

Weakly nonlocal Poisson brackets, Schouten brackets and supermanifolds

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on the occasion of his 70th birthday*
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Abstract

Poisson brackets between conserved quantities are a fundamental tool in the theory of integrable systems. The subclass of weakly nonlocal Poisson brackets occurs in many significant integrable systems. Proving that a weakly nonlocal differential operator defines a Poisson bracket can be challenging. We propose a computational approach to this problem through the identification of such operators with superfunctions on supermanifolds.

Introduction

In the geometric approach to the integrability of PDEs a central role is played by Poisson brackets [1, 4, 24]. A system of PDEs

$$u_t^i = f^i(t, x, u^j, u_x^j, u_{xx}^j, \dots) \quad (1)$$

with n unknown functions u^1, \dots, u^n (only two independent variables t, x for simplicity) admits a Hamiltonian formulation if there exists a differential operator P and a density $\mathcal{H} = \int h dx$ such that

$$u_t^i = P^{ij} \left(\frac{\delta \mathcal{H}}{\delta u^j} \right) \quad (2)$$

where $P = (P^{ij})$ is a Hamiltonian operator, i.e. a matrix of differential operators $P^{ij} = P^{ij\sigma} \partial_\sigma$, where $\partial_\sigma = \partial_x \circ \dots \circ \partial_x$ (total x -derivatives σ times), such that

$$\{F, G\}_P = \int \frac{\delta F}{\delta u^i} P^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j} dx \quad (3)$$

is a *Poisson bracket* (skew-symmetric and Jacobi). Such properties are equivalent to the conditions

1. $P^* = -P$, skew-adjointness of P , and
2. $[P, P] = 0$, where the (square) bracket is the (*variational*) *Schouten bracket* of differential operators.

Soon after the introduction of Hamiltonian operators in the theory of PDEs it was observed that Hamiltonian operators for many PDEs were indeed nonlocal (or pseudodifferential) operators. This fact was first described in [20] for the Krichever–Novikov equation. A wide class of such operators is constituted by the so-called weakly-nonlocal Hamiltonian operators [18]:

$$P^{ij} = P^{ij\sigma} \partial_\sigma + e^\alpha w_\alpha^i \partial_x^{-1} w_\alpha^j, \quad (4)$$

where e^α are constants and $w_\alpha^i = w_\alpha^i(u^i, u_x^i, \dots)$. The problem of defining the Schouten bracket of nonlocal operators in a non-ambiguous way has caught the interest of researchers, and it is still a lively topic. Indeed, the definition of (local) Hamiltonian is geometrically invariant. This implies that we can consider them as geometric properties of PDEs, like symmetries or conserved

quantities (see [14]). A rigorous approach to the computation of Schouten bracket for a wide class of nonlocal Hamiltonian operators has been proposed in [3], and it is based on the notion of nonlocal Poisson Vertex Algebra. In the case of weakly nonlocal Hamiltonian operators a computational solution to the problem has been recently proposed through the parallel development of an algorithm in the three different languages of distributions, operators, and Poisson Vertex Algebras [2].

The Poisson bracket construction for local operators can be rephrased by means of the well-known isomorphism between skew-symmetric linear differential operators and superfunctions on jets of superbundles [10] (see also [11, 12, 14]). Basically, n new *odd* dependent variables p_1, \dots, p_n are introduced such that the isomorphism reads on P as

$$P^{ij\sigma} \psi_i^1 \partial_\sigma \psi_j^2 \longrightarrow P^{ij\sigma} p_i p_{j\sigma}, \quad (5)$$

where ψ^1 and ψ^2 are the arguments of P as a differential operator. Here the product $p_i p_{j\sigma}$ is the Grassmann product. Then, the Schouten bracket between two operators can be written through a very elegant formula:

$$[P, Q] = \int \left((-1)^{|P|} \frac{\delta P}{\delta u^i} \frac{\delta Q}{\delta p_i} + \frac{\delta P}{\delta p_i} \frac{\delta Q}{\delta u^i} \right) dx. \quad (6)$$

This implies that in order to check that the expression vanishes one should calculate the Euler–Lagrange operator of the integrand and see if the result is zero. Note that skew-adjointness of the operators P and Q must hold for (6) to make sense; the condition of skew-adjointness can also be verified on the superfunctions that correspond to P and Q by a suitable formula, see [16].

When writing [2] we considered the possibility to extend the results of the paper to the formalism of superfunctions. Nonlocal terms in operators can be represented by introducing new *odd nonlocal variables*. Such a technique was first introduced for Hamiltonian operators in [13]. In the case of weakly nonlocal operators, this amounts at representing P in (4) as

$$P = P^{ij\sigma} p_i p_{j\sigma} + e^\alpha w_\alpha^i p_i r_\alpha, \quad (7)$$

where r_α are new nonlocal *odd* variables defined by $\partial_x r_\alpha = w_\alpha^j p_j$.

Unfortunately, a naive rephrasing of the arguments of [2] in terms of superfunctions does not work, as superfunctions do not allow to keep the role of the arguments ψ^1, ψ^2, ψ^3 distinct in the three-vectors.

Weakly nonlocal Poisson operators associated with conformally flat metrics can be interpreted as Jacobi structures. This beautiful observation allowed to find a generalization of the formula (6) to such operators in the framework of supermanifolds (see [17] for details).

In the present paper we consider general weakly nonlocal operators. Our starting observation is that it is possible to extend the formula (6) to weakly nonlocal superfunctions by a computational formula for the variational derivatives of nonlocal odd variables.

We illustrate this extension through examples that cover a large class of nonlocal Hamiltonian operators. More precisely, we prove the Hamiltonian property for pseudodifferential operators for the Krichever-Novikov equation and the modified KdV equation. Finally, we prove the well-known conditions that are equivalent to the Hamiltonian property for an important subclass of first-order weakly nonlocal operators introduced by Ferapontov [9].

In the Conclusions we outline some features of this computational approach that are promising in view of future investigations.

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1 Schouten bracket as Poisson bracket

We introduce new anticommuting (Grassmann) variables p_i , $i = 1, \dots, n$ and their x -derivatives (σ -times) $p_{i,\sigma}$. So, we work on the (infinite order) jet of a superbundle with even coordinates (x, u^i, u_σ^i) and odd coordinates $(p_i, p_{i,\sigma})$.

The formula (6) for the variational Schouten bracket between two operators, written as the superfunctions $P = P^{ij,\sigma} p_i p_{j,\sigma}$, $Q = Q^{ij,\sigma} p_i p_{j,\sigma}$, makes

use of the variational derivatives

$$\frac{\delta P}{\delta u^i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial P}{\partial u_\sigma^i} \right), \quad \frac{\delta P}{\delta p_i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial P}{\partial p_{i\sigma}} \right), \quad (8)$$

and similarly for Q . Note that the derivatives with respect to odd coordinates are odd derivatives, and total derivatives (7) are extended to odd variables as

$$\partial_\lambda = \frac{\partial}{\partial x^\lambda} + u_{\sigma,\lambda}^i \frac{\partial}{\partial u_\sigma^i} + p_{i\sigma,\lambda} \frac{\partial}{\partial p_{i\sigma}}. \quad (9)$$

The expression of the bracket at the right-hand side of (6) means that we are in an equivalence class: the integrand is defined up to total divergencies of superfunctions. Thus, in order to check that $[P, Q] = 0$ we need to compute the variational derivative of the superfunction of degree 3 which is the integrand in (6) and check that it is zero.

We would like to extend the Schouten bracket to weakly nonlocal operators. For simplicity, we will only consider the case of one non-vanishing coefficient e^α ; this means that we need to compute with superfunctions of the type

$$P = P^{ij,\sigma} p_i p_{j,\sigma} + W^i p_i r, \quad (10)$$

where r is a superfunction of degree 1 which is defined by the equation

$$r_x = Z^j p_j, \quad (11)$$

where W 's and Z 's are functions defined on the even part of the jet of the superbundle. Note that in the case of just one nonlocal variable r we have $W = Z$ in order to guarantee skew-adjointness; but, as in more general cases we might have several summands rearranged in such a way that $W \neq Z$, we prefer to give a more general formula.

Since the Poisson bracket for superfunctions is graded-bilinear, in order to be able to compute it for nonlocal superfunctions we just need a formula for the variational derivative of the nonlocal part of the superfunction

$$N = W^i p_i r. \quad (12)$$

Let us introduce the notation $W = W^i p_i$ and $Z = Z^i p_i$, and a new superfunction s of degree 1 defined by the equation $s_x = W^i p_i$. Denote the

Euler–Lagrange operator by \mathcal{E} . It is well-known [21] that if we have a density $N = \int n dx$, and if we denote by ℓ_N its linearization (or Fréchet derivative)¹:

$$\ell_N(\varphi) = \int \left(\frac{\partial n}{\partial u_\sigma^i} \partial_\sigma \varphi^i + (-1)^{|n|+1} \frac{\partial n}{\partial p_{j,\sigma}} \partial_\sigma \varphi^j \right) dx \quad (13)$$

then we have

$$\mathcal{E}(N) = \ell_N^*(1) = \left(\frac{\delta n}{\delta u^i}, \frac{\delta n}{\delta p_j} \right); \quad (14)$$

note that signs on the odd part cancel after taking the adjoint. We also recall the formula $\ell_{\Delta_1 \circ \Delta_2}(\varphi) = \ell_{\Delta_1}(\Delta_2(\varphi)) + \Delta_1 \circ \ell_{\Delta_2}(\varphi)$.

In order to compute the above expression, we also need the definition of the adjoint operator of a superdifferential operator ([21]; see also [11, 12])

$$\langle \psi, \Delta(\varphi) \rangle = (-1)^{|\Delta||\psi|} \langle \Delta^*(\psi), \varphi \rangle \quad (15)$$

where Δ is a differential operator between vector-valued functions, Δ^* is its adjoint, operating between covector-valued densities, and the angular brackets stand for the natural pairing with values in the variational cohomology (i.e., the result is a density up to total divergencies). We have

$$\begin{aligned} \langle \ell_N^*(1), \varphi \rangle &= \langle 1, \ell_N(\varphi) \rangle = \langle 1, \ell_{W,r}(\varphi) + W \partial_x^{-1}(\ell_Z(\varphi)) \rangle \\ &= \langle \ell_{W,r}^*(1) + \ell_{Z,s}^*(1), \varphi \rangle. \end{aligned} \quad (16)$$

In the above formulae, expressions like $\ell_{W,r}(\varphi)$ are linearizations of the term W with fixed r . Note that the parity of ∂_x^{-1} is 0 and that $\partial_x^{-1*} = -\partial_x^{-1}$. In coordinates, we have

$$\frac{\delta N}{\delta u^i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial W}{\partial u_\sigma^i} r \right) + (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Z}{\partial u_\sigma^i} s \right), \quad (17)$$

$$\frac{\delta N}{\delta p_i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial W}{\partial p_{i,\sigma}} r \right) + (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Z}{\partial p_{i,\sigma}} s \right), \quad (18)$$

The integrand at the right-hand side of (6) is a superfunction of degree 3; in order to check the vanishing of $[P, Q]$ we should be able to determine if the expression (6) is a total divergence. So, we need to compute the Euler–Lagrange operator of a 3-superfunction with nonlocal terms. The only problems come from the nonlocal terms. There might be two distinct types of nonlocal terms:

¹Here and in what follows absolute values of objects, like $|n|$, represent the parity of the object, 1 or 0.

1. $T_1 = Y_1^{i_1, \sigma_1; i_2, \sigma_2} p_{i_1, \sigma_1} p_{i_2, \sigma_2} r$; if we set $Y_1 = Y_1^{i_1, \sigma_1; i_2, \sigma_2} p_{i_1, \sigma_1} p_{i_2, \sigma_2}$ we can proceed in a way which is similar to what we did for the nonlocal bivector N (but keep into account the different gradings!), and get the formula

$$\ell_{T_1}^*(1) = \ell_{Y_1, r}^*(1) - \ell_{Z, y_1}^*(1). \quad (19)$$

In coordinates:

$$\frac{\delta T_1}{\delta u^i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Y_1}{\partial u_\sigma^i} r \right) - (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Z}{\partial u_\sigma^i} y_1 \right), \quad (20)$$

$$\frac{\delta T_1}{\delta p_i} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Y_1}{\partial p_{i, \sigma}} r \right) - (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial Z}{\partial p_{i, \sigma}} y_1 \right), \quad (21)$$

where $y_{1, x} = Y_1$.

2. $T_2 = Y_2^{i_1, \sigma_1} p_{i_1, \sigma_1} r s$; if we set $Y_2 = Y_2^{i_1, \sigma_1} p_{i_1, \sigma_1}$ we have

$$\langle \ell_{T_2}^*(1), \varphi \rangle = \langle 1, \ell_{T_2}(\varphi) \rangle \quad (22)$$

$$= \langle 1, \ell_{Y_2, r s}(\varphi) + Y_2 \ell_{r, s}(\varphi) + Y_2 r \ell_s(\varphi) \rangle \quad (23)$$

$$= \langle 1, \ell_{Y_2, r s}(\varphi) + Y_2 \partial_x^{-1} \ell_{Z, s}(\varphi) + Y_2 r \partial_x^{-1} \ell_W(\varphi) \rangle \quad (24)$$

$$= \langle \ell_{Y_2, r s}^*(1) - \ell_{Z, s y_2}^*(1) - \ell_{W, y_3}^*(1), \varphi \rangle \quad (25)$$

where y_2 is defined by $(y_2)_x = Y_2$ and y_3 by $(y_3)_x = Y_2 r$.

We stress that the above approach is purely computational, as in the proof of our formulae we make use of ∂_x^{-1} which has no ‘good’ geometrical definition. However, we can show that in concrete computations our formula reproduces known results; we hope to be able to provide a geometric justification of the formula in the future. Some new interesting developments in this direction can be found in [15], where nonlocal operators are treated in the framework of the geometry of jet spaces.

2 Examples of computation

In this Section we consider known examples of weakly nonlocal Hamiltonian operators defining Poisson brackets. We provide a systematic computational approach to the calculation of the Schouten bracket.

2.1 Krichever–Novikov equation

The Krichever–Novikov equation

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} \quad (26)$$

has the Hamiltonian operator $P = u_x \partial_x^{-1} u_x$ [20]. We rewrite it as $P = N = Wr = u_x pr$, where $r_x = u_x p$. The Schouten bracket is

$$[N, N] = 2 \frac{\delta N}{\delta u} \frac{\delta N}{\delta p}, \quad (27)$$

where

$$\frac{\delta N}{\delta u} = 2 \sum (-1)^k \partial_x^k \left(\frac{\delta W}{\delta u^{(k)}} r \right) = -2 \partial_x (pr) = -2 p_x r - 2 p^2 u_x = -2 p_x r, \quad (28)$$

and

$$\frac{\delta N}{\delta p} = 2 \sum (-1)^k \partial_x^k \left(\frac{\delta W}{\delta p^{(k)}} r \right) = -2 \partial_x (pr) = 2 u_x r. \quad (29)$$

Hence $[N, N] = -8 u_x p_x r^2 = 0$ (no need to compute the variational derivative of the 3-superfunction in this simple case).

2.2 Modified KdV

The modified KdV equation is

$$u_t = u^2 u_x + u_{xxx}; \quad (30)$$

it has the weakly nonlocal Hamiltonian operator [23]

$$P = \partial_x^3 + \frac{2}{3} u^2 \partial_x + \frac{2}{3} u u_x - \frac{2}{3} u_x \partial_x^{-1} u_x. \quad (31)$$

Let us set $P = L + N$, with $L = p_{xxx} p + \frac{2}{3} u^2 p_x p$ and $N = \frac{2}{3} u_x p r$. In the previous subsection we proved that $[N, N] = 0$, hence the Schouten bracket $[P, P]$ reduces to

$$[L + N, L + N] = [L, L] + 2[L, N] \quad (32)$$

By a direct computation we have

$$[L, L] = \frac{16}{3} u p p_x p_{xxx}. \quad (33)$$

Moreover

$$\begin{aligned}
[L, N] &= \frac{\delta L}{\delta u} \frac{\delta N}{\delta p} + \frac{\delta N}{\delta u} \frac{\delta L}{\delta p} \\
&= \left(\frac{4}{3} u p_x p \right) \left(\frac{4}{3} u_x r \right) + \left(-\frac{4}{3} p_x r \right) \left(-p_{xxx} - \frac{2}{3} u^2 p_x - \frac{2}{3} \partial_x(u^2 p) - \partial_x^3(p) \right) \\
&= \frac{16}{9} u u_x p_x p r - \frac{4}{3} r p_x p_{xxx} + \frac{16}{9} u u_x p_x r p + \frac{4}{3} p_x r p_{xxx} = -\frac{8}{3} r p_x p_{xxx}
\end{aligned}$$

Hence

$$[L + N, L + N] = \frac{16}{3} (u p p_x - r p_x) p_{xxx}. \quad (34)$$

The above expression yields, after integrating it by parts:

$$\begin{aligned}
\frac{16}{3} (u p p_x - r p_x) p_{xxx} &= -\frac{16}{3} \partial_x (u p p_x - r p_x) p_{xx} \\
&= \frac{16}{3} (-u_x p p_x + u_x p p_x) p_{xx} = 0. \quad (35)
\end{aligned}$$

More systematically, we can compute the Euler–Lagrange operator of (34). Let us set $T_L = u p p_x p_{xxx}$ and $T_N = -p_x p_{xxx} r$. We have:

$$\frac{\delta T_L}{\delta u} = p p_x p_{xxx} \quad (36)$$

$$\frac{\delta T_L}{\delta p} = -3u_{xx} p p_{xx} - 2u_x p p_{xxx} - u_{xxx} p p_x - 3u_x p_x p_{xx} \quad (37)$$

where the above computations have been done by the CDE package of Reduce [16, 19]. If y_1 is defined by $y_{1,x} = -p_x p_{xxx}$ we observe that, in this case $y_1 = -p_x p_{xx}$. We have

$$\frac{\delta T_N}{\delta u} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial(-p_x p_{xxx})}{\partial u_\sigma} r \right) - (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial(u_x p)}{\partial u_\sigma} y_1 \right) \quad (38)$$

$$= -\partial_x(p p_x p_{xx}) = -p p_x p_{xxx}, \quad (39)$$

and

$$\frac{\delta T_N}{\delta p} = (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial(-p_x p_{xxx})}{\partial p_\sigma} r \right) - (-1)^{|\sigma|} \partial_\sigma \left(\frac{\partial(u_x p)}{\partial p_\sigma} y_1 \right) \quad (40)$$

$$= -\partial_x(-p_{xxx} r) - \partial_{3x}(p_x r) - u_x(-p_x p_{xx}) \quad (41)$$

$$= 3u_{xx} p p_{xx} + 2u_x p p_{xxx} + u_{xxx} p p_x + 3u_x p_x p_{xx} \quad (42)$$

which yields the result. We stress that, without the explicit integration of y_1 , the simplification would have not been possible.

2.3 Example: first-order homogeneous weakly nonlocal operators

The class of first-order homogeneous Poisson brackets was introduced in [5]. This class is defined by first-order differential operators that are homogeneous with respect to x -derivatives. The main feature of such operators is that their ‘form’ is preserved by coordinate transformations of the type $\bar{u}^i = \bar{u}^i(u^j)$. This implies that the skew-symmetry and Jacobi property of the Poisson brackets translate into geometric properties of the coefficients of the differential operator.

The weakly nonlocal generalization of the above operators was introduced in a very special case in [7], and later in a much wider sense in [8]. The geometry of such operators is very rich and interesting; we invite the reader to have a look at [9] and references therein. We will consider a weakly nonlocal first-order operator P of the type:

$$P^{ij} = g^{ij} \partial_x + \Gamma_k^{ij} u_x^k + W_k^i u_x^k \partial_x^{-1} W_h^j u_x^h, \quad (43)$$

although more general operators are possible and natural [9]. Let us introduce a nonlocal odd variable r defined by $r_x = W_h^j u_x^h p_j$, and rewrite the operator P in odd variables:

$$P = g^{ij} p_{j,x} p_i + \Gamma_k^{ij} u_x^k p_j p_i + W_k^i u_x^k r p_i \quad (44)$$

Let us write $P = L + N$, where

$$L = g^{ij} p_{j,x} p_i + \Gamma_k^{ij} u_x^k p_j p_i, \quad N = W_k^i u_x^k r p_i. \quad (45)$$

We have to prove that the coefficients of P satisfy the following set of conditions (see [9])

$$g^{ij} = g^{ji}, \quad (46a)$$

$$g_{,k}^{ij} = \Gamma_k^{ij} + \Gamma_k^{ji}, \quad (46b)$$

$$g^{is} \Gamma_s^{jk} = g^{js} \Gamma_s^{ik}, \quad (46c)$$

$$g^{is} W_s^j = g^{js} W_s^i \quad (46d)$$

$$\nabla_i W_k^j = \nabla_k W_i^j, \quad (46e)$$

$$R_{kh}^{ij} = W_k^i W_h^j - W_k^j W_h^i. \quad (46f)$$

The skew-symmetry is equivalent to (46a) and (46b), and is assumed throughout the computation.

We use the formula

$$[P, P] = [L + N, L + N] = [L, L] + 2[L, N] + [N, N]. \quad (47)$$

Then, from (6) it is clear that we need the formulae:

$$\frac{\delta L}{\delta u^l} = 2\Gamma_l^{ji} p_{j,x} p_i + (\Gamma_{k,l}^{ij} - \Gamma_{l,k}^{ij}) u_x^k p_j p_i \quad (48)$$

$$\frac{\delta L}{\delta p_l} = -2g^{lj} p_{j,x} + (\Gamma_k^{jl} - \Gamma_k^{lj} - g_{k,l}^{jl}) u_x^k p_j = -2g^{lj} p_{j,x} - 2\Gamma_k^{lj} u_x^k p_j \quad (49)$$

$$\frac{\delta N}{\delta u^l} = 2(W_{l,k}^i - W_{k,l}^i) u_x^k p_i r + 2W_l^i W_k^j u_x^k p_i p_j + 2W_l^i p_{i,x} r \quad (50)$$

$$\frac{\delta N}{\delta p_l} = -2W_k^l u_x^k r \quad (51)$$

Using (47) we compute three expressions; they are defined up to total derivatives. We have:

$$\begin{aligned} [L, L] &= 2 \frac{\delta L}{\delta u^i} \frac{\delta L}{\delta p_i} = \\ &= 8\Gamma_l^{ji} g^{lm} p_i p_{j,x} p_{m,x} + (8\Gamma_l^{ji} \Gamma_k^{lm} + 4g^{lj} (\Gamma_{k,l}^{im} - \Gamma_{l,k}^{im})) u_x^k p_{j,x} p_m p_i \\ &\quad - 4(\Gamma_{h,l}^{ij} - \Gamma_{l,h}^{ij}) \Gamma_k^{lm} u_x^h u_x^k p_j p_i p_m \end{aligned}$$

$$\begin{aligned} [L, N] &= \frac{\delta N}{\delta u^i} \frac{\delta L}{\delta p_i} + \frac{\delta L}{\delta u^i} \frac{\delta N}{\delta p_i} = \\ &= (4g^{lj} (W_{l,k}^i - W_{k,l}^i) - 4\Gamma_k^{li} W_l^j + 4\Gamma_l^{ji} W_k^l) u_x^k p_i p_{j,x} r + \\ &\quad \left(4\Gamma_h^{lj} (W_{l,k}^i - W_{k,l}^i) - 2W_k^l (\Gamma_{h,l}^{ji} - \Gamma_{l,h}^{ji}) \right) u_x^h u_x^k p_i p_j r \\ &\quad - 4g^{lm} W_l^i W_k^j u_x^k p_i p_j p_{m,x} - 4\Gamma_h^{lm} W_l^i W_k^j u_x^k u_x^h p_i p_j p_m + 4g^{lj} W_l^i p_{i,x} p_{j,x} r \end{aligned}$$

Finally, taking into account that $r^2 = 0$ we obtain

$$[N, N] = 2 \frac{\delta N}{\delta u^i} \frac{\delta N}{\delta p_i} = -8W_l^i W_k^j W_m^l u_x^k u_x^m p_i p_j r.$$

Collecting all the terms together we get

$$\begin{aligned} [P, P] &= A^{ijh} p_i p_{j,x} p_{h,x} + B_k^{ijh} u_x^k p_i p_j p_{h,x} + C_{km}^{ijh} u_x^k u_x^m p_i p_j p_h + \\ &\quad D_k^{ij} u_x^k p_i p_{j,x} r + E_{kh}^{ij} u_x^k u_x^h p_i p_j r + F_{hk}^{ij} p_{i,x} p_{j,x} r \end{aligned}$$

that can be also written as

$$[P, P] = \tilde{A}^{ijh} p_i p_{j,x} p_{h,x} + \tilde{B}_k^{ijh} u_x^k p_i p_j p_{h,x} + \tilde{C}_{km}^{ijh} u_x^k u_x^m p_i p_j p_h + \tilde{D}_k^{ij} u_x^k p_i p_{j,x} r + \tilde{E}_{kh}^{ij} u_x^k u_x^h p_i p_j r + \tilde{F}^{ij} p_{i,x} p_{j,x} r$$

with

$$\begin{aligned} \tilde{A}^{ijh} &= \frac{1}{2}(A^{ijh} - A^{ihj}), \\ \tilde{B}_k^{ijh} &= \frac{1}{2}(B_k^{ijh} - B_k^{jih}) \\ \tilde{C}_{hk}^{ijm} &= \frac{1}{12}(C_{kh}^{ijm} - C_{kh}^{imj} - C_{kh}^{jim} + C_{kh}^{jmi} + C_{kh}^{mij} - C_{kh}^{mji}) + \\ &\quad \frac{1}{12}(C_{hk}^{ijm} - C_{hk}^{imj} - C_{hk}^{jim} + C_{hk}^{jmi} + C_{hk}^{mij} - C_{hk}^{mji}) \\ \tilde{D}_k^{ij} &= D_k^{ij}, \\ \tilde{E}_{kh}^{ij} &= \frac{1}{4}(E_{kh}^{ij} - E_{kh}^{ji}) + \frac{1}{4}(E_{hk}^{ij} - E_{hk}^{ji}), \\ \tilde{F}^{ij} &= \frac{1}{2}(F^{ij} - F^{ji}) \end{aligned}$$

We obtain

$$\begin{aligned} \tilde{A}^{ijh} &= 4(g^{lh}\Gamma_l^{ji} - g^{lj}\Gamma_l^{hi}) \\ \tilde{B}_k^{ijh} &= 4R_k^{ijh} + 4g^{li}(W_k^h W_l^j - W_k^j W_l^h) \\ \tilde{C}_{hk}^{ijm} &= -\frac{2}{3}\Gamma_k^{lm}(R_{lh}^{ij} - W_l^i W_h^j + W_l^j W_h^i) - \frac{2}{3}\Gamma_h^{lm}(R_{lk}^{ij} - W_l^i W_k^j + W_l^j W_k^i) \\ &\quad + \frac{2}{3}\Gamma_k^{li}(R_{lh}^{mj} - W_l^m W_h^j + W_l^j W_h^m) + \frac{2}{3}\Gamma_h^{li}(R_{lk}^{mj} - W_l^m W_k^j + W_l^j W_k^m) \\ &\quad - \frac{2}{3}\Gamma_k^{lj}(R_{lh}^{mi} - W_l^m W_h^i + W_l^i W_h^m) - \frac{2}{3}\Gamma_h^{lj}(R_{lk}^{mi} - W_l^m W_k^i + W_l^i W_k^m) \\ \tilde{D}_k^{ij} &= 2g^{lj}(\nabla_k W_l^i - \nabla_l W_k^i) - 2\Gamma_{mk}^i(g^{jl}W_l^m - g^{ml}W_l^j) \\ \tilde{E}_{hk}^{ij} &= 2\Gamma_h^{lj}(\nabla_k W_l^i - \nabla_l W_k^i) - 2\Gamma_h^{li}(\nabla_k W_l^j - \nabla_l W_k^j) + \\ &\quad 2\Gamma_k^{lj}(\nabla_h W_l^i - \nabla_l W_h^i) - 2\Gamma_k^{li}(\nabla_h W_l^j - \nabla_l W_h^j) + \\ &\quad W_k^l(R_{lh}^{ij} - W_l^i W_h^j + W_l^j W_h^i) + W_h^l(R_{lk}^{ij} - W_l^i W_k^j + W_l^j W_k^i) \\ &\quad - W_k^l(R_{lh}^{ji} - W_l^j W_h^i + W_l^i W_h^j) - W_h^l(R_{lk}^{ji} - W_l^j W_k^i + W_l^i W_k^j) \\ \tilde{F}^{ij} &= 2(g^{lj}W_l^i - g^{li}W_l^j) \end{aligned}$$

Let us set $T = [P, P]$. The system

$$\frac{\delta T}{\delta u^l} = 0, \quad \frac{\delta T}{\delta p_l} = 0, \quad (52)$$

yields the following conditions:

- $\tilde{B}_l^{ijh} = 0$, which is the coefficient of $p_i p_j p_{h,2x}$ in $\delta T / \delta u^l$;
- $2\tilde{A}^{ilj} = 0$, which is the coefficient of $p_i p_{j,2x}$ in $\delta T / \delta p_l$;
- $-\tilde{D}_k^{il} = 0$, which is the coefficient of $u_{2x}^k p_i r$ in $\delta T / \delta p_l$;
- $2\tilde{F}^{il} = 0$, which is the coefficient of $p_{i,2x} r$ in $\delta T / \delta p_l$.

The above conditions are equivalent to the conditions (46c), (46d), (46e), (46f), and imply the vanishing of the coefficients \tilde{C} and \tilde{E} .

We remark that the last step of the computation of the Schouten bracket is very straightforward: it is easy to derive the vanishing conditions from few selected coefficients in the variational derivative.

3 Conclusions

Weakly nonlocal hamiltonian operators arise naturally in the theory and applications of integrable systems [18].

In [2] we developed an algorithm to compute Schouten brackets of such operators using three different formalisms: distributions, pseudodifferential operators, Poisson vertex algebras. In this paper we propose an alternative approach based on the identification of weakly non local hamiltonian operators with superfunctions on supermanifolds. This approach requires to define variational derivative for nonlocal variables. This allows to extend the known formula for the Schouten bracket of local operators in a straightforward way.

Finding necessary and sufficient conditions for the vanishing of the bracket is not immediate, the main difficulty being that the nonlocal odd variables that arise in the computations should be checked in order to see if they can be integrated (see Section 2.2). However this problem can be easily fixed and the implementation of the main result on a computer algebra program seems possible. For instance, the Reduce package CDE [16, 22] already allows to use local and nonlocal variables and contains an implementation of the Schouten bracket for local operators in terms of odd variables.

A set of software packages for the symbolic calculation of the Schouten bracket adapted to all the above formalisms will be the subject of our future work.

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