/2} \to M$ can be endowed with the twisted flat P-relative connection $\widehat{\nabla}^P = \widehat{\nabla}^{\mathcal{L}} \otimes 1 + 1 \otimes \widehat{\nabla}^B$, where $\widehat{\nabla}^{\mathcal{L}}$ is the P-relative connection induced by $\nabla^{\mathcal{L}}$. The space $\Gamma_P(M, \mathcal{L} \otimes N_P^{1/2})$ of polarized sections of $\mathcal{L}$ is the space of sections of the line bundle $\mathcal{L} \otimes N_P^{1/2} \to M$ that are covariantly constant for the P-relative connection $\widehat{\nabla}^P$:

$$\Gamma_P(M, \mathcal{L} \otimes N_P^{1/2}) = \{s \in \Gamma(M, \mathcal{L} \otimes N_P^{1/2}) : \widehat{\nabla}^P_V s = 0, \text{for every } V \in \Gamma(M, P)\}.$$

It is well known that a complex line bundle $K \to M$ admits a square root if, and only if, its first Chern class $c_1(K) \in H^2(M; \mathbb{Z})$ is even. If P is a real polarization which is reducible over the submersion $\pi \colon M \to Q$, then the symplectic isomorphism $P \to \pi^* T^* Q$ implies that $N_P = \pi^* K_Q$, where $K_Q = \Lambda^n_{\mathbb{C}} T^* Q \to Q$ is the determinant of the complexified cotangent bundle of Q. Hence P admits a metalinear structure if and only if K_Q admits a square root $K_Q^{1/2} \to Q$. In particular, if Q is orientable it follows that P admits a metalinear structure.

In the reducible case, if $\pi^* K_Q^{1/2} \to M$ is a metalinear structure, then any connection ∇^Q on $K_Q^{1/2} \to Q$ endows the line bundle $\mathcal{L} \otimes \pi^* K_Q^{1/2} \to M$ with the connection $\widetilde{\nabla} = \nabla^{\mathcal{L}} \otimes 1 + 1 \otimes \pi^*(\nabla^Q)$. Taking into account Proposition 3.2 we immediately get the following key result.

Theorem 3.3. *Let P be a polarization which is reducible over the submersion $\pi \colon M \to Q$ and let $K_Q^{1/2} \to Q$ be a metalinear structure. Then, every linear connection ∇^Q on $K_Q^{1/2} \to Q$ yields the same space of polarized sections*

$$\Gamma_P(M, \mathcal{L} \otimes N_P^{1/2}) = \{s \in \Gamma(M, \mathcal{L} \otimes N_P^{1/2}) : \widetilde{\nabla}_V s = 0, \forall\, V \in \Gamma(M, P)\}. \tag{20}$$

Proof. For any connection ∇^Q on $K_Q^{1/2} \to Q$, the π-relative connection on $\mathcal{L} \otimes \pi^* K_Q^{1/2} \to M$ induced by the twisted connection $\nabla^{\mathcal{L}} \otimes 1 + 1 \otimes \pi^*(\nabla^Q)$ coincides with the flat π-relative connection $\widehat{\nabla}^P = \widehat{\nabla}^{\mathcal{L}} \otimes 1 + 1 \otimes \widehat{\nabla}^B$. □

Corollary 3.4. *If, in addition to the above hypotheses, the prequantum structure $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ is reducible over the submersion $\pi \colon M \to Q$ and $(L, \langle\,,\rangle, \nabla)$ is a reduction, then the space of polarized sections turns out to be*

$$\Gamma_P(M, \mathcal{L} \otimes N_P^{1/2}) = \Gamma(Q, L \otimes K_Q^{1/2}). \tag{21}$$

Proof. Since the line bundle $\mathcal{L} \otimes N_P^{1/2} = \pi^*(L \otimes K_Q^{1/2})$ is a pullback, its space of global sections is given by

$$\Gamma(M, \mathcal{L} \otimes N_P^{1/2}) = \mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(Q)} \Gamma(Q, L \otimes K_Q^{1/2}).$$

Taking into account now that both partial covariant derivatives $\nabla^{\mathcal{L}}$ and $\pi^*(\nabla)$ are a pullback, we immediately obtain the required identification. □

Remark 3.5. As every connection $\widetilde{\nabla}$ in the family of connections $\widetilde{\nabla} = \nabla^{\mathcal{L}} \otimes 1 + 1 \otimes \pi^*(\nabla^Q)$ parameterized by the connections ∇^Q of $K_Q^{1/2}$, yields the same space of polarized sections $\Gamma_P(M, \mathcal{L} \otimes N_P^{1/2})$, we can quantize observables by using the connection $\widetilde{\nabla}$ instead of the π-relative connection $\widehat{\nabla}^P$. While the space of polarized sections does not change, we will see that the quantization of classical observables changes in a substantial way.

3.2 The modified quantization method

In this Subsection we propose a modification of the standard GQ scheme which will make a fundamental use of the geometric structure that is present on the space of polarized sections. More precisely, we will show that while we will use the same Hilbert space as GQ, we will determine quantizable observables in the more general situation of a reducible quantum structure.

Then, we make the key remark that the representation of classical observables as quantum operators on that space *is not unique*. More precisely, the representation can be modified with respect to the standard prescription of GQ (see Subsection 2.2) by choosing any connection ∇^Q on the bundle of half-forms $K_Q^{1/2} \to Q$. Indeed, a connection on $K_Q^{1/2} \to Q$ is usually defined in an natural way from a linear connection on $\tau_Q^\vee \colon T^*Q \to Q$, so we will assume that a linear connection ∇^Q on $\tau_Q^\vee$ is given.

The most important example that we have in mind is the geometric quantization of the cotangent bundle T^*Q of an orientable Riemannian manifold (Q, g). This will be described in next section. However, in principle other examples are possible and thus it is worth to describe the modified quantization procedure in a more general situation.

The main idea is that the action of Hamiltonian vector fields of quantizable observables on half-forms must be defined through ∇^Q-preserving flows.

3.2.1 Reducible quantum structures and Liouville manifolds

Here we will show that there is a close relationship between reducible quantum structures and Liouville manifolds. We start by giving the necessary definitions.

Definition 3.6. Let (M, ω) be a symplectic manifold. We say that a quantum structure $\mathcal{Q} = (\mathcal{L}, \langle\, ,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ is reducible over a submersion $\pi \colon M \to Q$ if both the prequantum structure $(\mathcal{L}, \langle\, ,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}})$ and the polarization P are reducible over π and the prequantum structure has a reduction $(L, \langle\, ,\rangle, \nabla)$ such that $\nabla^{\mathcal{L}}$ and $\pi^*\nabla$ have isomorphic P-polarized sections. In this case we say that $(L, \langle\, ,\rangle, \nabla)$ is a reduction of the quantum structure $\mathcal{Q}$.

Proposition 3.7. *If a quantization structure* $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ *is reducible over a submersion* $\pi\colon M \to Q$ *and* $(L, \langle\,,\rangle, \nabla)$ *is a reduction, then there exists a* 1*-form* $\alpha \in \Omega^1(M)$ *vanishing on* $V(M/Q)$ *and a closed* 2*-form* $\bar{\omega} \in \Omega^2(Q)$ *such that*

$$\omega = \pi^*\bar{\omega} + d\alpha. \tag{22}$$

Moreover $\bar{\omega} = i\hbar R[\nabla]$.

Proof. As in the proof of Proposition 2.4 one has $\nabla^{\mathcal{L}} - \pi^*\nabla = -\frac{i}{\hbar}\alpha$ for a certain 1-form $\alpha \in \Omega(M)$. Given a vertical vector field $V \in \Gamma(M, V(M/Q)) = \Gamma(M, P)$ one has

$$\nabla^{\mathcal{L}}_V s - \pi^*\nabla_V s = -\frac{i}{\hbar}\alpha(V)s$$

and therefore, $\nabla^{\mathcal{L}}$ and $\pi^*\nabla$ have the same P-polarized sections if and only if $\alpha(V) = 0$. The last claim follows from Proposition 2.4. □

According to (see [30, pag. 234-235]), a *reducible Liouville manifold* is a pair $((M,\omega), P)$ formed by a symplectic manifold (M,ω) and a real polarization P which is reducible over a submersion $\pi\colon M \to Q$ such that there exists a 1-form $\alpha \in \Omega^1(M)$ that vanishes on P and $\omega - d\alpha = \pi^*\bar{\omega}$ for some $\omega \in \Omega^2(Q)$. Any such $\alpha \in \Omega^1(Q)$ is called a (generalized) *Liouville* 1*-form* for the Liouville manifold.

We recall that a polarization on a symplectic manifold yields a Liouville manifold if and only if a certain characteristic class $\mathcal{E}(P) \in H^1(M, \mathcal{V}_P)$, the Euler obstruction of P, vanishes. The class is defined in terms of the π-relative local system $\mathcal{V}_P := V_{\widehat{\nabla}^B}(M/Q)$ defined by the parallel sections of $V(M/Q)$ with respect to the Bott connection $\widehat{\nabla}^B$. In any reducible Liouville manifold $((M,\omega), P)$ with Liouville form α it holds that $(M, d\alpha)$ is a symplectic manifold. Therefore, any Liouville manifold is not compact.

After the above considerations, Proposition 3.7 can be reformulated as follows.

Proposition 3.8. *If a quantization* $\mathcal{Q} = (\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ *is reducible over a submersion* $\pi\colon M \to Q$ *and* $(L, \langle\,,\rangle, \nabla)$ *is a reduction, then* $((M,\omega), \alpha, P)$ *is a Liouville manifold, where* α *is determined by* $\nabla^{\mathcal{L}} - \pi^*\nabla = -\frac{i}{\hbar}\alpha$.

We also have the following result.

Theorem 3.9. *A quantization* $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ *is reducible over a submersion* $\pi\colon M \to Q$ *if and only if the* π*-relative connection* $\pi_{M/Q}(\nabla^{\mathcal{L}})$ *is an absolute parallelism relative to* π*; that is, if and only if the holonomy groups of* $\pi_{M/Q}(\nabla^{\mathcal{L}})$ *along the fibers of* π *are trivial.*

Proof. If we have a reduction $(L, \langle\,,\rangle, \nabla)$, then $\nabla^{\mathcal{L}} - \pi^*\nabla = -\frac{i}{\hbar}\alpha$ and therefore by Proposition 3.7 one has

$$\pi_{M/Q}(\nabla^{\mathcal{L}}) - \pi_{M/Q}(\pi^*\nabla) = -\frac{i}{\hbar}\pi_{M/Q}(\alpha) = 0.$$

Now taking into account Lemma 2.2 we get $\pi_{M/Q}(\nabla^{\mathcal{L}}) = \widehat{\nabla}^{\pi}$ and therefore $\pi_{M/Q}(\nabla^{\mathcal{L}})$ is an absolute parallelism relative to π, see [29]. For the other implication suppose that $\pi_{M/Q}(\nabla^{\mathcal{L}})$ is an absolute parallelism on $\mathcal{L}$, then it follows from Theorem 2.5 that there exist a line bundle $L \to Q$ such that $\mathcal{L} \simeq \pi^*L$ and $\pi_{M/Q}(\nabla^{\mathcal{L}}) = \widehat{\nabla}^{\pi}$ and again by Lemma 2.2 we have $\pi_{M/Q}(\nabla^{\mathcal{L}}) = \pi_{M/Q}(\pi^*\nabla)$ which shows that the 1-form $\nabla^{\mathcal{L}} - \pi^*\nabla$ vanishes on $V(M/Q)$. Taking into account Proposition 3.7 this finishes the proof. □

From now on we assume that $(\mathcal{L}, \langle\,,\rangle^{\mathcal{L}}, \nabla^{\mathcal{L}}, P)$ *is a quantum structure reducible over a submersion* $\pi\colon M \to Q$, with a reduction $(L, \langle\,,\rangle, \nabla)$ and α is a Liouville 1-form.

3.2.2 Quantizable observables

As we have seen in Corollary 3.4, in the reducible case the space of polarized sections is defined in a unique way. This allows us to define the Hilbert space of the quantum theory.

From now on *we assume that* P *admits a metalinear structure with a line bundle of half-forms* $K_Q^{1/2} \to Q$. The Hilbert space of the quantum theory is, as in standard GQ, built out of the space of polarized sections. According to Corollary 3.4, *the Hilbert space of quantum states* $\mathcal{H}_P$ *is the* L^2*-completion of compactly supported sections in* $\Gamma(Q, L \otimes K_Q^{1/2})$.

We recall that in standard GQ, quantizable observables are determined by requiring that they should preserve the space of polarized sections. Since in our scheme this space is the same as in GQ, modified quantizable observables must satisfy the same condition as in GQ: they are functions $f \in \mathcal{O}_P \subset \mathcal{C}^\infty(M)$ such that $[X_f, P] \subset P$. We recall that $\mathcal{O}_P$ is a Lie subalgebra of the Poisson algebra $(C^\infty(M), \{\ ,\ \})$.

Let $((M, \omega), P)$ be a reducible Liouville manifold with associated submersion $\pi\colon M \to Q$. We denote the space of vector fields on Q and M by $\mathfrak{X}(Q)$ and $\mathfrak{X}(M)$, respectively. We observe that for any Liouville 1-form α there is a natural lift mapping $\ell_\alpha : \mathfrak{X}(Q) \to \mathfrak{X}(M)$ such that given $X \in \mathfrak{X}(Q)$, its lift is the unique vector field $X^{\ell_\alpha} \in \mathfrak{X}(M)$ that projects to X and satisfies $i_{X^{\ell_\alpha}} d\alpha = -d(\alpha(X^{\ell_\alpha}))$. In the case of a cotangent bundle $(M = T^*Q, \omega_0)$, this gives the well known cotangent lift.

Taking into account that any point of M has a coordinate neighbourhood with fibered coordinates $\{q^i, p_i\}_{i=1}^n$ for the submersion $\pi\colon M \to Q$ such that $\alpha = p_i dq^i$, see [30, Proposition 3.5], it is easy to prove the following:

Theorem 3.10. *Let $((M,\omega), P)$ be a reducible Liouville manifold with associated submersion $\pi\colon M \to Q$. If α is a Liouville 1-form α with associated lift $\ell_\alpha : \mathfrak{X}(Q) \to \mathfrak{X}(M)$, then one has*

$$\mathcal{O}_P = \{f \in C^\infty(M)\colon f = \pi^* f^0 + \alpha(X^{\ell_\alpha}), f^0 \in C^\infty(Q), X \in \mathfrak{X}(Q)\}.$$

Moreover, for any $f^0 \in C^\infty(Q)$ and $X \in \mathfrak{X}(Q)$, the associated Hamiltonian vector fields $X_{\pi^* f^0}$, $X_{\alpha(X^{\ell_\alpha})}$ on (M,ω) verify that

$$X_{\pi^* f^0}, \text{ and } X_{\alpha(X^{\ell_\alpha})} - X^{\ell_\alpha} \text{ are } \pi\text{-vertical.}$$

Therefore, for any $f = \pi^* f^0 + \alpha(X^{\ell_\alpha}) \in \mathcal{O}_P$ the Hamiltonian vector field $X_f \in \mathfrak{X}(M)$ is projectable and its projection is $X \in \mathfrak{X}(Q)$.

In the reducible case the space of quantizable observables $\mathcal{O}_P$ can be identified with $C^\infty(Q) \times \mathfrak{X}(Q)$ and thus we can transfer to it the Lie algebra structure of $\mathcal{O}_P$. An easy computation proves the following result.

Proposition 3.11. *Let $((M,\omega), P)$ be a reducible Liouville manifold with associated submersion $\pi\colon M \to Q$ and Liouville form α. Let $\bar\omega \in \Omega^2(Q)$ is a closed form such that $\omega = d\alpha + \pi^*\bar\omega$, then the Lie algebra structure of $\mathcal{O}_P$ is given by*

$$\{f_1, f_2\} = \{(f_1^0, X_1), (f_2^0, X_2)\} = (X_1 f_2^0 - X_2 f_1^0 - \bar\omega(X_1, X_2), [X_1, X_2]),$$

for any $f_1 = (f_1^0, X_1), f_2 = (f_2^0, X_2) \in C^\infty(Q) \times \mathfrak{X}(Q)$.

3.2.3 Quantization of classical observables

Here, quantum operators are modified according to our guiding idea. For any quantizable function $f \in \mathcal{O}_P$ we denote by $\tilde{X}_f^{\nabla^Q}$ the vector field on the tensor product bundle $\mathcal{L} \otimes N_P^{1/2}$ naturally induced by the vector fields: $X_f^{\mathcal{L}} = h^{\nabla^{\mathcal{L}}}(X_f) + i\frac{f}{\hbar}E^{\mathcal{L}}$, defined in (11), and $\widehat{X}_f^{\nabla^Q} := h^{\pi^*(\nabla^Q)}(X_f)$.

Definition 3.12 (Modified geometric quantization)**.** A quantizable observable $f \in \mathcal{O}_P$ yields a quantum operator

$$\widehat{f}\colon \Gamma(M, \mathcal{L} \otimes N_P^{1/2}) \to \Gamma(M, \mathcal{L} \otimes N_P^{1/2}) \tag{23}$$

that on a decomposable half-form $\tilde\psi = s' \otimes \sqrt{\nu} \in \Gamma(M, \mathcal{L} \otimes N_P^{1/2})$ acts by

$$\widehat{f}(\tilde\psi) = i\hbar L_{\tilde{X}_f^{\nabla^Q}} \tilde\psi = i\hbar (L_{X_f^{\mathcal{L}}}\, s' \otimes \sqrt{\nu} + s' \otimes L_{\widehat{X}_f^{\nabla^Q}}\, \pi^* \sqrt{\nu}). \tag{24}$$

In the above definition we operate by projectable vector fields. Indeed, given $f \in \mathcal{O}_P$ such that $f = \pi^* f^0 + \alpha(X^{\ell_\alpha})$ with $f^0 \in C^\infty(Q)$ and $X \in \mathfrak{X}(Q)$, one has

$$\begin{aligned} X_f^{\mathcal{L}} &= h^{\nabla^{\mathcal{L}}}(X_f) + i\frac{f}{\hbar}E^{\mathcal{L}} = h^{\pi^*\nabla - \frac{i}{\hbar}\alpha}(X_f) + i\frac{f}{\hbar}E^{\mathcal{L}} = \\ &= h^{\pi^*\nabla}(X_f) - \frac{i}{\hbar}\alpha(X_f)E^{\mathcal{L}} + i\frac{f}{\hbar}E^{\mathcal{L}}. \end{aligned}$$

Since α vanishes on vertical vector fields, by Theorem 3.10 we have $\alpha(X_f) = \alpha(X^{\ell_\alpha})$ and therefore

$$X_f^{\mathcal{L}} = h^{\pi^*\nabla}(X_f) + i\frac{\pi^* f^0}{\hbar}E^{\mathcal{L}}. \tag{25}$$

Moreover, since X_f projects to X, and the Euler vector field $E^{\mathcal{L}}$ of $\mathcal{L}$ projects to the Euler vector field E^L of L, it follows that $h^{\pi^*\nabla}(X_f)$ projects to $h^\nabla(X)$ and $i\frac{\pi^* f^0}{\hbar}E^{\mathcal{L}}$ projects to $i\frac{f^0}{\hbar}E^L$. Hence we have proved that $X_f^{\mathcal{L}} \in \mathfrak{X}(\mathcal{L})$ projects to the vector field $X_f^L := h^\nabla(X) + i\frac{f^0}{\hbar}E \in \mathfrak{X}(L)$. In a similar way $\widehat{X}_f^{\nabla^Q} = h^{\pi^*(\nabla^Q)}(X_f) \in \mathfrak{X}(N_P^{1/2})$ projects to $X^{\nabla^Q} := h^{\nabla^Q}(X) \in \mathfrak{X}(K_Q^{1/2})$. We denote by $X_f^{\nabla^Q}$ the vector field on the tensor product bundle $L \otimes K_Q^{1/2}$ naturally induced by X_f^L and X^{∇^Q}. The above discussion proves the following:

Proposition 3.13. *For any $f = \pi^* f^0 + \alpha(X^{\ell_\alpha}) \in \mathcal{O}_P$ with $f^0 \in C^\infty(Q)$ and $X \in \mathfrak{X}(Q)$ the action of its associated operator $\widehat{f}$ on a decomposable half-form $\tilde{\psi} = \pi^*\psi$ with $\psi = s \otimes \sqrt{\nu} \in \mathcal{H}_P$ is given by*

$$\widehat{f}(\tilde{\psi}) = i\hbar\pi^*(L_{X_f^{\nabla^Q}}\psi) = i\hbar\pi^*(L_{X_f^L}s \otimes \sqrt{\nu} + s \otimes L_{X^{\nabla^Q}}\sqrt{\nu}) \tag{26}$$

$$= i\hbar\pi^*((\nabla_X s + i\frac{f^0}{\hbar}s) \otimes \sqrt{\nu} + s \otimes \nabla_X^Q\sqrt{\nu}). \tag{27}$$

Therefore the modified quantum operator $\widehat{f}$ defined in (24) *preserves $\mathcal{H}_P$ (i.e $\widehat{f}(\mathcal{H}_P) \subset \mathcal{H}_P$), and its action on wave functions $\psi \in \mathcal{H}_P$ is given by $\widehat{f}(\psi) = i\hbar L_{X_f^{\nabla^Q}}\psi$.*

Taking into account the identification $\mathcal{O}_P = C^\infty(Q) \times \mathfrak{X}(Q)$ proved in Theorem 3.10, for every $f = (f^0, X) \in \mathcal{O}_P$ we define an operator

$$\rho_f^L := L_{X_f^L} \colon \Gamma(Q, L) \to \Gamma(Q, L), \quad \rho_f^L(s) = \nabla_X s + i\frac{f^0}{\hbar}s. \tag{28}$$

Analogously we define
(29)
$$\rho_f \colon \Gamma(Q, L \otimes K_Q^{1/2}) \to \Gamma(Q, L \otimes K_Q^{1/2}), \quad \rho_f(\psi) = \rho_f^L(s) \otimes \sqrt{\nu} + s \otimes \nabla_X^Q\sqrt{\nu}.$$

The action of the quantum operator associated to $f = (f^0, X) \in \mathcal{O}_P$ simply reads $\widehat{f}(\psi) = i\hbar\rho_f(\psi)$. In order for the modified geometric quantization to be physically acceptable we have to check that the fundamental commutation identity (14) still holds. As we will shortly see this is true if we require the flatness of ∇^Q, regarded as a connection on $K_Q^{1/2}$. For the same reason we also need to check that the quantum operators are symmetric. In order to do this, since they are differential operators and the space of compactly supported C^∞ wave functions $\Gamma_0(Q, L \otimes K_Q^{1/2})$ is dense in the Hilbert space $\mathcal{H}_P$, it is enough to check that they are formally self-adjoint.

Theorem 3.14. *Let $f_1 = (f_1^0, X_1)$, $f_2 = (f_2^0, X_2) \in \mathcal{O}_P$, and suppose that the connection ∇^Q on $K_Q^{1/2} \to Q$ is flat. Then*

$$\widehat{\{f_1, f_2\}} = i\hbar[\widehat{f}_1, \widehat{f}_2]. \tag{30}$$

For every $f = (f^0, X) \in \mathcal{O}_P$ the associated quantum operator $\widehat{f}\colon \mathcal{H}_P \to \mathcal{H}_P$ is symmetric if the linear operator $A_X = L_X - \nabla^Q_X$ vanishes on the line bundle $|K_Q|$ of densities. If in addition X is complete then $\widehat{f}$ is self-adjoint.

Proof. One easily checks that $[X^L_{f_1}, X^L_{f_2}] = X^L_{\{f_1,f_2\}}$. Then it is straightforward to prove that for a decomposable wave function $\psi = s \otimes \sqrt{\nu}$ one has

$$\begin{aligned}
[\widehat{f}_1, \widehat{f}_2](\psi) =& (i\hbar)^2(L_{X^L_{\{f_1,f_2\}}} s \otimes \sqrt{\nu} + s \otimes [\nabla^Q_{X_1}, \nabla^Q_{X_2}]\sqrt{\nu}) \\
=& (i\hbar)^2(\rho^L_{\{f_1,f_2\}} s \otimes \sqrt{\nu} + s \otimes \nabla^Q_{[X_1,X_2]}\sqrt{\nu}) \\
=& (i\hbar)^2\rho_{\{f_1,f_2\}}(\psi) = i\hbar\widehat{\{f_1, f_2\}}(\psi).
\end{aligned}$$

If $f = (f^0, X) \in \mathcal{O}_P$ and $\psi_1 = s_1 \otimes \sqrt{\nu_1}$, $\psi_2 = s_2 \otimes \sqrt{\nu_2} \in \Gamma(Q, L \otimes K_Q^{1/2})$, then a straightforward computation shows

$$\langle \widehat{f}(\psi_1), \psi_2\rangle_Q = \int_Q \langle \widehat{f}(\psi_1), \psi_2\rangle = \langle \psi_1, \widehat{f}(\psi_2)\rangle_Q + \int_Q \nabla^Q_X(\langle s_1, s_2\rangle\langle\sqrt{\nu_1}, \sqrt{\nu_2}\rangle_P), \tag{31}$$

where $\nabla^Q_X(\langle s_1, s_2\rangle\langle\sqrt{\nu_1}, \sqrt{\nu_2}\rangle_P)$ denotes the covariant derivative of the density $\eta = \langle s_1, s_2\rangle\langle\sqrt{\nu_1}, \sqrt{\nu_2}\rangle_P$. If $A_X = 0$ then we have $\int_Q \nabla^Q_X \eta = \int_Q L_X\eta$. Now if ψ_1 is compactly supported then the same is true for η and it is well known that in this case $\int_Q L_X\eta = 0$, and so the proof is finished. □

In the first part of the previous proof the role played by the flatness of ∇^Q is clearly fundamental. Notice however that, for several good reasons, the flatness condition on ∇^Q should not be regarded as a strong requirement:

- in GQ this prescription is fulfilled as we act on half-forms by a Lie derivative that behaves like the covariant derivative of a flat connection [31, p. 185].
- in the main example of quantization of the phase space of a particle moving on a Riemannian manifold the flatness of ∇^Q holds true even if we use the connection induced by the Levi-Civita connection (see Remark 2.1).

The most delicate condition is the divergence-free requirement on the vector field X. Here we make the following important remarks:

- The most important class of examples for applications in Physics is the quantization of natural mechanical systems. Here, as we will see in the next section, the basic examples of physical observables satisfy the divergence-free property. Moreover, it is well-known that the space of divergence-free vector fields is infinite-dimensional, and therefore this means that the space of observables that can be straightforwardly quantized is large.
- In natural mechanical systems the Riemannian metric is supposed to play an important role and as a matter of fact it enters in the Schrödinger operator. This means that extending the role of the metric should be quite a natural assumption. Just as an example, every Killing (*i.e.* metric preserving) vector field X is divergence-free, and in view of the above remark this is a physically meaningful class of vector fields.

The quantization of non straightforwardly quantizable observables in our modified quantization method is performed by changing the BKS method according to our guiding principles.

Definition 3.15 (Modified BKS method)**.** A non straightforwardly quantizable observable $H \in \mathcal{C}^\infty(M)$ yields a quantum operator $\widehat{H}$ which is defined on decomposable wave functions $\psi = s \otimes \eta \in \mathcal{H}$ as

$$\widehat{H}(\tilde{\psi}) = i\hbar \frac{d}{dt} \mathcal{H}_{0t} \circ \tilde{\phi}^{\nabla^Q,H}_{t,*}(\tilde{\psi})\big|_{t=0},$$

where

$$\tilde{\phi}^{\nabla^Q,H}_{t,*}(\tilde{\psi}) = \phi^{H,L}_{t,*}(\psi) \otimes \phi^{\nabla^Q,H}_{t,*}(\sqrt{\nu}),$$

$\tilde{\phi}^{\nabla^Q,H}_t$ is the flow of $\tilde{X}^{\nabla^Q}_H$ and $\phi^{\nabla^Q,H}_t$ is the flow of $\widehat{X}^{\nabla^Q}_H$.

4 Modified geometric quantization on Riemannian manifolds

In this section we specialize our constructions to the case of a classical mechanical system for a particle, modelled by a Riemannian manifold (Q, g). Its phase space is the symplectic manifold (T^*Q, ω_0), where $\omega_0 = d\theta_0$ is the canonical symplectic form, the differential of the canonical Liouville form θ_0.

We derive the quantum operators for the most common observables with our modified quantization method. This leads to interesting consequences that will be analyzed in the conclusive section.

We assume that Q is orientable, so that $\wedge^n T^*Q$ is a trivial real line bundle. We will denote by (q^i) local coordinates on Q; consequently (q^i, p_i) will denote coordinates on T^*Q.

We refer to [24] for a detailed derivation of the energy operator in this situation within the framework of the standard GQ (see p. 120, Section 7.2, or p. 180, Section 10.1 for the case of a nonzero electromagnetic field).

We assume that the topological conditions for the existence of a prequantum structure on (T^*Q, ω_0) are fulfilled (see Section 2.2). It can be proved that the topological triviality of the typical fiber of $T^*Q \to Q$ implies that *all prequantum structures are reducible*. So, we may assume without loss of generality a quantum bundle $\mathcal{L} \to T^*Q$ endowed with a connection which are the pull-back, respectively, of a Hermitian complex line bundle $L \to Q$ and a connection ∇ on this bundle.

The polarization P that we choose is the vertical one, *i.e.* $P = VT^*Q$ which is locally spanned by the vector fields $\partial/\partial p_i$. Since Q is orientable, its determinant bundle admits a square root $K_Q^{1/2} = \sqrt{\wedge^n_{\mathbb{C}} T^*Q}$, and there exists a metaplectic structure: $N_P^{1/2} = \tau_Q^\vee(K_Q^{1/2})$.

The canonical bundle $N_P^{1/2}$ is endowed with the pull-back $(\tau_Q^\vee)^*(\nabla^g)$ of the canonical connection $\nabla^Q = \nabla^g$ on $K_Q^{1/2}$ defined by the Levi-Civita connection on $\tau_Q^\vee \colon T^*Q \to Q$ using the natural procedures described in Subsection 2.1 (see (6)).

The Hilbert space $\mathcal{H}_P$ of quantum states is given in both GQ and the modified GQ by the L^2-completion of compactly supported polarized sections in $\Gamma_P(T^*Q, \mathcal{L}) = \Gamma(Q, L \otimes K_Q^{1/2})$ (see Corollary 3.4).

Proposition 4.1. *If $\psi = s \otimes \sqrt{\nu_g}$ is a wave function, (q^i, p_i) are canonical coordinates and $X = f^i \frac{\partial}{\partial q^i}$ is a divergence-free vector field, then the modified* *geometric quantization procedure yields*

$$\widehat{q^i}(\psi) = i\hbar q^i s \otimes \sqrt{\nu_g}, \tag{32}$$

$$\widehat{f^i p_i}(\psi) = -i\hbar f^i \frac{\partial}{\partial q^i}\psi^0 b_0 \otimes \sqrt{\nu_g}, \tag{33}$$

Proof. According to Definition 3.12 and Proposition 3.13 the quantum operator $\widehat{f}$ acts on a wave function $\psi = s \otimes \sqrt{\nu_g}$ as

$$\hat{f}(\psi) = i\hbar(L_{X_f^L} s \otimes \sqrt{\nu_g} + s \otimes L_{X^{\nabla g}} \sqrt{\nu_g}). \tag{34}$$

Using (2), we have

$$L_{X^{\nabla g}} \sqrt{\nu_g} = \nabla^g_X \sqrt{\nu_g} = 0.$$

So only the first summand in (34) contributes to the quantum operator and it is easy to prove the result. □

Remark 4.2. The natural observables like positions q^i and momenta p_i fulfill the divergence-free condition. We would like to stress that also the angular momentum fulfills the same conditions, and hence is also straightforwardly quantizable in the modified quantization (see, *e.g.*, [22]).

The most interesting result concerns the quantization of the Hamiltonian H.

Theorem 4.3. *The modified BKS method yields the quantum energy operator*

$$\widehat{H}(\psi) = \left(-\frac{\hbar^2}{2}\Delta(s) + Vs\right) \otimes \sqrt{\nu_g}. \tag{35}$$

Proof. We shall evaluate the pairing (15). Following [31, p. 202], we have

$$(\tilde{\psi}_1, \mathcal{H}_{0t}(\tilde{\psi}_2)) = \int_{T^*Q} \psi_1 \overline{\psi_2} e^{-itH/\hbar} \sqrt{(\nu_g, \phi^{g,H}_{t,*}\nu_g)} \tag{36}$$

where $\phi^{g,H}_t$ denotes the flow of $X_H^{\nabla^g}$. Since in the modified BKS procedure we replaced the flow of X_H by the flow of $X_H^{\nabla^g}$ we have that

$$\sqrt{(\nu_g, \phi^{g,H}_{t,*}\nu_g)} = \nu_g \tag{37}$$

as the parallel transport preserves the metric volume.

The left-hand side of (37) was the source of the scalar curvature in the quantum energy operator obtained through the BKS procedure. This means that in the modified BKS procedure the scalar curvature term is not present as the time derivative of ν_g vanishes. □

Corollary 4.4. *Let $\psi = s \otimes \sqrt{\nu_g} \in \mathcal{H}_P$. Then*

$$\Delta(s) \otimes \sqrt{\nu_g} = \tilde{\Delta}(\psi),$$

where Δ and $\tilde{\Delta}$ are the Bochner Laplacians of the line bundles $L \to Q$ and $L \otimes K_Q^{1/2} \to Q$, respectively.

Proof. The proof is easily achieved by recalling that the Bochner Laplacian is just the double covariant derivative followed by a metric contraction. Since the covariant derivative in the tensor product bundle $L \otimes K_Q^{1/2} \to Q$ annihilates the factor $\sqrt{\nu_g}$, the result follows. □

Remark 4.5. In a Riemannian manifold (Q, g) the scalar curvature r_g measures the volume distorsion (to the second order) of balls of small radius in Q compared to Euclidean balls of the same radius [10, p. 168, Theorem 3.98]. However if we move the volume along the flow of the parallel transport, there is no distorsion.

5 Discussion

In the above sections we have demonstrated that the wide set of choices that build up one representation in GQ can be further enlarged if we allow for the possibility to choose a connection on the polarization bundle such that the induced connection on the half-form bundle is flat. The additional restriction that we have obtained for the coefficients of momenta in quantizable observables, the divergence free condition, does not seem to be too strong since the basic observables of position, momenta and angular momenta fulfill the condition. However, there are many other theories that yield a quantum energy operator with scalar curvature as a multiplicative operator (see the Introduction for details). In particular we may mention Feynman path integral [7, 5, 1] or Weyl quantization [18, 17].

It is not possible to prove in one single paper that a modification of Feynman path integral or Weyl quantization might yield a Schrödinger operator with no scalar curvature term. However, after the above results, it seems to us that there exists a possibility that the Feynman or Weyl quantization procedures might be modified along the above ideas to yield $k = 0$, or even any other nonzero value of k like in [14], as was already remarked in [7].

The technical remark 4.5 can help understanding how to carry on a quantization procedure in a way that would preserve the metric volume.

We deliberately ignored the important topic of complex polarizations. The possibility that a polarization has a natural geometric structure inherited from a metric or by a natural connection in some related space is concrete, and it would be at least interesting to explore it by analogy with our previous guidelines in order to see if any interesting consequences would appear.

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