# Bi-Hamiltonian structure of the Oriented Associativity Equation

Maxim V. Pavlov[1], Raffaele F. Vitolo[2]
[1]Lebedev Physical Institute of Russian Academy of Sciences,
Moscow, Leninskij Prospekt, 53;
[2]Department of Mathematics and Physics,
University of Salento, Lecce, Italy
and INFN, Section of Lecce
http://poincare.unisalento.it/vitolo

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**Abstract**

The Oriented Associativity equation plays a fundamental role in the theory of Integrable Systems. In this paper we prove that the equation, besides being Hamiltonian with respect to a first-order Hamiltonian operator, has a third-order non-local homogeneous Hamiltonian operator belonging to a class which has been recently studied, thus providing a highly non-trivial example in that class and showing intriguing connections with algebraic geometry.

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## 1 Introduction

The Associativity equation, or Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation, plays a fundamental role in the geometric theory of Integrable Systems. Its solutions define Frobenius manifolds, which correspond to integrable systems; Frobenius manifolds are an important part of the theory of quantum cohomology and Gromov–Witten invariants. These relationships were shown by B. Dubrovin in his seminal paper [3].

The nonlinear partial differential system of equations

$$\frac{\partial^2 c^i}{\partial a^j \partial a^m}\frac{\partial^2 c^m}{\partial a^k \partial a^n} = \frac{\partial^2 c^i}{\partial a^k \partial a^m}\frac{\partial^2 c^m}{\partial a^j \partial a^n} \tag{1}$$

on $N$ unknown functions $(c^i)$ of $N$ independent variables $(a^j)$ was introduced in [13] as a generalization of the Associativity equations. Its solutions define $F$-manifolds, which are still in correspondence with integrable systems. The far-reaching implication of this generalization are an active subject of study: flat and bi-flat $F$-manifolds have interesting connections with Painlevé equations [1, 18, 19]; see also the papers [15, 16]) devoted to coisotropic deformations. We call the system (1) the *Oriented Associativity equation*.

The Oriented Associativity equation admits the scalar linear spectral problem

$$\frac{\partial^2 h}{\partial a^i \partial a^j} = \lambda \frac{\partial^2 c^m}{\partial a^i \partial a^j}\frac{\partial h}{\partial a^m} \tag{2}$$

(see, for instance, [3, 23]) that ensure that the equation is integrable as it possesses a Lax pair.

We observe that the Associativity equation [3] can be obtained from (1) by the potential reduction $c^i = \eta^{im}\partial F/\partial a^m$, where $\eta^{ks}$ is a constant nondegenerate symmetric matrix.

In this paper, we will *prove the existence of a bi-Hamiltonian formalism for the Oriented Associativity equation* (1) in the case $N = 3$.

The above result has strong analogies with the known results on the Associativity (WDVV) equation. Indeed, the Associativity equation can be written as $N-2$ commuting hydrodynamic-type systems of conservation laws [9]. For $N = 3$ the (only) system was shown to be bi-Hamiltonian in [8]. Further investigations shown a similar situation in the case $N = 4$: the Hamiltonian operators were, in both cases, a first-order homogeneous operator (found in [9] for $N = 4$) and a third-order homogeneous operator (found in [24] for $N = 4$). First-order homogeneous operators [4] can be written as $A_1^{ij} = g^{ij}\partial_x$, where $(g^{ij})$ is a constant matrix, in a suitable coordinate system; third-order homogeneous operators [5] have a more complicated structure, and can be brought to the form

$$A_2^{ij} = \partial_x \left( g^{ij}\partial_x + c^{ij}_k u^k_x \right) \partial_x, \tag{3}$$

where $u^k$ are dependent variables. Such operators have been extensively studied quite recently [10, 11].

A first-order homogeneous Hamiltonian operator for the simplest case ($N = 3$) of Oriented Associativity equation (written in the form of a hydrodynamic-type system) was found [23] with the same method as in the Associativity case. It was natural to conjecture that a third-order homogeneous Hamiltonian operator might exist.

In this paper, we will *prove that the Oriented Associativity equation in hydrodynamic-type form admits a* ***non-local*** *third-order homogeneous Hamiltonian operator* of a class that was recently introduced by M. Casati, E.V. Ferapontov and the authors of this paper in [2]. We stress that the proof is achieved in the simplest case $N = 3$ only.

The significance of the result is high: indeed, it is known that the Associativity equations (in hydrodynamic form) in the cases $N = 3$ and $N = 4$ discussed above correspond to linear line congruences, which are algebraic varieties in the Plücker variety all lines of a projective space [12]. Their third-order Hamiltonian operators correspond to quadratic line complexes, which are algebraic varieties of lines in a projective space of different dimension with respect to the previous lines [10, 11].

The Oriented Associativity equation (in hydrodynamic form) can also be interpreted as a line congruence, even if we still do not know if the congruence is linear. The third-order non-local homogeneous Hamiltonian operator that we find as the main result in this paper also defines a quadratic line complex. It is thus clear that the strong links between the Associativity equation and projective-geometric varieties are preserved for the Oriented Associativity equation. We believe that such structures play an important role in the rich geometry of such equations, with lots of interesting Mathematics yet to be discovered.

The computation related to finding the non-local Hamiltonian operator is highly non-trivial, and it is made possible by a systematic use of computer algebra systems, in particular Reduce and its package `CDE` for computations with Hamiltonian operators [17, 25].

## 2 The Oriented Associativity equation

It is shown in [23] that the Oriented Associativity equations can be regarded as the compatibility conditions of $N-1$ commuting flows of type

$$a^i_{t^k} = \partial_x \frac{\partial c^i}{\partial a^k}, \qquad k = 2, \dots, N; \tag{4}$$

under a further condition on the unknowns $c^i$, by analogy with the Associativity equation: we set $\partial c^i / \partial a^1 \partial a^k = \delta^i_k$. This specifies completely the dependence of $c^i$ on $a^1$: $c^1 = (1/2)(a^1)^2 + u^1$, $c^k = a^1 a^k + u^k$ where $u^i = u^i(a^2, \dots, a^N)$ are new unknown functions. The equations (4) can be rewritten in terms of the new unknowns. In the case $N = 3$, if we set $u^1 = u$, $u^2 = v$, $u^3 = w$, the compatibility conditions of the 2 commuting flows in (4) lead to the system of three quadratic second-order equations

$$\begin{aligned} u_{xx} &= v_{xt} w_{xx} - v_{xx} w_{xt} + w_{xt}^2 - w_{xx} w_{tt}, \\ u_{xt} &= v_{tt} w_{xx} - v_{xt} w_{xt}, \\ u_{tt} &= v_{xt}^2 - v_{xx} v_{tt} + v_{tt} w_{xt} - v_{xt} w_{tt}, \end{aligned} \tag{5}$$

after renaming $a^2 = x$, $a^3 = t$. The system (5) is the Oriented Associativity equation in the simplest case $N = 3$. It is endowed by the Lax pair

$$\begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}_x = \lambda \begin{pmatrix} 0 & 1 & 0 \\ u_{xx} & v_{xx} & w_{xx} \\ u_{xt} & v_{xt} & w_{xt} \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix}_t = \lambda \begin{pmatrix} 0 & 0 & 1 \\ u_{xt} & v_{xt} & w_{xt} \\ u_{tt} & v_{tt} & w_{tt} \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix} \tag{7}$$

Let us introduce a new set of field variables $q^1 = u_{xx}, q^2 = u_{xt}, q^3 = v_{xx}, q^4 = v_{xt}$, $q^5 = w_{xx}, q^6 = w_{xt}$. Then, the quadratic system (5) becomes the six component hydrodynamic type system of conservation laws

$$\begin{aligned} q^1_t &= q^2_x, & q^2_t &= \partial_x \frac{q^2 q^6 + q^1 q^4 - q^2 q^3}{q^5}, \\ q^3_t &= q^4_x, & q^4_t &= \partial_x \frac{q^2 + q^4 q^6}{q^5}, \\ q^5_t &= q^6_x, & q^6_t &= \partial_x \frac{(q^6)^2 - q^3 q^6 + q^4 q^5 - q^1}{q^5}. \end{aligned} \tag{8}$$

It is possible to prove that the system (8) has three further conservation laws:

$$\partial_t v^k = \partial_x \frac{(v^k)^2 - q^3 v^k - q^1}{q^5}, \quad k = 1, 2, 3, \tag{9}$$

where $v^i$ are the roots of the characteristic polynomial $\lambda^3 - (q^3 + q^6)\lambda^2 + (q^3 q^6 - q^4 q^5 - q^1)\lambda + q^1 q^6 - q^2 q^5$ of one of the matrices of the Lax pair (6). By Viète's theorem we have $q^3 + q^6 = v^1 + v^2 + v^3$, so that only two of the densities $v^i$ are new.

# 3 First-order Hamiltonian structure

The hydrodynamic-type system (8) admits a first-order homogeneous Hamiltonian operator. This class of Hamiltonian operators was first introduced in [4]. Operators in this class always admit a coordinate system in which they can be presented as $A = h^{ij}\partial_x$, where $h^{ij}$ is a constant matrix. The results in this sections were found in [23], using techniques that are analogous to those used in [8].

We can change the coordinates in the above hydrodynamic-type system to the new coordinates $(u^k)$ defined by the Viète formulae:

$$\begin{aligned}
&u^1 = v^1, \qquad && u^2 = v^2,\\
&u^3 = v^3, \qquad && u^4 = q^4,\\
&u^5 = q^5, \qquad && u^6 = 2q^3 - (v^1 + v^2 + v^3).
\end{aligned} \tag{10}$$

They are related with $(q^i)$ by the formulae

$$\begin{aligned}
q^1 &= \frac{1}{4}(u^1 + u^2 + u^3)^2 - (u^1 u^2 + u^1 u^3 + u^2 u^3) - \frac{1}{4}(u^6)^2 - u^4 u^5,\\
q^2 &= \frac{2u^1 u^2 u^3 + (u^1 + u^2 + u^3 - u^6) q^1}{2u^5},\\
q^3 &= \frac{1}{2}(u^1 + u^2 + u^3 + u^6),\\
q^4 &= u^4,\\
q^5 &= u^5,\\
q^6 &= \frac{1}{2}(u^1 + u^2 + u^3 - u^6).
\end{aligned} \tag{11}$$

Note that the inverse formulae contain cubic roots, and have a much more complicated expression. In the coordinates $(u^i)$ a Hamiltonian formulation of the system becomes immediate:

$$u^i_t = h^{ik}\partial_x \frac{\partial H}{\partial u^k}, \quad \text{where} \quad h^{ik} = -\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{12}$$

the Hamiltonian density is $H = q^2$ and the momentum density is $P = q^1 = \frac{1}{2}h_{ik}u^i u^k$.

# 4 Third-order nonlocal operators and systems of conservation laws

After the results in [23], we might be tempted to conjecture that, by analogy with the Associativity equation, also the Oriented Associativity equation is endowed by two local homogeneous Hamiltonian operators, of first order and third order. Strictly speaking, this is not true. We recall that the conditions for an operator of the form (3) to be Hamiltonian (provided $\det(g^{ij}) \neq 0$) are

$$g_{ij} = g_{ji}, \tag{13a}$$

$$c_{nkm} = \frac{1}{3}(g_{mn,k} - g_{kn,m}) \tag{13b}$$

$$g_{ij,k} + g_{jk,i} + g_{ki,j} = 0, \tag{13c}$$

$$c_{nml,k} + c^s_{ml}c_{snk} = 0. \tag{13d}$$

where $(g_{ij})^{-1} = (g^{ij})$ and $c_{ijk} = g_{iq}g_{jp}c^{pq}_k$. By repeating the procedure that led to the results in [24], we found a candidate $(g^{ij})$ for a leading term of a third order homogeneous Hamiltonian operator. However, that candidate fulfills (13a), (13b) and (13c) but it does not fulfill (13d). After the results in [2], we conjectured that the third-order homogeneous Hamiltonian operator $B = (B^{ij})$ might be nonlocal, of the type

$$B^{ij} = \partial_x \circ F^{ij} \circ \partial_x = \partial_x(g^{ij}\partial_x + c^{ij}_k q^k_x + c^\alpha w^i_{\alpha k} q^k_x \partial_x^{-1} w^j_{\alpha h} q^h_x)\partial_x \tag{14}$$

and $w^i_{\alpha k} = w^i_{\alpha k}(q^j)$, with $c^\alpha \in \mathbb{R}$. In such an ansatz $F^{ij}$ has the same structure as Ferapontov–Mokhov nonlocal first-order homogeneous operators [21, 6, 7]. However, the two compositions with $\partial_x$ change the conditions for the operator to be Hamiltonian to [2]

$$w_{\alpha ij} + w_{\alpha ji} = 0, \tag{15a}$$

$$w_{\alpha ij,l} - c^s_{ij} w_{\alpha sl} = 0, \tag{15b}$$

$$c_{nml,k} + c^s_{ml} c_{snk} + c^\alpha w_{\alpha ml} w_{\alpha nk} = 0, \tag{15c}$$

in addition to (13a), (13b), (13c) (of course, (15c) is a modification of (13d)), where $w_{ij} = g_{is} w^s_j$. We remain with the problem of determining the tensors $w^i_{\alpha j}$. It is known that in Ferapontov–Mokhov case they are matrices of commuting flows with respect to the hydrodynamic-type system of which the operator is Hamiltonian. In this case, that is false: the condition of compatibility between $B$ and the Oriented Associativity equation (8) can be derived by the condition that the Hamiltonian operator maps conserved quantities into symmetries. It was shown in [14] that such a condition is equivalent to finding solutions $B$ to the equation

$$\ell_E(B(\mathbf{p})) = 0, \tag{16}$$

over the *adjoint system* (or cotangent covering)

$$\begin{cases} E = 0, \\ \ell^*_E(\mathbf{p}) = 0, \end{cases} \tag{17}$$

where $\mathbf{p}$ is an auxiliary (vector) variable, $E^i = u^i_t - (V^i)_x = 0$ is the initial equation, with $V^i = V^i(\mathbf{u})$ the vector of fluxes, $\ell_E$ is the formal linearization (Fréchet derivative) of $E$ and $\ell^*_E$ its adjoint operator.

It is easier to compute the condition (16) in potential coordinates $b^i_x = q^i$. We have

$$B^{ij} = -(g^{ij}(\mathbf{b}_x)\partial_x + c^{ij}_k(\mathbf{b}_x) b^k_{xx} + c^\alpha w^i_{\alpha k}(\mathbf{b}_x) b^k_{xx} \partial_x^{-1} w^j_{\alpha h}(\mathbf{b}_x) b^h_{xx}) \tag{18}$$

and $E^i = b^i_t - V^i(\mathbf{b}_x)$. Let us introduce the notation

$$\frac{\partial V^i}{\partial b^k_x} = V^i_k, \quad \frac{\partial V^i}{\partial b^k_x \partial b^h_x} = V^i_{kh}, \quad g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial b^k_x}, \quad c^{ij}_{k,h} = \frac{\partial c^{ij}_k}{\partial b^h_x},$$

and similarly for other derivatives. We have

$$\ell_F(\boldsymbol{\varphi}) = \partial_t \varphi^i - V^i_j \partial_x \varphi^j, \quad \ell^*_F(\boldsymbol{\psi}) = -\partial_t \psi_k + \partial_x (V^i_k \psi_i). \tag{19}$$

so that the adjoint system is

$$b^i_t = V^i(\mathbf{b}_x) \tag{20}$$

$$p_{k,t} = V^i_{kh} b^h_{xx} p_i + V^i_k p_{i,x} \tag{21}$$

If we assume that $w^i_{\alpha j}(\mathbf{b}_x) b^j_{xx}$ are symmetries of the system (20) then we can prove that they yield conservation laws on the adjoint system whose densities and fluxes are, respectively:

$$r_{\alpha t} = V^i_j w^j_{\alpha k} b^k_{xx} p_i, \quad r_{\alpha x} = w^i_{\alpha k} b^k_{xx} p_i. \tag{22}$$

The potential variables $r_\alpha$ allow us to represent the operator as

$$B^i(\mathbf{p}) = -g^{ij} p_{j,x} - c^{ij}_k b^k_{xx} p_j - c^\alpha w^i_{\alpha k} b^k_{xx} r_\alpha. \tag{23}$$

**Lemma 1** *The condition $\ell_F(B(\mathbf{p})) = 0$ is equivalent to the conditions:*

$$-g^{ij} V^h_j + V^i_j g^{jh} = 0 \tag{24a}$$

$$-g^{ih}_{,k} V^k_l - g^{ij} 2V^h_{jl} - c^{ij}_l V^h_j + V^i_j g^{jh}_{,l} + V^i_j c^{jh}_l = 0 \tag{24b}$$

$$-g^{ik} V^h_{kl} - c^{ih}_k V^k_l + V^i_k c^{kh}_l = 0. \tag{24c}$$

$$\begin{aligned}
&-g^{ij} V^h_{jlm} - \frac{1}{2}\big(c^{ih}_{m,j} V^j_l + c^{ih}_{l,j} V^j_m\big) - c^{ih}_k V^k_{lm} \\
&\quad - \frac{1}{2}\big(c^{ij}_m V^h_{jl} + c^{ij}_l V^h_{jm}\big) + \frac{1}{2}\big(V^i_j c^{jh}_{m,l} + V^i_j c^{jh}_{l,m}\big) \\
&\quad - c^\alpha \frac{1}{2}\big(V^h_k (w^i_{\alpha l} w^k_{\alpha m} + w^i_{\alpha m} w^k_{\alpha l}) + V^i_j (w^j_{\alpha l} w^h_{\alpha m} + w^j_{\alpha m} w^h_{\alpha l})\big) = 0
\end{aligned} \tag{24d}$$

$$\begin{aligned}
&-w^i_{\alpha h,k} V^k_m - w^i_{\alpha m,k} V^k_h - w^i_{\alpha k} V^k_{m,h} \\
&\quad - w^i_{\alpha k} V^k_{h,m} + V^i_k w^k_{\alpha m,h} + V^i_k w^k_{\alpha h,m} = 0
\end{aligned} \tag{24e}$$

$$-w^i_{\alpha k} V^k_h + V^i_k w^k_{\alpha h} = 0 \tag{24f}$$

**Proof.** We have:

$$\begin{aligned}
&\ell_F(B(\mathbf{p}))^i = \\
&= -g^{ij}_{,k} b^k_{xt} p_{j,x} - g^{ij} p_{j,xt} - c^{ij}_{k,h} b^h_{xt} b^k_{xx} p_j - c^{ij}_k b^k_{xxt} p_j - c^{ij}_k b^k_{xx} p_{j,t} \\
&\quad + V^i_j \big(g^{jk}_{,h} b^h_{xx} p_{k,x} + g^{jk} p_{k,xx} + c^{jh}_{k,l} b^l_{xx} b^k_{xx} p_h + c^{jh}_k b^k_{xxx} p_h + c^{jh}_k b^k_{xx} p_{h,x}\big) \\
&\quad - c^\alpha w^i_{\alpha k,h} b^h_{xt} b^k_{xx} r_\alpha - c^\alpha w^i_{\alpha k} b^k_{xxt} r_\alpha - c^\alpha w^i_{\alpha k} b^k_{xx} V^h_l w^l_{\alpha m} b^m_{xx} p_h \\
&\quad + V^i_j c^\alpha \big(w^j_{\alpha k,h} b^h_{xx} b^k_{xx} r_\alpha + w^j_{\alpha k} b^k_{xxx} r_\alpha + w^j_{\alpha k} b^k_{xx} w^h_{\alpha l} b^l_{xx} p_h\big).
\end{aligned}$$

After replacing the derivatives $b^h_{xt}$, $b^h_{xxt}$ and $p_{j,xt}$ using the equations (20) and (21) we obtain a polynomial in $p_j$, $b^k_{xx}$ and higher $x$-derivatives; its coefficient shall vanish, they are are the conditions of the statement. ■

**Remark 2** *A direct computation shows that the two flows $V^i$ and $w^j_{\alpha k}u^k_{xx}$ commute if and only if the conditions* (24e) *and* (24f) *hold true. Moreover, by arguments that are similar to those of [12, Theorem 1] it can be proved that* (24d) *is a consequence of the other equations* (24) *and* (13), (15).

# 5 Third-order Hamiltonian structure

It is known[10] that $g_{ij}$ shall be polynomials of second degree in the variables $(q^k)$. Then, equations (24a), (24b), (24c) are easily solved with respect to $(g_{ij})$, where $V^i$ is the vector of fluxes of the Oriented Associativity equation (8). We obtain the unique solution:

$$(g_{ij}) = \begin{pmatrix} 2 & 0 & q^3 & -q^5 & 2q^4 & 0 \\ 0 & 0 & 2q^5 & 0 & -q^3+2q^6 & -q^5 \\ q^3 & 2q^5 & -2(q^1+q^4q^5) & q^5(q^3-q^6) & -q^2+q^4(q^3-q^6) & 2q^4q^5 \\ -q^5 & 0 & q^5(q^3-q^6) & 2(q^5)^2 & -q^1+q^3(-q^3+2q^6)-2q^4q^5-(q^6)^2 & q^5(-q^3+q^6) \\ 2q^4 & -q^3+2q^6 & -q^2+q^4(q^3-q^6) & -q^1+q^3(-q^3+2q^6)-2q^4q^5-(q^6)^2 & 2(q^4)^2 & -q^2+q^4(-q^3+q^6) \\ 0 & -q^5 & 2q^4q^5 & q^5(-q^3+q^6) & -q^2+q^4(-q^3+q^6) & -2q^4q^5 \end{pmatrix} \tag{25}$$

The metric $(g_{ij})$ turns out to be a Monge metric of a quadratic line complex, as it solves the equation (13c), but it *does not* fulfill (13d). Hence, we shall compute suitable tensors $w^i_{\alpha j}$.

A direct computation of $w^i_{\alpha j}$ as symmetries of (20) is very heavy. Since we have at our disposal a Lax pair, we can compute a sequence of homogeneous conserved quantities with a standard technique in the theory of integrable systems; see [24] for details. Then, we can transform them into symmetries using the Hamiltonian operator. We rewrite (6) in terms of $(q^i)$ and get

$$\begin{pmatrix} \psi \\ \psi^1 \\ \psi^2 \end{pmatrix}_x = \lambda \begin{pmatrix} 0 & 1 & 0 \\ q^1 & q^3 & q^5 \\ q^2 & q^4 & q^6 \end{pmatrix} \begin{pmatrix} \psi \\ \psi^1 \\ \psi^2 \end{pmatrix}. \tag{26}$$

By eliminating $\psi^1$, $\psi^2$ from (26) we obtain the single linear PDE

$$\begin{aligned}\big(&-q^1_x\lambda^2q^5+q^5_x\lambda^2q^1+\lambda^3q^1q^5q^6-\lambda^3q^2(q^5)^2\big)\psi+\\ &\big(-q^3_x\lambda q^5+q^5_x\lambda q^3-\lambda^2q^1q^5+\lambda^2q^3q^5q^6-\lambda^2q^4(q^5)^2\big)\psi_x\\ &+\big(-q^5_x-\lambda q^3q^5-\lambda q^5q^6\big)\psi_{2x}+q^5\psi_{3x}=0\end{aligned}$$

The substitution $\psi=\exp\int r dx$ yields a nonlinear ordinary differential equation on the function $r$ and its first and second order derivatives. This function $r$ plays the role of a generating function of conservation law densities with respect to the parameter $\lambda$ for the system (8). The expansion of $r$ at infinity (i.e. $\lambda\to\infty$)

$$r=\lambda h_{-1}+h_0+\frac{h_1}{\lambda}+\frac{h_2}{\lambda^2}+...,$$

in the above equation leads to a sequence of differential relationships between the coefficients $h_{-1}$, $h_0$, $h_1$,... The leading term (the coefficient of $\lambda^3$) coincides with the characteristic equation of the eigenvalues of the matrix in (26). Thus, starting from $h_{-1}=u^k$, $k=1,\,2,\,3$, the expansion of $r$ with respect to $\lambda$ has three branches of conservation law densities, that we denote by $h_{ik}$, $i=0,\,1$,...Such densities are quasihomogeneous polynomials of degrees $\deg h_{ik}=i+1$ with respect to the grading $\deg u=0$, $\deg\partial_x=1$, and their coefficients are expressible via rational functions of $(u^k)$.

Using Reduce [17] we found all expressions of $h_{ik}$, for $k=1,\,2,\,3$ and $i=0$: they are of the form $h_{0k}=c_{ki}(u)u^i_x$, where

$$\begin{aligned}h_{01}=\frac{1}{S_1}\Big(&-4u^4_x(u^5)^2+u^5_x((u^6)^2-2u^6u^1+(u^1)^2-(u^2)^2\\ &+2u^2u^3-(u^3)^2)+2u^6_xu^5(-u^6+u^1)+2u^1_xu^5(-2u^1+u^2+u^3)\\ &+2u^2_xu^5(u^2-u^3)+2u^3_xu^5(-u^2+u^3)\Big)\end{aligned}\tag{27}$$

$$\begin{aligned}h_{02}=\frac{1}{S_2}\Big(&4u^4_x(u^5)^2+u^5_x(-(u^6)^2+2u^6u^2+(u^1)^2-2u^1u^3\\ &-(u^2)^2+(u^3)^2)+2u^6_xu^5(u^6-u^2)+2u^1_xu^5(-u^1+u^3)+\\ &2u^2_xu^5(-u^1+2u^2-u^3)+2u^3_xu^5(u^1-u^3)\Big)\end{aligned}\tag{28}$$

$$
\begin{aligned}
h_{03} = \frac{1}{S_3}\Big( & -4u^4_x(u^5)^2 + u^5_x((u^6)^2 - 2u^6u^3 - (u^1)^2 + 2u^1u^2 \\
& - (u^2)^2 + (u^3)^2) + 2u^6_x u^5(-u^6 + u^3) + 2u^1_x u^5(u^1 - u_2) \\
& + 2u^2_x u^5(-u^1 + u^2) + 2u^3_x u^5(u^1 + u^2 - 2u^3)\Big)
\end{aligned} \tag{29}
$$

where $S_1 = 4u^5((u^1)^2 - u^1u^2 - u^1u^3 + u^2u^3)$, $S_2 = 4u^5(u^1u^2 - u^1u^3 - (u^2)^2 + u^2u^3)$ $S_3 = 4u^5(u^1u^2 - u^1u^3 - u^2u^3 + (u^3)^2)$.

We observe that the above conserved densities are not independent. It holds:

$$
h_{01} + h_{02} + h_{03} = 0. \tag{30}
$$

We get (higher) commuting flows from the above conserved densities by the formula in potential coordinates

$$
b^i_t = h^{ik}\partial_x^{-1}\frac{\delta h_{0i}}{\delta b^k} = h^{ik}\left(-\frac{\partial h_{0i}}{\partial b^k_x} + \partial_x \frac{\partial h_{0i}}{\partial b^k_{xx}}\right) = w^i_j(\mathbf{b}_x)b^j_{xx}. \tag{31}
$$

In particular, from the independent densities $h_{01}$ and $h_{02}$ we obtain the commuting flows $w^i_{1j}(\mathbf{b}_x)b^j_{xx}$ and $w^i_{2j}(\mathbf{b}_x)b^j_{xx}$. We observe that they are commuting flows for the system (20) in potential coordinates, and they define higher commuting flows $(w^i_{\alpha j}u^j_x)_x$ for the system (10). So, they fulfill the conditions (24e) and (24f) as they are invariant conditions. However, we need to check the Hamiltonian property of the operator $B$ in coordinates $(q^i)$.

In principle, it is possible to invert the coordinate change (11) and express the commuting flows $(w^i_{\alpha j}u^j_x)_x$ in coordinates $(q^i)$.

**Lemma 3** *The change of coordinate formula for the flow $(w^i_{\alpha j}(\mathbf{u})u^j_x)_x$ into the flow $(w^i_{\alpha j}(\mathbf{q})q^j_x)_x$ is*

$$
\frac{\partial q^k}{\partial u^h} w^h_i(\mathbf{u}) \frac{\partial u^i}{\partial q^j} = w^k_j(\mathbf{q}).
$$

We remain with the computational problem of expressing $(u^i)$ in terms of $(q^k)$. This would lead to complicated expressions involving roots, so we will write down the matrix $w_{\alpha kj}$ for the two flows $(\alpha = 1, 2)$ using the coordinates

$(u^i)$ as parameters for $(q^k)$. The two matrices $w_{\alpha kj}$ turn out to be skew-symmetric, so that the condition (15a) is satisfied. All coefficients $w_{\alpha kj}$ have denominators that contain factors of

$$\Delta = \sqrt{|\det(g_{ij})|} = (u^1 - u^2)(u^1 - u^3)(u^2 - u^3)u^5; \tag{32}$$

if we introduce the notation $\tilde{w}_{\alpha kj} = \Delta w_{\alpha kj}$, then the only nonzero components of the two matrices $\tilde{w}_{\alpha kj}$ (for $k < j$) are

$$\begin{aligned}
\tilde{w}_{113} &= \tfrac{-(u^2+u^3-u^1-u^6)(u^2-u^3)u^5}{2}\\
\tilde{w}_{114} &= (u^2 - u^3)(u^5)^2\\
\tilde{w}_{125} &= \tfrac{-(u^2+u^3-u^1-u^6)(u^2-u^3)u^5}{2}\\
\tilde{w}_{126} &= (u^2 - u^3)(u^5)^2\\
\tilde{w}_{134} &= (u^2 - u^3)(u^5)^2u^1\\
\tilde{w}_{135} &= \tfrac{-(4u^4u^5+(u^6)^2-2u^6u^1+(u^1)^2-(u^2)^2+2u^2u^3-(u^3)^2)(u^6+u^1-u^2-u^3)(u^2-u^3)}{8}\\
\tilde{w}_{145} &= \tfrac{-(4u^4u^5+(u^6)^2-4u^6u^1-(u^1)^2+2u^1u^2+2u^1u^3-(u^2)^2+2u^2u^3-(u^3)^2)(u^2-u^3)u^5}{4}\\
\tilde{w}_{146} &= (u^2 - u^3)(u^5)^2u^1\\
\tilde{w}_{156} &= \tfrac{-(4u^4u^5+(u^6)^2-2u^6u^1+(u^1)^2-(u^2)^2+2u^2u^3-(u^3)^2)(u^6+u^1-u^2-u^3)(u^2-u^3)}{8}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{w}_{213} &= \tfrac{-(u^6-u^1+u^2-u^3)(u^1-u^3)u^5}{2}\\
\tilde{w}_{214} &= -(u^1 - u^3)(u^5)^2\\
\tilde{w}_{225} &= \tfrac{-(u^6-u^1+u^2-u^3)(u^1-u^3)u^5}{2}\\
\tilde{w}_{226} &= -(u^1 - u^3)(u^5)^2\\
\tilde{w}_{234} &= -(u^1 - u^3)(u^5)^2u^2\\
\tilde{w}_{235} &= \tfrac{(4u^4u^5+(u^6)^2-2u^6u^2-(u^1)^2+2u^1u^3+(u^2)^2-(u^3)^2)(u^6-u^1+u^2-u^3)(u^1-u^3)}{8}\\
\tilde{w}_{245} &= \tfrac{-((u^2-u^3)^2+(u^1)^2-2(u^2+u^3)u^1-(u^6-4u^2)u^6-4u^4u^5)(u^1-u^3)u^5}{4}\\
\tilde{w}_{246} &= -(u^1 - u^3)(u^5)^2u^2\\
\tilde{w}_{256} &= \tfrac{(4u^4u^5+(u^6)^2-2u^6u^2-(u^1)^2+2u^1u^3+(u^2)^2-(u^3)^2)(u^6-u^1+u^2-u^3)(u^1-u^3)}{8}
\end{aligned}$$

We end this paper by exhibiting the third-order Hamiltonian operator for the Oriented Associativity equation in hydrodynamic form.

**Theorem 4** *The Oriented Associativity equation in hydrodynamic form (8) admits the non-local third-order homogeneous Hamiltonian operator*

$$B^{ij} = \partial_x\Big(g^{ij}\partial_x + c^{ij}_k q^k_x + c^1 w^i_{1k} q^k_x \partial_x^{-1} w^j_{1h} q^h_x + c^2 (w^i_{1k} q^k_x \partial_x^{-1} w^j_{2h} q^h_x + w^i_{2k} q^k_x \partial_x^{-1} w^j_{1h} q^h_x) + c^3 w^i_{2k} q^k_x \partial_x^{-1} w^j_{2h} q^h_x\Big)\partial_x \quad (33)$$

*where $g^{ij}$ is the inverse of the Monge metric (25), $c^{ij}_k$ are defined through $g^{ij}$, $w^i_{1j}$ and $w^i_{2j}$ are the matrices of the two commuting flows that we found above and*

$$c^1 = 2, \quad c^2 = 1, \quad c^3 = 2. \quad (34)$$

**Proof.** It is only necessary to check the conditions

$$c_{ijk,l} + g^{pq} c_{pjk} c_{qil} + c^1 w_{1jk} w_{1il} + c^2 (w_{1jk} w_{2il} + w_{1il} w_{2jk}) + c^3 w_{2jk} w_{2il} = 0, \quad (35)$$

$$w_{aij,s} \frac{\partial u^s}{\partial q^l} - g^{pq} c_{pij} w_{aql} = 0, \qquad a = 1, 2 \quad (36)$$

which yield the Hamiltonian property with the given values of the constants $c^i$. ∎

**Remark 5** *The operator $B$ can be rewritten as*

$$B^{ij} = L^{ij} - \partial_x\big(w^i_{1k} q^k_x\big)\partial_x^{-1}\partial_x\big(w^j_{1h} q^h_x\big) - \partial_x\big((w^i_{1k} + w^i_{2k}) q^k_x\big)\partial_x^{-1}\partial_x\big((w^j_{1h} + w^j_{2h}) q^h_x\big) - \partial_x\big(w^i_{2k} q^k_x\big)\partial_x^{-1}\partial_x\big(w^j_{2h} q^h_x\big) \quad (37)$$

*where $L^{ij}$ is a local operator. This shows that $B$ is also a new and highly non-trivial example in the class of weakly nonlocal operators, introduced in [22]. We stress that the coefficients $\partial_x(w^i_{\alpha k} q^k_x)$ are higher commuting flows of the Oriented Associativity equation.*

**Remark 6** *In the Associativity case,* i.e. *on Frobenius manifolds, it is given a constant nondegenerate matrix $\eta_{\alpha\beta}$ and $c^i$ are the components of the gradient of a potential function $F$: $c^i = \eta^{im}\partial F/\partial a^m$. If one is interested in quasihomogeneous solutions of the Associativity equations then in the generic* *case $\eta_{\alpha\beta}$ can be transformed into the antidiagonal identity [3]. Then, the Associativity equation reduces to*

$$f_{ttt} = f_{xxt}^2 - f_{xtt} f_{xxx}.$$

*Introducing new variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$, the compatibility condition for the Associativity equation becomes the hydrodynamic-type system*

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x.$$

*It is easy to show that there exists a unique Monge metric that fulfills* (24a), (24b) *and* (24c), *namely*

$$(g_{ij}) = \begin{pmatrix} -2a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

*This defines a local operator ($w^i_j = 0$), as it fulfills the condition* (13d), *and was discovered in [8] by different methods.*

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