Published in
Arch. Math. (Brno),
34 n. 4 (1998), 483–504
`http://www.emis.de/journals/AM`

A new infinite order formulation of variational sequences

Raffaele Vitolo[1]

Dept. of Mathematics "E. De Giorgi", Università di Lecce,

via per Arnesano, 73100 Italy

email: `Raffaele.Vitolo@unile.it`

Abstract

The theory of variational bicomplexes is a natural geometrical setting for the calculus of variations on a fibred manifold. It is a well–established theory although not spread out very much among theoretical and mathematical physicists. Here, we present a new approach to infinite order variational bicomplexes based upon the finite order approach due to Krupka. In this approach the information related to the order of jets is lost, but we have a considerable simplification both in the exposition and in the computations. We think that our infinite order approach could be easily applied in concrete situations, due to the conceptual simplicity of the scheme.

Key words: Fibred manifold, jet space, infinite order jet space, variational bicomplex, variational sequence.

1991 MSC: 58A12, 58A20, 58E30, 58G05.

[1] This paper has been partially supported by Fondazione F. Severi, GNFM of CNR, MURST, University of Florence.

Introduction

The theory of variational bicomplexes can be regarded as a natural geometrical setting for the calculus of variations on a fibred manifold [1, 2, 9, 10, 11, 14, 16, 17, 18, 19, 20]. The geometric objects which appear in the calculus of variations find a place on the vertices of the bicomplex, and are put in relation by the morphisms of the bicomplex. Such morphisms are closely related to the differential of forms on the jet spaces of the starting fibred manifold. Moreover, the global inverse problem is solved in this context.

Some formulations [2, 11, 14, 16, 17, 18, 19, 20] of variational bicomplexes are carried on by means of infinite order jet techniques. Roughly speaking, the vertices of variational bicomplexes are spaces of forms defined on jet spaces of any order. These spaces have a natural splitting which turns out to be very useful from a technical viewpoint. The formulation in [1] is partially carried on by means of finite order jet spaces.

The finite order variational bicomplex has been introduced by Krupka [9] by a very simple construction from a conceptual viewpoint. Namely, it is produced when one quotients the de Rham sequence on a finite order jet space by means of an intrinsically defined subsequence arising from the fibring. In this way, one obtains a bicomplex where the horizontal morphisms are either the differentials of forms or quotient morphisms, and the vertical morphisms are either inclusions or natural projections on quotient spaces. For an intrinsic analysis of this theory, see [21].

The above formulation can help in keeping trace of the order of the geometric objects involved at each vertex of the bicomplex. This fundamental feature depends on the fact that Krupka's formulation uses finite order jet. Of course, this feature is lost in infinite order approaches. But, in order to show the connection between the quotient sequence and the calculus of variations one has to face several technical difficulties. The most difficult point is that the spaces of forms on a finite order jets do not split as their analogues in the infinite order case. This is one of the main obstacles that one meets when giving a representation of the quotient sequence by means of forms (see [21]).

In this paper, we present a new approach to infinite order variational bicomplexes which is inspired by Krupka's finite order approach. We think that this infinite order approach has both the advantages of the conceptual simplicity of Krupka's scheme and the advantages in computations due to the use of spaces of forms at infinite order.

Indeed, as it is proved in [22], our infinite order approach turn out to be the direct limit of Krupka's finite order approach. Anyway, in this paper we show that we can directly formulate our infinite order approach without passing through the finite order one and the direct limit.

In the first section we introduce jet spaces, the contact structure [12, 15] and the sheaves of forms on jets, and evaluate their direct limit.

In the second section we define the infinite order variational bicomplex, which is inspired by the finite order approach due to Krupka.

In the third and the fourth sections we give in two steps an isomorphism of the infinite order variational sequence with a sequence of presheaves which are the direct limit on some sheaves of forms on jet bundles. Here, the first variation formula [7] plays an essential role.

In the last section, we interpret the sequence that we found in the above two sections in terms of geometric objects and operators of the calculus of variations.

We end the introduction with some mathematical conventions. In this paper, manifolds are connected and C^∞, and maps between manifolds are C^∞. Morphisms of fibred manifolds (and hence bundles) are morphisms over the identity of the base manifold, unless otherwise specified.

We make use of definitions and results on presheaves and sheaves from [23]. In particular, we are concerned only with (pre)sheaves of $I\!R$–vector spaces, hence '(pre)sheaf morphism' stands for morphism of (pre)sheaves of $I\!R$–vector spaces.

If $\mathcal{P}$ be a presheaf, then we denote by $\overline{\mathcal{P}}$ the sheaf generated (in the sense of [23]) by $\mathcal{P}$. We denote by $\mathcal{P}_U$ the set of sections of a (pre)sheaf $\mathcal{P}$ over a topological space X defined on the open subset $U \subset X$. We recall that a sequence of (pre)sheaves over X is said to be exact if it is locally exact (see [23] for a more precise definition). If $\mathcal{A}$, $\mathcal{B}$ are two sub(pre)sheaves of a sheaf $\mathcal{P}$, then the wedge product $\mathcal{A} \wedge \mathcal{B}$ is defined to be the sub(pre)sheaf of sections of $\overset{2}{\wedge}\mathcal{P}$ generated by wedge products of sections of $\mathcal{A}$ and $\mathcal{B}$.

Let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a family of (pre)sheaves and $\{\iota^m_n : \mathcal{P}_n \to \mathcal{P}_m\}_{n,m\in\mathbb{N},n\leq m}$ be a family of injective (pre)sheaf morphisms such that, for all $n, m, p \in \mathbb{N}$, $n \leq m \leq p$, we have $\iota^p_m \circ \iota^m_n = \iota^p_n$ and $\iota^n_n = \mathrm{id}_{\mathcal{P}_n}$. We say $\{\mathcal{P}_n, \iota^m_n\}$ to be an *injective system* of (pre)sheaves. We define the *direct limit* of the injective system to be the presheaf

$$\mathcal{P} := \lim_{\rightarrow} \mathcal{P}_n := \bigsqcup_{n\in\mathbb{N}} \mathcal{P}_n / \sim ,$$

where $\sim$ is the equivalence relation defined as follows. For each $s \in \mathcal{P}_n$ and $s' \in \mathcal{P}_{n'}$, if $n \leq n'$, then $s \sim s'$ if and only if $\iota^{n'}_n(s) = s'$. Note that, in general, the direct limit of an injective system of sheaves needs not to be a sheaf. Let $\{\mathcal{P}_n, \iota^m_n\}$ and $\{\mathcal{Q}_n, \epsilon^m_n\}$ be two injective systems of (pre)sheaves, and suppose that we have a family of (pre)sheaf morphisms $\{f_n : \mathcal{P}_n \to \mathcal{Q}_n\}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}_m \arrow[r, "f_m"] & \mathcal{Q}_m \\
\mathcal{P}_n \arrow[r, "f_n"] \arrow[u, "\iota^m_n"] & \mathcal{Q}_n \arrow[u, "\epsilon^m_n"']
\end{tikzcd}$$

Then, the presheaf morphism

$$f := \lim_{\rightarrow} f_n : \mathcal{P} \to \mathcal{Q} : [s] \mapsto [f_m(s)] ,$$

where $s \in \mathcal{P}_m$ for some m, is well defined.

Acknowledgements. I would like to thank I. Kolář, D. Krupka, M. Modugno, and J. Štefánek for stimulating discussions.

Commutative diagrams have been drawn by P. Taylor's `diagrams` macro package.

1 Jet spaces

In this section we recall some basic facts on jet spaces. Namely, we start with a natural splitting of the cotangent bundle of jet spaces. Then, we study some natural sheaves of forms on jet spaces, and introduce the horizontal and vertical differential of forms. Finally, we evaluate the direct limit of the sheaves and morphisms.

As our framework, we assume a fibred manifold

$$\pi : \boldsymbol{Y} \to \boldsymbol{X}$$

with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n+m$. We deal with the vertical bundle $V\boldsymbol{Y} := \ker T\pi \to \boldsymbol{Y}$. Moreover, for $0 \leq r$, we are concerned with the r–th jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fibrings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$ for $0 \leq s \leq r$. A detailed account of the theory of jets can be found in [12, 11, 15].

Charts on $\boldsymbol{Y}$ adapted to the fibring are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and label base coordinates; Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i), respectively, the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart. We denote multi–indices of dimension n by underlined letters such as $\underline{p} = (p_1, \ldots, p_n)$, with $0 \leq p_1, \ldots, p_n$; we identify a standard index λ with the multi–index $\underline{\lambda}$ defined by $\underline{\lambda}_\lambda = 1$ and $\underline{\lambda}_\mu = 0$ if $\mu \neq \lambda$. We also set $|\underline{p}| := p_1 + \cdots + p_n$ and $\underline{p}! := p_1! \ldots p_n!$. The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^0, y^i_{\underline{p}})$, with $0 \leq |\underline{p}| \leq r$; in particular, if $|\underline{p}| = 0$, then we set $y^i_{\underline{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the fibre coordinates are denoted by $(\partial^{\underline{p}}_i)$ and $(d^i_{\underline{p}})$, $0 \leq |\underline{p}| \leq r, 1 \leq i \leq m$, respectively.

A section $s : \boldsymbol{X} \to \boldsymbol{Y}$ can be naturally prolonged to a section $j_r s : \boldsymbol{X} \to J_r\boldsymbol{Y}$, with coordinate expression $y^i_{\underline{p}} \circ j_r s = \partial_{\underline{p}}\, y^i \circ s$. If $\boldsymbol{Z} \to \boldsymbol{X}$ is another fibred manifold and $f : \boldsymbol{Y} \to \boldsymbol{Z}$ is a morphism over $\mathrm{id}_{\boldsymbol{X}}$, then f can be naturally prolonged to a morphism $J_r f : J_r\boldsymbol{Y} \to J_r\boldsymbol{Z}$ over $\mathrm{id}_{J_{r-1}\boldsymbol{Y}}$ by means of the characterisation $(J_r f)\circ j_r s = j_r(f \circ s)$ for any section $s : \boldsymbol{X} \to \boldsymbol{Y}$. A vertical vector field $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ can be naturally prolonged to a vertical vector field $u_r : J_r\boldsymbol{Y} \to VJ_r\boldsymbol{Y}$ by prolonging its flow in the above way and by considering the natural isomorphism $J_rV\boldsymbol{Y} \simeq VJ_r\boldsymbol{Y}$. The coordinate expressions are given later.

Splitting of the cotangent bundle

We recall the natural inclusion $J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T^*\boldsymbol{X} \subset T^*J_r\boldsymbol{Y}$ and projection $T^*J_r\boldsymbol{Y} \to V^*J_r\boldsymbol{Y}$. We have no natural complementary maps; so, $T^*J_r\boldsymbol{Y}$ has no natural splitting into the direct sum of 'vertical' and 'horizontal' tangent subspaces over $J_r\boldsymbol{Y}$. On the other hand, we obtain such a natural splitting over $J_{r+1}\boldsymbol{Y}$ by means of the "contact maps" on jet spaces (see [12]). Namely, for $r \geq 0$, we consider the natural injective fibred morphism over $J_{r+1}\boldsymbol{Y} \to J_r\boldsymbol{Y}$

$$д_{r+1} : J_{r+1}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_r\boldsymbol{Y} \,,$$

and the complementary surjective fibred morphism

$$\vartheta_{r+1} : J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} TJ_r\boldsymbol{Y} \to VJ_r\boldsymbol{Y} \,,$$

whose coordinate expression are

$$\begin{aligned} д_{r+1} &= d^\lambda \otimes д_{r+1\lambda} = d^\lambda \otimes (\partial_\lambda + y^j_{\underline{p}+\lambda} \partial^{\underline{p}}_j) \,, \qquad 0 \leq |\underline{p}| \leq r, \\ \vartheta_{r+1} &= \vartheta^j_{\underline{p}} \otimes \partial^{\underline{p}}_j = (d^j_{\underline{p}} - y^j_{\underline{p}+\lambda} d^\lambda) \otimes \partial^{\underline{p}}_j \,, \qquad 0 \leq |\underline{p}| \leq r \,. \end{aligned}$$

The transpose of the maps $д_{r+1}, \vartheta_{r+1}$ are the fibred morphism over $J_{r+1}\boldsymbol{Y} \to J_r\boldsymbol{Y}$

$$\begin{aligned} &д^*_{r+1} : J_{r+1}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T^*J_r\boldsymbol{Y} \to T^*\boldsymbol{X} \,, \\ &\vartheta^*_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to T^*J_{r-1}\boldsymbol{Y} \,. \end{aligned}$$

We have the remarkable vector bundles

$$\begin{aligned} &\operatorname{im} д^*_{r+1} \simeq J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \,, \\ &\operatorname{im} \vartheta^*_r \subset J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} \subset T^*J_r\boldsymbol{Y} \,. \end{aligned}$$

Thus we obtain the natural splitting of $T^*J_r\boldsymbol{Y}$ over $J_{r+1}\boldsymbol{Y}$ [12]

$$J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \operatorname{im} д^*_{r+1} \oplus \operatorname{im} \vartheta^*_{r+1} \,, \tag{1}$$

given by

$$\alpha = (д_{r+1} \lrcorner \alpha) + (\vartheta_{r+1} \lrcorner \alpha) \,.$$

Sheaves of forms

We are concerned with some distinguished sheaves of forms on jet spaces (see also [15]). Note that we consider sheaves on $J_r\boldsymbol{Y}$ with respect to the topology generated by open sets of the kind $(\pi^r_0)^{-1}(\boldsymbol{U})$, where $\boldsymbol{U} \subset \boldsymbol{Y}$ is open in $\boldsymbol{Y}$. This is suggested by the topological triviality of the fibre of $J_{r+1}\boldsymbol{Y} \to J_r\boldsymbol{Y}$ [9].

Let us introduce the sheaves which will play a basic role throughout the paper. Let $0 \leq k, h$.

i. For $0 \leq r$, we consider the standard sheaf $\overset{k}{\Lambda}_r$ of k–forms $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_r\boldsymbol{Y}$ on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, we consider the subsheaf $\overset{k}{\Lambda}_{(r,s)} \subset \overset{k}{\Lambda}_r$ of local fibred morphisms over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ of the type $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^* J_s\boldsymbol{Y}$. Pullback by π^r_s provides the natural inclusion $\overset{k}{\Lambda}_s \subset \overset{k}{\Lambda}_{(r,s)}$. Of course, if $s = r$, then $\overset{k}{\Lambda}_{(r,r)} = \overset{k}{\Lambda}_r$.

iii. For $0 \leq r+1$, we consider the subsheaf $\overset{(k,h)}{\Lambda}{}_{r+1} \subset \overset{k}{\Lambda}_{r+1}$ of local fibred morphisms over $J_{r+1}\boldsymbol{Y}$ of the type

$$\alpha : J_{r+1}\boldsymbol{Y} \to \overset{k}{\wedge} \mathrm{im}\, \vartheta^*_{r+1} \underset{\boldsymbol{X}}{\wedge} \overset{h}{\wedge} T^*\boldsymbol{X} \,.$$

iv. For $0 \leq s < r+1$, we consider the subsheaf $\overset{(k,h)}{\Lambda}{}_{(r+1,s)} \subset \overset{k}{\Lambda}_{(r+1,s)}$ of local fibred morphisms over $J_{r+1}\boldsymbol{Y} \to J_{s+1}\boldsymbol{Y}$ of the type

$$\alpha : J_{r+1}\boldsymbol{Y} \to \overset{k}{\wedge} \mathrm{im}\, \vartheta^*_{s+1} \underset{\boldsymbol{X}}{\wedge} \overset{h}{\wedge} T^*\boldsymbol{X} \,.$$

Of course, if $s = r$ then $\overset{(k,h)}{\Lambda}{}_{(r+1,r)} = \overset{(k,h)}{\Lambda}{}_{r+1}$.

The fibred splitting (1) yields a fundamental sheaf splitting.

Lemma 1.1. *We have the splitting*

$$\overset{1}{\Lambda}_{(r+1,r)} = \overset{(0,1)}{\Lambda}{}_{r+1} \oplus \overset{(1,0)}{\Lambda}{}_{r+1} \,,$$

where the projection on the first factor and on the second factor are given, respectively, by

$$\overset{1}{h} = д^*_{r+1} : \overset{1}{\Lambda}_{(r+1,r)} \to \overset{(0,1)}{\Lambda}{}_{r+1} : \alpha \mapsto д_{r+1} \lrcorner\, \alpha \,,$$
$$\overset{1}{v} = \vartheta^*_{r+1} : \overset{1}{\Lambda}_{(r+1,r)} \to \overset{(1,0)}{\Lambda}{}_{r+1} : \alpha \mapsto \vartheta_{r+1} \lrcorner\, \alpha \,.$$

If $\alpha \in \overset{1}{\Lambda}_{(r+1,r)}$ has the coordinate expression $\alpha = \alpha_\lambda d^\lambda + \alpha^{\underline{p}}_i d^i_{\underline{p}}$ $(0 \leq \underline{p} \leq r)$, then

$$\overset{1}{h}(\alpha) = (\alpha_\lambda + y^i_{\underline{p}} \alpha^{\underline{p}}_i)\; d^\lambda \,, \qquad \overset{1}{v}(\alpha) = \alpha^{\underline{p}}_i \vartheta^i_{\underline{p}} \,.$$

Proposition 1.1. *The above splitting of* $\overset{1}{\Lambda}_{(r+1,r)}$ *induces the splitting*

$$\overset{k}{\Lambda}_{(r+1,r)} = \bigoplus_{l=0}^{k} \overset{(k-l,l)}{\Lambda}{}_{r+1} \,,$$

where the projections are given by

$$\overset{(k-l,l)}{\square}(\vartheta^*_{r+1}, д^*_{r+1}) : \overset{k}{\Lambda}_{(r+1,r)} \to \overset{(k-l,l)}{\Lambda}{}_{r+1}$$

(for $\overset{(k-l,l)}{\square}$ *see Appendix).*

Let us study explicitly the projection maps. We denote with $\overset{k}{h}$ the projection of the above splitting on the summand with the highest degree of the horizontal factor (which, of course, cannot be greater than n). In other words, we have

$$\overset{k}{h} : \overset{k}{\Lambda}_{(r+1,r)} \to \begin{cases} \overset{(0,n)}{\Lambda}_{r+1} & \text{if } k \leq n \\ \overset{(k-n,n)}{\Lambda}_{r+1} & \text{if } k > n\,; \end{cases}$$

We denote also the projection complementary to $\overset{k}{h}$ by $\overset{k}{v} := Id - \overset{k}{h}$.

Now, we evaluate the coordinate expression of $\overset{k}{h}$. Let $\alpha \in \overset{k}{\Lambda}_{(r+1,r)}$.

If $0 < k \leq n$, then we have the coordinate expression

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h}{}_{\lambda_{h+1} \dots \lambda_k}\, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_h}_{\underline{p}_h} \wedge d^{\lambda_{h+1}} \wedge \dots \wedge d^{\lambda_k}$$

where the coordinate functions are sections of $\overset{0}{\Lambda}_{r+1}$, and the indices'range is $0 \leq |\underline{p}_j| \leq r$, $0 \leq h \leq k$. We remark that the indices λ_j are suppressed if $h = k$, and the indices $\frac{\underline{p}_j}{i_j}$ are suppressed if $h = 0$. We have

$$\overset{k}{h}(\alpha) = y^{i_1}_{\underline{p}_1+\lambda_1} \dots y^{i_h}_{\underline{p}_h+\lambda_h} \alpha^{\underline{p}_1 \dots \underline{p}_h}_{i_1 \dots i_h}{}_{\lambda_{h+1} \dots \lambda_k} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_k}\,.$$

If $k > n$, then we have the coordinate expression

$$\alpha = \alpha^{\underline{p}_1 \dots \underline{p}_{k-n+l}}_{i_1 \dots i_{k-n+l}}{}_{\lambda_{l+1} \dots \lambda_n}\, d^{i_1}_{\underline{p}_1} \wedge \dots \wedge d^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_{l+1}} \wedge \dots \wedge d^{\lambda_n}\,,$$

where the coordinate functions are sections of $\overset{0}{\Lambda}_{r+1}$, and the indices'range is $0 \leq |\underline{p}_j| \leq r$, $0 \leq l \leq n$. We remark that, the indices λ_j are suppressed if $l = n$. We have

$$\begin{aligned} \overset{k}{h}(\alpha) = \sum y^{j_1}_{\underline{q}_1+\lambda_1} \dots y^{j_l}_{\underline{q}_l+\lambda_l} \alpha^{\underline{p}_1 \widehat{\dots} \underline{p}_{k-n+l} \underline{q}_1 \dots \underline{q}_l}_{i_1 \widehat{\dots}\, i_{k-n+l}\, j_1 \dots j_l}{}_{\lambda_{l+1} \dots \lambda_n} \\ \vartheta^{i_1}_{\underline{p}_1} \wedge \widehat{\dots} \wedge \vartheta^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \wedge d^{\lambda_1} \wedge \dots \wedge d^{\lambda_n}\,, \end{aligned}$$

where the sum is over the subsets

$$\left\{ {}^{j_1}_{\underline{q}_1} \dots {}^{j_l}_{\underline{q}_l} \right\} \subset \left\{ {}^{i_1}_{\underline{p}_1} \dots {}^{i_{k-n+l}}_{\underline{p}_{k-n+l}} \right\},$$

and $\widehat{\dots}$ stands for suppressed indexes (and corresponding contact forms) belonging to one of the above subsets.

Example 1.1. Here we evaluate the coordinate expressions of the projection h in the case $k = 2$. Suppose that $\alpha \in \overset{k}{\Lambda}_{(2,1)}$ has the coordinate expression

$$\begin{aligned} \alpha =& \alpha_{\mu\lambda}\, d^{\mu} \wedge d^{\lambda} + \alpha_{i\lambda}\, d^{i} \wedge d^{\lambda} + \alpha^{\mu}_{i\,\lambda}\, d^{i}_{\mu} \wedge d^{\lambda} + \\ & \alpha_{ji}\, d^{j} \wedge d^{i} + \alpha^{\mu}_{j\,i}\, d^{j}_{\mu} \wedge d^{i} + \alpha^{\mu\lambda}_{j\,i}\, d^{j}_{\mu} \wedge d^{i}_{\lambda}\,. \end{aligned}$$

If $n = 1$, then $\lambda = \mu = 1$, $d^\mu \wedge d^\lambda = 0$ and

$$\begin{aligned} h(\alpha) = & (\alpha_{i\lambda} + 2y^j_\lambda \alpha_{ji} - y^j_{\mu+\lambda} \alpha^{\mu}_{j\,i}) \, \vartheta^i \wedge d^\lambda + \\ & (\alpha^{\mu}_{j\,\lambda} + y^i_\lambda \alpha^{\mu}_{j\,i} + 2y^i_{\lambda+\nu} \alpha^{\mu\nu}_{j\,i}) \, \vartheta^j_\mu \wedge d^\lambda \, . \end{aligned}$$

If $n \geq 2$, then

$$\begin{aligned} h(\alpha) = & (\alpha_{\mu\lambda} + y^i_\mu \alpha_{i\lambda} + y^i_{\mu+\nu} \alpha^{\nu}_{i\,\lambda} + \\ & y^j_\mu y^i_\lambda \alpha_{ji} + y^j_{\mu+\nu} y^i_\lambda \alpha^{\nu}_{j\,i} + y^j_{\mu+\nu} y^i_{\lambda+\rho} \alpha^{\nu\rho}_{j\,i}) \, d^\mu \wedge d^\lambda \, . \quad \square \end{aligned}$$

Horizontal and vertical differential

The exterior differential d together with the contact maps yield two derivations ('of degree one along π^{r+1}_r') of $\overset{k}{\Lambda}_r$ (see [15]). Namely, we define the *horizontal* and *vertical differentials* to be the sheaf morphisms

$$d_h := i_{\text{д}_{r+1}} \circ d - d \circ i_{\text{д}_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \, , \qquad d_v := i_{\vartheta_{r+1}} \circ d - d \circ i_{\vartheta_{r+1}} : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \, ,$$

It can be proved (see [15]) that d_h and d_v fulfill the properties

$$\begin{gathered} d_h^2 = d_v^2 = 0 \ , \qquad d_h \circ d_v + d_v \circ d_h = 0 \ , \\ d_h + d_v = (\pi^{r+1}_r)^* \circ d \ , \\ (j_{r+1}s)^* \circ d_v = 0 \ , \qquad d \circ (j_r s)^* = (j_{r+1}s)^* \circ d_h \ . \end{gathered}$$

The action of d_h and d_v on functions $f : J_r\boldsymbol{Y} \to I\!R$ and one–forms on $J_r\boldsymbol{Y}$ uniquely characterises d_h and d_v. We have the coordinate expressions

$$\begin{gathered} d_h f = (\text{д}_{r+1})_\lambda . f d^\lambda = (\partial_\lambda f + y^i_{\underline{p}+\lambda} \partial^{\underline{p}}_i f) d^\lambda \, , \\ d_h d^\lambda = 0 \, , \qquad d_h d^i_{\underline{p}} = -d^i_{\underline{p}+\lambda} \wedge d^\lambda \, , \qquad d_h \vartheta^i_{\underline{p}} = -\vartheta^i_{\underline{p}+\lambda} \wedge d^\lambda \, , \\ d_v f = \partial^{\underline{p}}_i f \vartheta^i_{\underline{p}} \ , \\ d_v d^\lambda = 0 \, , \qquad d_v d^i_{\underline{p}} = d^i_{\underline{p}+\lambda} \wedge d^\lambda \, , \qquad d_v \vartheta^i_{\underline{p}} = 0 \, . \end{gathered}$$

Direct limit

The sheaf injections π^r_s $(r \geq s)$ provide several inclusions between the sheaves of forms previously introduced. This yields several injective systems, whose direct limit is studied here.

We define the presheaves on $\boldsymbol{Y}$

$$\overset{k}{\Lambda} := \lim_{\to} \overset{k}{\Lambda}_r \, , \qquad \overset{(k,h)}{\Lambda} := \lim_{\to} \overset{(k,h)}{\Lambda}_{(r+1,r)} \, .$$

By simple counterexamples, it can be proved that the above presheaves are not sheaves in general, because the gluing axiom fails to be true.

Remark 1.1. For any equivalence class $[\alpha] \in \overset{k}{\Lambda}$ there exists a distinguished representative $\beta \in \overset{k}{\Lambda}_r$ whose order r is minimal. The same holds for $\overset{(0,k)}{\Lambda}$ and $\overset{(k,0)}{\Lambda}$. Accordingly, we shall often indicate by $\beta \in \overset{k}{\Lambda}$ (without brackets) such a minimal section. □

Lemma 1.2. *We have* $\lim\limits_{\to} \overset{k}{\Lambda}_{(r+1,r)} = \lim\limits_{\to} \overset{k}{\Lambda}_r \equiv \overset{k}{\Lambda}$.

Proof. In fact, we have the inclusions $\overset{k}{\Lambda}_r \subset \overset{k}{\Lambda}_{(r+1,r)} \subset \overset{k}{\Lambda}_{r+1}$ $\boxed{QED}$

Theorem 1.1. *We have the natural splitting*

$$\overset{k}{\Lambda} = \bigoplus_{l=0}^{k} \overset{(k-l,l)}{\Lambda} .$$

Proof. It comes from the above lemma and the splitting of proposition 1.1. $\boxed{QED}$

Remark 1.2. The above splitting represents one of the major differencies between the finite order and the infinite order case. As we shall see, in the infinite order formulations one has to deal with quotients of $\overset{k}{\Lambda}$ by sheaves of contact forms. The above splitting allows us to identify such quotients with 'more concrete' spaces (see section 3). The situation is much more complicated in the finite order case for the lack of such a splitting. In fact, the inclusion $\overset{k}{\Lambda}_r \subset \overset{k}{\Lambda}_{(r+1,r)}$ is a proper inclusion, and we are in the bad situation described in remark 5.4. Nevertheless, by means of the splitting of proposition 1.1, we are able to recover in the finite order case almost all features of infinite order formulations, but in a much more difficult way (see [21]). □

Proposition 1.2. *The sheaf morphisms* d, d_h, d_v, $\overset{k}{h}$, *admit direct limits. Namely, such direct limits turn out to be the presheaf morphisms*

$$d : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d\alpha] ,$$

$$d_h : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d_h\alpha] , \quad d_v : \overset{k}{\Lambda} \to \overset{k+1}{\Lambda} : [\alpha] \mapsto [d_v\alpha] ,$$

$$\overset{k}{h} : \overset{k}{\Lambda}_{(r+1,r)} \to \begin{cases} \overset{(0,n)}{\Lambda}{}_{r+1} : [\alpha] \mapsto [\overset{k}{h}] & \textit{if } k \leq n \\ \overset{(k-n,n)}{\Lambda}{}_{r+1} : [\alpha] \mapsto [\overset{k}{h}] & \textit{if } k > n \, ; \end{cases}$$

Note that the map $\overset{k}{h}$ of the above proposition turns out to be the projection of the splitting of theorem 1.1 on the factor with the highest horizontal degree; in other words, the direct limit of the projection is the projection of the splitting of the direct limit.

We observe that we did not indicate the degree of d, d_h and d_v. This is both for a matter of 'tradition' and not to make too heavy the notation.

Finally, next proposition analyses the relationship of d_h and d_v with the splitting of the above theorem.

Proposition 1.3. *We have*

$$d_h(\overset{(0,k)}{\Lambda}) \subset \overset{(0,k+1)}{\Lambda}\,, \qquad d_v(\overset{(0,k)}{\Lambda}) \subset \overset{(1,k)}{\Lambda}\,,$$
$$d_h(\overset{(k,0)}{\Lambda}) \subset \overset{(k,1)}{\Lambda}\,, \qquad d_v(\overset{(k,0)}{\Lambda}) \subset \overset{(k+1,0)}{\Lambda}\,.$$

PROOF. From the action of d_h, d_v on functions and local coordinate bases of forms. $\boxed{QED}$

2 Infinite order variational sequence

In this section, we introduce a new infinite order approach to variational sequences. This infinite order approach is based on the finite order approach by Krupka [9]. Indeed, as it is proved in [22], this infinite order approach turn out to be the direct limit of Krupka's finite order approach.

The de Rham exact sheaf sequence on $J_r\boldsymbol{Y}$ passes to direct limits. More precisely, it yields the following exact presheaf sequence

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda} \xrightarrow{\ d\ } \overset{1}{\Lambda} \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } \overset{k}{\Lambda} \xrightarrow{\ d\ } \dots$$

which is said to be the *infinite order de Rham sequence*. Of course, this sequence does not become trivial after a certain value of k.

Now, we introduce an exact natural subsequence of the de Rham sequence, which is of particular importance in the variational calculus, although being defined independently (see [9]).

We consider the restriction $\overset{k}{h}|_{\overset{k}{\Lambda}_r}$ of the projection $\overset{k}{h}$ to the subsheaf $\overset{k}{\Lambda}_r \subset \overset{k}{\Lambda}_{(r+1,r)}$. We introduce a new subsheaf of $\overset{k}{\Lambda}_r$. Namely, following Krupka [9, 10] we set $\overset{k}{\Theta}_r$ to be the sheaf generated (in the sense of [23]) by the presheaf $\ker \overset{k}{h}|_{\overset{k}{\Lambda}_r} + d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r}$. In other words, we set

$$\overset{k}{\Theta}_r := \ker \overset{k}{h}|_{\overset{k}{\Lambda}_r} + \overline{d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r}}\,.$$

Remark 2.1. Of course $\ker \overset{k}{h}|_{\overset{k}{\Lambda}_r}$ is a sheaf. But, in general, the gluing axiom fails to be true for $d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r}$. Anyway, in the particular case when $\dim \boldsymbol{X} = 1$ and $k > 1$, the sum $\ker \overset{k}{h}|_{\overset{k}{\Lambda}_r} + d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r}$ turns out to be a direct sum, and $d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r}$ turns out to be a sheaf. $\square$

Remark 2.2. If $0 \leq k \leq n$, then $d \ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r} \subset \ker \overset{k}{h}|_{\overset{k}{\Lambda}_r}$, so that $\overset{k}{\Theta}_r = \ker \overset{k}{h}|_{\overset{k}{\Lambda}_r}$. Moreover, we have

$$\ker \overset{k}{h}|_{\overset{k}{\Lambda}_r} = \{\alpha \in \overset{k}{\Lambda}_r \mid (j_r s)^* \alpha = 0 \text{ for every section } s : \boldsymbol{X} \to \boldsymbol{Y}\}\,.$$

This shows that for $k \leq n$ the sheaf $\overset{k}{\Theta}_r$ consists of forms which do not give contribution to action–like functionals [9, 15, 21]. □

Thus, we have the injective system of sheaves $\{\overset{k}{\Theta}_s, \pi^{r*}_s\}$. We define the presheaves on $\boldsymbol{Y}$

$$\overset{k}{\Theta} := \lim_{\to} \overset{k}{\Theta}_r\,.$$

It is clear that $\overset{k}{\Theta}$ is a subpresheaf of $\overset{k}{\Lambda}$. Thus, we say the following natural subsequence

$$0 \longrightarrow \overset{1}{\Theta} \xrightarrow{d} \overset{2}{\Theta} \xrightarrow{d} \dots \xrightarrow{d} \overset{k}{\Theta} \xrightarrow{d} \dots$$

to be the *infinite order contact subsequence* of the infinite order de Rham sequence.

Theorem 2.1. *The infinite order contact subsequence is exact.*

Proof. First, we observe that remark 1.1 still holds in the case of $\overset{k}{\Theta}$. So, to any $[\alpha] \in \overset{k}{\Theta}$ such that $\alpha \in \overset{k}{\Theta}_r$ we apply the *contact homotopy operator* [9], which is the restriction of the standard homotopy operator of Poincaré's lemma to $\overset{k}{\Theta}_r \subset \overset{k}{\Lambda}_r$ to find a local potential $[\beta] \in \overset{k-1}{\Theta}$ of $[\alpha]$. $\boxed{QED}$

Now, we introduce a bicomplex by quotienting the infinite order de Rham sequence by the infinite order contact subsequence. We obtain a new sequence, the *infinite order variational sequence*, which turns out to be exact.

Theorem 2.2. *The following diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta} \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta} \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{k}{\Theta} \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & I\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda} \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda} \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda} \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{k}{\Lambda} \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & I\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}/\overset{1}{\Theta} \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}/\overset{2}{\Theta} \arrow[r, "\mathcal{E}_2"] \arrow[d] & \dots \arrow[r, "\mathcal{E}_{k-1}"] & \overset{k}{\Lambda}/\overset{k}{\Theta} \arrow[r, "\mathcal{E}_k"] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 & & 0 &
\end{tikzcd}$$

where $\mathcal{E}_k$ are the quotient morphisms and the vertical arrows are natural inclusions or quotient projections, is commutative. Moreover, rows and columns are exact presheaf sequences.

PROOF. We have to prove only the exactness of the bottom row of the diagram. But this follows from the exactness of the other rows and of the columns. $\boxed{QED}$

Definition 2.1. We say the bottom row of the above diagram to be the *infinite order variational sequence* associated with the fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ (see [9]). $\square$

The above construction is yielded naturally just by the differential structure and the fibring of the underlying manifold. On the other hand our attention to the bottom line is inspired by the variational calculus.

It is possible to prove [22] that the infinite order variational sequence is the direct limit of the injective system of Krupka's finite order variational sequences. $\square$

We have an interesting result about the exactness of the infinite order variational sequence. Let us consider the cochain complex of global sections

$$0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \overset{0}{\Lambda}_{\boldsymbol{Y}} \xrightarrow{d} (\overset{1}{\Lambda}/\overset{1}{\Theta})_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_1} \dots \xrightarrow{\mathcal{E}_{k-1}} (\overset{k}{\Lambda}/\overset{k}{\Theta})_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_k} \dots$$

and denote by H^k_{IVS} the k–th cohomology group of the above cochain complex.

Corollary 2.1. *For all $k \geq 0$ there is a natural isomorphism*

$$H^k_{IVS} \simeq H^k_{de\ Rham}\boldsymbol{Y}$$

PROOF. This comes from the analogous result about the finite order variational sequence [9, 22]. $\boxed{QED}$

3 Representation of the variational sequence

In this section we provide a sequence which is isomorphic to the variational sequence and is more easily interpreted in terms of the calculus of variations.

First of all, we analyse the case $0 < k \leq n$.

Proposition 3.1. *Let $0 < k \leq n$. Then, we have the natural isomorphism*

$$I_k : \overset{k}{\Lambda}/\overset{k}{\Theta} \to \overset{(0,k)}{\Lambda} : [\alpha] \to \overset{k}{h}(\alpha)\,.$$

PROOF. By remark 2.2 the above map is well defined. Clearly, if $\overset{k}{h}(\alpha) = \overset{k}{h}(\beta)$, then $\alpha - \beta \in \ker \overset{k}{h} \equiv \overset{k}{\Theta}$, so the map is injective. Moreover, the map is surjective because $\overset{k}{h}$ is surjective on $\overset{(0,k)}{\Lambda}$. $\boxed{QED}$

Next, for $k > n$ we study the quotient spaces $\overset{k}{\Lambda}/\overset{k}{\Theta}$.

We denote by $\overline{d_h(\overset{(k-n,n-1)}{\Lambda}{}_r)}$ (see Introduction) the sheaf generated by the presheaf $d_h(\overset{(k-n,n-1)}{\Lambda}{}_r)$. This means that the sheaf $\overline{d_h(\overset{(k-n,n-1)}{\Lambda}{}_r)}$ consists of sections α which are of the local type $\alpha = d_h\beta$ with $\beta \in \overset{(k-n,n-1)}{\Lambda}{}_r$.

Moreover, we set

$$\overline{d_h}(\overset{(k-n,n-1)}{\Lambda}) := \lim_{\rightarrow} \overline{d_h(\overset{(k-n,n-1)}{\Lambda}{}_r)}\,,$$

where we introduced the symbol $\overline{d_h}$ for evident practical reasons.

Lemma 3.1. *Let $k > n$. Then, we have the isomorphism*

$$\overset{k}{h}(\overset{k}{\Theta}) \to \overline{d_h}(\overset{(k-n,n-1)}{\Lambda}) : \gamma \mapsto \overset{k}{h}(\gamma)\,.$$

Proof. It comes from the following inclusions

$$\overline{\overset{k}{h}(d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_r})} \subset \overline{d_h(\overset{(k-n,n-1)}{\Lambda}{}_{r+1})} \subset \overline{\overset{k}{h}(d\ker \overset{k-1}{h}|_{\overset{k}{\Lambda}_{r+1}})}\,. \quad \boxed{QED}$$

Proposition 3.2. *Let $k > n$. Then, we have the natural isomorphisms*

$$\tilde{I}_k : \overset{k}{\Lambda}/\overset{k}{\Theta} \to \overset{(k-n,n)}{\Lambda}\big/\overline{d_h}(\overset{(k-n,n-1)}{\Lambda}) : [\alpha] \to [\overset{k}{h}(\alpha)]\,.$$

Theorem 3.1. *The infinite order variational sequence is isomorphic to the following sequence*

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda} \xrightarrow{\tilde{\mathcal{E}}_0} \overset{(0,1)}{\Lambda} \xrightarrow{\tilde{\mathcal{E}}_1} \dots \xrightarrow{\tilde{\mathcal{E}}_{n-1}} \overset{(0,n)}{\Lambda} \xrightarrow{\tilde{\mathcal{E}}_n}$$
$$\xrightarrow{\tilde{\mathcal{E}}_n} \overset{(1,n)}{\Lambda}\big/\overline{d_h}(\overset{(1,n-1)}{\Lambda}) \xrightarrow{\tilde{\mathcal{E}}_{n+1}} \dots \xrightarrow{\tilde{\mathcal{E}}_{n+i-1}} \overset{(i,n)}{\Lambda}\big/\overline{d_h}(\overset{(1,n-1)}{\Lambda}) \xrightarrow{\tilde{\mathcal{E}}_{n+i}} \dots$$

where $\tilde{\mathcal{E}}_k = d_h$ if $0 \le k \le n-1$, and $\tilde{\mathcal{E}}_k([\alpha]) = [d_v(\alpha)]$ if $k > n$.

Proof. If $0 \le k \le n-1$ then we have

$$\tilde{\mathcal{E}}_k(\overset{k}{h}(\alpha)) = I_k(\mathcal{E}_k([\alpha])) = I_k([d\alpha]) = I_k([(d_h + d_v)(\overset{k}{h}(\alpha) + \overset{k}{v}(\alpha))]) = d_h\overset{k}{h}(\alpha)\,,$$

where the last passage is due to proposition 1.3.

If $k \ge n$ then we have

$$\tilde{\mathcal{E}}_k([\overset{k}{h}(\alpha)]) = I_k(\mathcal{E}_k([\alpha])) = I_k([d\alpha]) = [\overset{k}{h}(d\alpha)]\,,$$

and

$$\overset{k}{h}(d\alpha) = \overset{k}{h}((d_h + d_v)(\overset{k}{h}(\alpha) + \overset{k}{v}(\alpha))) = d_v(\overset{k}{h}(\alpha)) + d_h(\overset{k}{v}(\alpha))\,,$$

hence the result. $\boxed{QED}$

4 Representation of the 'shortened' variational sequence

As far as we know, there is no interpretation of the k–th degree terms of the variational sequence in terms of the calculus of variations, for $k \geq n+3$. For this reason, we restrict our interest to a 'shortened' version of the representation of the variational sequence of the previous section. Namely, we consider the subsequence

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda} \xrightarrow{d_h} \overset{(0,1)}{\Lambda} \xrightarrow{d_h} \overset{(0,2)}{\Lambda} \xrightarrow{d_h} \dots$$

$$\dots \xrightarrow{\quad d_h \quad} \overset{(0,n)}{\Lambda} \xrightarrow{\tilde{\mathcal{E}}_n} \overset{(1,n)}{\Lambda} \Big/ \overline{d_h}(\overset{(1,n-1)}{\Lambda}) \xrightarrow{\tilde{\mathcal{E}}_{n+1}} \tilde{\mathcal{E}}_{n+1}(\overset{(1,n)}{\Lambda} \Big/ \overline{d_h}(\overset{(1,n-1)}{\Lambda})) \longrightarrow 0\,,$$

of the infinite order variational sequence.

The task of the next subsections is to give a natural isomorphism between the two quotient spaces of the above 'shortened' variational sequence and some presheaves of forms on jet spaces. In this way, we are able to give explicit coordinate expressions for the morphisms $\tilde{\mathcal{E}}_n$ and $\tilde{\mathcal{E}}_{n+1}$.

Euler morphism

Here we find an isomorphism of $\overset{(1,n)}{\Lambda} \Big/ \overline{d_h}(\overset{(1,n-1)}{\Lambda})$ with a direct limit of an injective system of sheaves of forms on jet bundles. To this aim, we use a result by Kolář [7].

To proceed further, we introduce new notation and recall a few results from the theory of jets [12].

On any coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$ (with coordinates adapted to the fibring) we set

$$\epsilon := d^1 \wedge \dots \wedge d^n\,, \qquad \epsilon_\lambda := i_{\partial_\lambda}\epsilon\,.$$

We recall the natural inclusion $i_{r,s} : J_{r+s}\boldsymbol{Y} \to J_r J_s \boldsymbol{Y}$ which is characterised by $i_{r,s} \circ j_{r+s}t = j_r(j_s t)$ for any section $t : \boldsymbol{X} \to \boldsymbol{Y}$.

Now, consider the fibrings $J_r\boldsymbol{Y} \to \boldsymbol{X}$ and $\mathrm{pr}_1 : \boldsymbol{X} \times I\!R \to \boldsymbol{X}$. For $s \geq 0$ there is the well–known isomorphism $J_s(\boldsymbol{X} \times I\!R) \simeq T^*_s\boldsymbol{X} \times I\!R$, where $T^*_s\boldsymbol{X}$ is the s–th order cotangent bundle of $\boldsymbol{X}$. Let $f : J_r\boldsymbol{Y} \to I\!R$ be a map. Then we define the *formal derivative* of f to be the function

$$\mathcal{D}_{(s)}f := J_s f \circ i_{s,r} : J_{s+r}\boldsymbol{Y} \to T^*_s\boldsymbol{X} \times I\!R\,.$$

Let $(z_{\underline{p}})$ be local coordinates on $T^*_s\boldsymbol{X}$, $0 \leq |\underline{p}| \leq s$. Then we set

$$\mathcal{D}_{\underline{p}}f := z_{\underline{p}} \circ \mathcal{D}_{(s)}f\,.$$

The definition of prolongation yields $\mathcal{D}_{\underline{p}}f \circ j_{r+|\underline{p}|}s = \partial_{\underline{p}}(f \circ j_r s)$; of course, this equality uniquely characterises $\mathcal{D}_{\underline{p}}$. We can easily verify that $\mathcal{D}_{\underline{p}} \circ \mathcal{D}_{\underline{q}} = \mathcal{D}_{\underline{p}+\underline{q}}$. In the particular case when $|\underline{p}| = 1$ (so that we can identify $\underline{p} = \lambda$) then we have the coordinate expression

$$\mathcal{D}_\lambda f = (\text{д}_{r+1})_\lambda . f = \partial_\lambda f + y^i_{\underline{q}+\lambda}\partial^{\underline{q}}_i f \qquad 0 \leq |\underline{q}| \leq s\,,$$

which coincides with the standard first order formal derivative expression. The coordinate expression of $\mathcal{D}_{\underline{p}} f$ can be easily derived from the inductive formula $\mathcal{D}_{\underline{p}+\lambda} = \mathcal{D}_\lambda \mathcal{D}_{\underline{p}}$. A Leibnitz' rule holds for $\mathcal{D}_{\underline{p}}$ (see [15]); if $g \in \overset{0}{\Lambda}_r$, then we have

$$\mathcal{D}_{\underline{p}}(fg) = \sum_{\underline{q}+\underline{t}=\underline{p}} \frac{\underline{p}!}{\underline{q}!\underline{t}!} \mathcal{D}_{\underline{q}} f \, \mathcal{D}_{\underline{t}} g \, .$$

If a vertical vector field $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ has the expression $u = u^i \partial_i$, then its natural prolongation $u_r : J_r\boldsymbol{Y} \to VJ_r\boldsymbol{Y}$ has the expression $u_r = \mathcal{D}_{\underline{p}} u^i \partial_i^{\underline{p}}$.

Theorem 4.1. *(First variation formula [7]) Let* $\alpha \in \overset{1}{\Lambda}_r \wedge \overset{(0,n)}{\Lambda}_r \subset \overset{(1,n)}{\Lambda}_r$. *Then there is a unique pair of elements*

$$E_\alpha \in \overset{(1,0)}{\Lambda}_{(2r,0)} \wedge \overset{(0,n)}{\Lambda}_{2r} \, , \qquad F_\alpha \in \overset{(1,0)}{\Lambda}_{(2r,r)} \wedge \overset{(0,n)}{\Lambda}_{2r} \, ,$$

such that

i. $(\pi^{2r}_r)^* \alpha = E_\alpha - F_\alpha$;

ii. F_α *is locally of the form* $F_\alpha = d_h p_\alpha$, *with* $p_\alpha \in \overset{(1,0)}{\Lambda}_{(2r-1,r-1)} \wedge \overset{(0,n)}{\Lambda}_{2r}$.

Remark 4.1. Thus, E_α and F_α are uniquely defined. However, it is possible to determine a global p_α fulfilling the above conditions, but p_α is not uniquely determined unless $\dim \boldsymbol{X} = 1$ or $r = 1$. For $r = 2$, we are able to characterise a unique p_α by means of an additional requirement (see [7] for a complete discussion). $\square$

In coordinates, if $\alpha = \alpha_i^{\underline{p}} \vartheta^i_{\underline{p}} \wedge \epsilon$, then we have the well–known expression

$$E_\alpha = (-1)^{|\underline{p}|} \mathcal{D}_{\underline{p}} \alpha_i^{\underline{p}} \vartheta^i \wedge \epsilon \, . \tag{2}$$

Proposition 4.1. *We have the injective sheaf morphism*

$$I_{n+1} : \overset{(1,n)}{\Lambda} / \overline{d_h}(\overset{(1,n-1)}{\Lambda}) \to \overset{n+1}{\Lambda} : [\alpha] \mapsto E_\alpha \, .$$

Proof. The morphism I_{n+1} is well–defined. In fact, it is easily seen that it does not depend on the representative of the equivalence class $[\alpha]$. Moreover, due to the uniqueness of the decomposition in the first variation formula, E_α annihilates sections $[\alpha] \in \overline{d_h}(\overset{(1,n-1)}{\Lambda})$.

The morphism is also injective. In fact, suppose that $E_\alpha = E_\beta$. Then by the first variation formula we have $\alpha - \beta = F_\beta - F_\alpha$, hence $[\alpha - \beta] = 0$. QED

The final step is to characterise the image of I_{n+1}. Let us define the following presheaf

$$\overset{(1,0)}{\Lambda}_{(\cdot,0)} := \lim_{\rightarrow} \overset{(1,0)}{\Lambda}_{(r,0)} \, .$$

The claimed result is given by the following theorem.

Theorem 4.2. *We have the sheaf isomorphism*

$$I_{n+1} : \overset{(1,n)}{\Lambda} / \overline{d_h}(\overset{(1,n-1)}{\Lambda}) \to \boldsymbol{\mathcal{E}},$$

where $\boldsymbol{\mathcal{E}}$ *is the presheaf*

$$\boldsymbol{\mathcal{E}} := \overset{(1,0)}{\Lambda}_{(\cdot,0)} \wedge \overset{(0,n)}{\Lambda} .$$

PROOF. The image of I_{n+1} is characterised by the first variation formula. Namely, we have

$$\boldsymbol{\mathcal{E}} = (\overset{(1,n)}{\Lambda} + d_h(\overset{(1,n-1)}{\Lambda})) \cap (\overset{(1,0)}{\Lambda}_{(\cdot,0)} \wedge \overset{(0,n)}{\Lambda}) .$$

But we have the inclusion $(\overset{(1,0)}{\Lambda}_{(\cdot,0)} \wedge \overset{(0,n)}{\Lambda}) \subset (\overset{(1,n)}{\Lambda} + d_h(\overset{(1,n-1)}{\Lambda}))$, hence the result. $\boxed{QED}$

Helmholtz morphism

Here we find an isomorphism of $\tilde{\mathcal{E}}_{n+1}(\overset{(1,n)}{\Lambda} / \overline{d_h}(\overset{(1,n-1)}{\Lambda}))$ with a direct limit of an injective system of sheaves of forms on jet bundles. To this aim, we make use of the second variation formula [21].

Lemma 4.1. *We have the natural injection*

$$\tilde{\mathcal{E}}_{n+1}(\overset{(1,n)}{\Lambda} / \overline{d_h}(\overset{(1,n-1)}{\Lambda})) \to \overset{(2,n)}{\Lambda} / \overline{d_h}(\overset{(2,n-1)}{\Lambda}) : [d_v \alpha] \mapsto [d_v E_{h(\alpha)}] .$$

PROOF. It is a direct consequence of the first variation formula and $d_v d_h = -d_h d_v$. $\boxed{QED}$

Lemma 4.2. *Let* $\beta \in \overset{(1,0)}{\Lambda}_s \wedge \overset{(1,0)}{\Lambda}_{(s,0)} \wedge \overset{(0,n)}{\Lambda}_s$. *Then, there is a unique element*

$$\tilde{H}_\beta \in \overset{(1,0)}{\Lambda}_{(2s,s)} \otimes \overset{(1,0)}{\Lambda}_{(2s,0)} \wedge \overset{(0,n)}{\Lambda}_{2s}$$

such that, for all $u : \boldsymbol{Y} \to V\boldsymbol{Y}$, *we have*

$$E_{\hat{\beta}} = u_{2s} \lrcorner \tilde{H}_\beta ,$$

where $\hat{\beta} := i_{u_s}\beta$.

PROOF. Let $\boldsymbol{U} \subset \boldsymbol{Y}$ be an open coordinate subset, and suppose that we have the expression on $\boldsymbol{U}$

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \epsilon , \qquad 0 \le |\underline{p}| \le s .$$

Then we have the coordinate expression

$$E_{\hat{\beta}} = \mathcal{D}_{\underline{p}} u^i \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} \mathcal{D}_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\;\;i} \right) \vartheta^j \wedge \epsilon \,.$$

Let us set

$$\tilde{H}_\beta := \left(\beta^{\underline{p}}_{i\,j} - \sum_{|\underline{q}|=0}^{s-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|} \frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!} \mathcal{D}_{\underline{q}} \beta^{\underline{p}+\underline{q}}_{j\;\;i} \right) \vartheta^i_{\underline{p}} \otimes \vartheta^j \wedge \epsilon \,.$$

Then, by the arbitrariness of u, $\tilde{H}_\beta$ is the unique morphism fulfilling the conditions of the statement on $\boldsymbol{U}$. By this uniqueness, we deduce that $\tilde{H}_\beta$ is intrinsic. QED

Theorem 4.3. *(Second variation formula [21]).*
Let $\beta \in \overset{(1,0)}{\Lambda}_s \wedge \overset{(1,0)}{\Lambda}_{(s,0)} \wedge \overset{(0,n)}{\Lambda}_s$. *Then, there is a unique pair of elements*

$$H_\beta \in \overset{(1,0)}{\Lambda}_{(2s,s)} \wedge \overset{(1,0)}{\Lambda}_{(2s,0)} \wedge \overset{(0,n)}{\Lambda}_{2s} \,, \quad G_\beta \in \overset{(2,0)}{\Lambda}_{(2s,s)} \wedge \overset{(0,n)}{\Lambda}_{2s} \,,$$

such that

i. $\pi^{2s\,*}_s \beta = H_\beta - G_\beta$

ii. $H_\beta = 1/2\, A(\tilde{H}_\beta)$*, where A is the antisymmetrisation map.*

Moreover, G_β is locally of the type $G_\beta = d_h q_\beta$, where $q_\beta \in \overset{(2,n-1)}{\Lambda}_{2s-1}$.

Proof. It is clear that G_β is uniquely determined by β and the choice $H_\beta = 1/2\, A(\tilde{H}_\beta)$.

Let us denote by $L_{\mathcal{D}_\lambda}$ the Lie derivative with respect to the field $(д_{r+1})_\lambda$. We denote by $L_{\mathcal{D}_{\underline{p}}}$ the iterated Lie derivative. It can be easily seen [15] by induction on $|\underline{p}|$ that, on a coordinate open subset $\boldsymbol{U} \subset \boldsymbol{Y}$, we have

$$\beta = \beta^{\underline{p}}_{i\,j} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \epsilon = \beta^{\underline{p}}_{i\,j} L_{\underline{p}}(\vartheta^i) \wedge \vartheta^j \wedge \epsilon = (-1)^{|\underline{p}|} \vartheta^i \wedge L_{\underline{p}}(\beta^{\underline{p}}_{i\,j} \vartheta^j) \wedge \epsilon + 2 d_h q_\beta \,,$$

which yields the thesis by the Leibnitz' rule. QED

Remark 4.2. Thus, H_β and G_β are uniquely defined. However, in general, we do not know whether it is possible to determine a global q_β fulfilling the above conditions. If $\dim \boldsymbol{X} = 1$, then there exists a unique q_β fulfilling the above conditions. Moreover, if $s = 2$, we are able to characterise a unique q_β by means of an additional requirement [8]. □

Proposition 4.2. *We have the injective morphism*

$$I_{n+2} : \tilde{\mathcal{E}}_{n+1}(\overset{(1,n)}{\Lambda} / \overline{d_h}(\overset{(1,n-1)}{\Lambda})) \to \overset{(1,0)}{\Lambda} \wedge \overset{(1,0)}{\Lambda}_{(\cdot,0)} \wedge \overset{(0,n)}{\Lambda} : [d_v \alpha] \mapsto H_{d_v E_\alpha} \,.$$

PROOF. I_{n+2} is well defined due to the uniqueness of the decomposition of the second variation formula. The injectivity of I_{n+2} follows from the above theorem, because if $d_v E_\alpha$ and $d_v E_\beta$ fulfill $H_{d_v E_\alpha} = H_{d_v E_\beta}$, then we have (locally)

$$d_v E_\alpha - d_v E_\beta = G_{d_v E_\beta} - G_{d_v E_\alpha} \,. \quad \boxed{\scriptstyle QED}$$

Let us set $\boldsymbol{\mathcal{H}} := \operatorname{im} I_{n+2}$. We have no characterisation of $\boldsymbol{\mathcal{H}}$. But the above proposition allows us to select a distinguished presheaf containing $\boldsymbol{\mathcal{H}}$. More precisely, we can state the following theorem.

Theorem 4.4. *The sheaf* $\tilde{\mathcal{E}}_{n+1}(\overset{(1,n)}{\Lambda}/\overline{d_h}(\overset{(1,n-1)}{\Lambda}))$ *is isomorphic to the image*

$$\boldsymbol{\mathcal{H}} \subset \overset{(1,0)}{\Lambda} \wedge \overset{(1,0)}{\Lambda}_{(\cdot,0)} \wedge \overset{(0,n)}{\Lambda}$$

of the injective morphism I_{n+2}.

5 Variational sequence and the calculus of variations

We can summarise the results of the above sections in the following theorem. Let us set $\boldsymbol{\mathcal{L}} := \overset{(0,n)}{\Lambda}$.

Theorem 5.1. *The shortened infinite order variational sequence is isomorphic to the exact sequence*

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda} \xrightarrow{d_h} \overset{(0,1)}{\Lambda} \xrightarrow{d_h} \dots \xrightarrow{d_h} \overset{(0,n-1)}{\Lambda} \xrightarrow{d_h} \boldsymbol{\mathcal{L}} \xrightarrow{\mathcal{E}} \boldsymbol{\mathcal{E}} \xrightarrow{\mathcal{H}} \boldsymbol{\mathcal{H}} \longrightarrow 0\,,$$

where the maps $\mathcal{E}$ *and* $\mathcal{H}$ *are defined as the maps which make the following diagram commuting*

$$\begin{tikzcd}
\boldsymbol{\mathcal{L}} \arrow[r, "\mathcal{E}_n"] \arrow[dr, "\mathcal{E}"] & \overset{n+1}{\Lambda}/\overset{n+1}{\Theta} \arrow[r, "\mathcal{E}_{n+1}"] \arrow[d, "I_{n+1}\circ\tilde{I}_{n+1}"] & \mathcal{E}_{n+1}(\overset{n+1}{\Lambda}/\overset{n+1}{\Theta}) \arrow[r] \arrow[d, "I_{n+2}\circ\tilde{I}_{n+2}"] & 0 \\
 & \boldsymbol{\mathcal{E}} \arrow[r, "\mathcal{H}"] & \boldsymbol{\mathcal{H}} \arrow[r] & 0
\end{tikzcd}$$

Remark 5.1. The natural representation of the quotient sequence as a sequence of sheaves of 'concrete' forms yields a clear interpretation in terms of the calculus of variations. □

We have the following coordinate expressions.
If $L \in \boldsymbol{\mathcal{L}}$, with $L = f\,\epsilon$, then

$$\mathcal{E}(L) = (-1)^{|\underline{p}|}\mathcal{D}_{\underline{p}}\partial_i^{\underline{p}} f\,\vartheta^i \wedge \epsilon\,.$$

If $E \in \boldsymbol{\mathcal{E}}$, with $E = E_i\,\vartheta^i \wedge \epsilon$, then

$$\mathcal{H}(E) = \frac{1}{2}\left(\partial_i^{\underline{p}} E_j - \sum_{|\underline{q}|=0}^{2r+1-|\underline{p}|} (-1)^{|\underline{p}+\underline{q}|}\frac{(\underline{p}+\underline{q})!}{\underline{p}!\underline{q}!}\mathcal{D}_{\underline{q}}\partial_j^{\underline{p}+\underline{q}} E_i\right)\vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge \epsilon\,.$$

We say $L \in \boldsymbol{\mathcal{L}}$ to be a *Lagrangian type morphism*. We observe that, due to the exactness of the variational sequence, a variationally trivial Lagrangian is locally of the form $d_h\alpha$, where $\alpha \in \overset{(0,n-1)}{\Lambda}$.

We say the map $\mathcal{E} : \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{E}}$ to be the *Euler operator*. We say $\mathcal{E}(L) \in \boldsymbol{\mathcal{E}}$ to be the *Euler morphism associated with* L. We say $E \in \boldsymbol{\mathcal{E}}$ to be an *Euler type morphism.*

We say the map $\mathcal{H} : \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{H}}$ to be the *Helmholtz operator*. We say $\mathcal{H}(E) \in \boldsymbol{\mathcal{H}}$ to be the *Helmholtz morphism associated with* E. We observe that, due to the exactness of the variational sequence, if $\mathcal{H}(E) = 0$ then there exists (locally) a Lagrangian L such that $\mathcal{E}(L) = 0$. Moreover, if $H^{n+1}_{\text{de Rham}}(\boldsymbol{Y}) = 0$, then it is possible to find a global Lagrangian fulfilling the above condition.

We recall from the above section that

$$\begin{aligned} &I_{n+1}\circ\tilde{I}_{n+1}([\alpha]) = E_{\hat{\alpha}}\,,\\ &I_{n+2}\circ\tilde{I}_{n+2}(\mathcal{E}_{n+1}([\alpha])) = H_{d_v(E_{\hat{\alpha}})}\,, \end{aligned}$$

where $\hat{\alpha} := \overset{n+1}{h}(\alpha)$.

We say $p_{\hat{\alpha}}$ (see theorem 4.1) to be a (local) *momentum* associated with the Euler morphism induced by α.

Let $L \in \boldsymbol{\mathcal{L}}$ be a Lagrangian type morphism. Then we say $\theta := L + p_{dL}$ to be a *Poincaré–Cartan form* associated with the Lagrangian L. It is evident that the well–known problem of the uniqueness of the Poincaré–Cartan form is equivalent to the problem of the uniqueness of the momentum for dL.

We say $q_{d_vE_{\hat{\alpha}}}$ (see theorem 4.3) to be a (local) *momentum* associated with the Helmholtz morphism induced by α.

Remark 5.2. Our names given to the above objects ($q_{d_vE_{\hat{\alpha}}}$ excepted) are justified by the fact that this objects turn out to be just the homonymous objects of the standard calculus of variations on fibred manifolds. As for $q_{d_vE_{\hat{\alpha}}}$, it is a new object introduced in [21] (see also [8]) whose interpretation in terms of the calculus of variaitons is still unknown. □

Remark 5.3. In the direct approach to Lagrangian formalism one starts with a Lagrangian $L \in \boldsymbol{\mathcal{L}}$ and fills in the further vertices of the bicomplex (in the direction bottom-up, left-right) by means of the maps of the variational sequence and by the surjectivity of the projections. Of course, the objects in the center and top row need not to be unique.

In the inverse approach to Lagrangian formalism one starts with a Euler type morphism $E \in \mathcal{E}$ and finds, under the Helmholtz closure condition, a local Lagrangian, which is defined up to the horizontal differential of a form $\alpha \in \overset{(0,n-1)}{\Lambda}$. Clearly, this form yields the filling in procedure as in the direct case; but, now, some objects are defined up to a gauge.

In [13] we studied the Lagrangian formalism for the mechanics of one particle associated with a geometric model of Galilei spacetime. Namely, a metric, a connection and a spacetime 2–form yield directly a global *dynamical* 2*–form* $\omega \in \overset{2}{\Lambda}$ and a global $E \in \mathcal{E}$. Thus, we are able to fill in the bicomplex starting equivalently with ω or E. Therefore, the objects recovered on left (Lagrangian, Poincaré–Cartan form and momentum) are defined only locally and up to a gauge. We proved that this approach is of fundamental importance for the quantisation of mechanics. We hope that it could be of the same importance for the quantisation of fields. □

Appendix: direct sums and exterior products

Let V be a vector space such that $\dim V = n$. We define the *box product* (see also [6]) of r linear morphisms $a_1, \ldots, a_r : V \to V$ is defined to be the linear map

$$\overset{r}{\Box} a_i : \overset{r}{\wedge} V \to \overset{r}{\wedge} V :$$
$$v_1 \wedge \ldots \wedge v_r \mapsto \sum_{\sigma \in S_r} |\sigma| a_1(v_{\sigma(1)}) \wedge \ldots \wedge a_r(v_{\sigma(r)}) \,.$$

where S_r is the set of all permutation of order r. The box product fulfills

$$\overset{r}{\Box} a_i = \overset{r}{\Box} a_{\sigma(i)} \qquad \forall\, \sigma \in S_r \,;$$

in particular, if $a_1 = \cdots = a_r = a$, then $\overset{r}{\Box} a = r! \overset{r}{\wedge} a$. So, $\overset{r}{\Box}$ yields a map $\overset{k}{\odot}(\mathrm{End}(V)) \to \mathrm{End}(\overset{k}{\wedge} V)$.

We have a remarkable feature of the box product. Suppose that $V = W_1 \oplus W_2$, with $p_1 : V \to W_1$ and $p_2 : V \to W_2$ the related projections. Then, we have the splitting

$$\overset{m}{\wedge} V = \bigoplus_{k+h=m} \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \,, \tag{3}$$

where $\overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2$ is the subspace of $\overset{m}{\wedge} V$ generated by the wedge products of elements of $\overset{k}{\wedge} W_1$ and $\overset{h}{\wedge} W_2$. The projections related to the above splitting turn out to be the maps

$$\overset{(k,h)}{\Box} (p_1, p_2) : \overset{m}{\wedge} V \to \overset{k}{\wedge} W_1 \wedge \overset{h}{\wedge} W_2 \,,$$

where $\overset{(k,h)}{\square}(p_1, p_2) := \frac{1}{k!h!} \overset{k+h}{\square} a_i$, with $a_i = p_1$ if $1 \leq i \leq k$ and $a_i = p_2$ if $k+1 \leq i \leq k+h$.

Remark 5.4. Let $V' \subset V$ be a vector subspace, and set $W'_1 := p_1(V')$, $W'_2 := p_2(V')$. Then we have

$$V' \subset W'_1 \oplus W'_2 \,,$$

but the inclusion, in general, is not an equality.

References

[1] I. M. Anderson and T. Duchamp, On the existence of global variational principles, *Amer. Math. J.*, **102** (1980), 781–868.

[2] M. Bauderon, Le problème inverse du calcul des variations, *Ann. de l'I.H.P.*, **36**, n. 2 (1982), 159–179.

[3] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, GTM **82** (Springer–Verlag, Berlin, 1982).

[4] P. Dedecker and W. M. Tulczyjew, Spectral sequences and the inverse problem of the calculus of variations, in Internat. Coll. on Diff. Geom. Methods in Math. Phys., Aix–en–Provence, 1979; Lecture Notes in Mathematics **836** (Springer–Verlag, Berlin, 1980), 498–503.

[5] M. Ferraris and M. Francaviglia, Global Formalism in Higher Order Calculus of Variations, *Diff. Geom. and its Appl., Part II*, Proc. of the Conf. (University J. E. Purkyne, Brno, 1984), 93–117.

[6] W. Greub, *Multilinear Algebra* (Springer–Verlag, 1978).

[7] I. Kolář, A geometrical version of the higher order Hamilton formalism in fibred manifolds, *Jour. Geom. Phys.* **1**, n. 2 (1984), 127–137.

[8] I. Kolář and R. Vitolo, On the Helmholtz operator for Euler morphisms, preprint 1997.

[9] D. Krupka, Variational sequences on finite order jet spaces, *Diff. Geom. and its Appl.*, Proc. of the Conf. (World Scientific, New York, 1990) 236–254.

[10] D. Krupka, Topics in the calculus of variations: finite order variational sequences, *Diff. Geom. and its Appl.*, Proc. of the Conf., Opava (Czech Republic), (1993) 473–495.

[11] B. A. Kuperschmidt, Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalism, Lecture Notes in Math. **775**: Geometric Methods in Mathematical Physics (Springer, Berlin, (1980)) 162–218.

[12] L. Mangiarotti and M. Modugno, Fibered Spaces, Jet Spaces and Connections for Field Theories, *Int. Meet. on Geometry and Physics*, Proc. of the Conf. (Pitagora Editrice, Bologna, 1983) 135–165.

[13] M. Modugno, R. Vitolo: Quantum connection and Poincaré–Cartan form, *Conference in honour of A. Lichnerowicz*, Frascati, ottobre 1995; ed. G. Ferrarese, Pitagora, Bologna.

[14] P. J. Olver, C. Shakiban, A Resolution of the Euler Operator, *Proc. Am. Math. Soc.*, **69** (1978), 223–229.

[15] D. J. Saunders, *The Geometry of Jet Bundles* (Cambridge Univ. Press, 1989).

[16] F. Takens, A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979), 543–562.

[17] W. M. Tulczyjew, The Lagrange Complex, *Bull. Soc. Math. France* **105** (1977), 419–431.

[18] W. M. Tulczyjew, The Euler–Lagrange Resolution, *Internat. Coll. on Diff. Geom. Methods in Math. Phys.*, Aix–en–Provence, 1979; Lecture Notes in Mathematics **836** (Springer–Verlag, Berlin, 1980), 22–48.

[19] A. M. Vinogradov, On the algebro–geometric foundations of Lagrangian field theory, *Soviet Math. Dokl.* **18** (1977), 1200–1204.

[20] A. M. Vinogradov, A spectral sequence associated with a non–linear differential equation, and algebro–geometric foundations of Lagrangian field theory with constraints, *Soviet Math. Dokl.* **19** (1978), 144–148.

[21] R. Vitolo, Finite order Lagrangian bicomplexes, *Math. Proc. of the Camb. Phil. Soc.*, to appear **124** n. 3, 1998.

[22] R. Vitolo, On different geometric formulations of Lagrangian formalism, preprint 1997, to appear on *Diff. Geom. and Appl.*.

[23] R. O. Wells, *Differential Analysis on Complex Manifolds*, GTM **65** (Springer–Verlag, Berlin, 1980).