Homogeneous Hamiltonian operators and the theory of coverings

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Dedicated to the memory of A.M. Vinogradov, with gratitude.

Abstract

A new method (by Kersten, Krasil'shchik and Verbovetsky), based on the theory of differential coverings, allows to relate a system of PDEs with a differential operator in such a way that the operator maps conserved quantities into symmetries of the system of PDEs. When applied to a quasilinear first-order system of PDEs and a Dubrovin–Novikov homogeneous Hamiltonian operator the method yields conditions on the operator and the system that have interesting differential and projective geometric interpretations.

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1 Introduction

Hamiltonian methods have become a standard part of the theory of nonlinear Partial Differential Equations (PDEs) and integrable systems (see, for example, [24]). Determining if a certain PDE (or a system of PDEs) has a Hamiltonian formulation yields important information on its integrability.

A fundamental contribution to the problem of finding out Hamiltonian formulations for PDEs has been presented in [14]. The necessary condition for a PDE (or a system of PDEs) to admit a Hamiltonian formulation is presented as the existence of a *shadow of symmetry in a covering* of the given PDE. The shadow of symmetry can be identified with a differential operator that maps cosymmetries of the PDE into symmetries of the PDE. The above operators are called *variational bivectors* on the PDE. Such a definition is more general than the property of being a Hamiltonian operator.

The concept of covering is due to A.M. Vinogradov [36] (see also [37, 17]); shadows of symmetries are a natural object in the theory of coverings. In [14] two natural coverings were associated with any PDE, the *tangent covering* and the *cotangent covering*. The invariance of the coverings under a wide class of coordinate changes is proved in [19]. Hamiltonian operators (local and non-local) can be found as shadows of symmetries on the cotangent covering. Finding a shadow of symmetry is a calculation of the same nature as the calculation of generalized symmetries for a system of PDEs.

When looking for Hamiltonian operators for a certain PDE (or system of PDEs) it is much easier to look for variational bivectors first. Indeed, the equation that variational bivectors fulfill is a linear overdetermined system in their coefficients, and that has higher chances of being solved with respect to a direct search for a Hamiltonian formulation of the PDE, which is generally nonlinear (see (2) below).

Differential-geometric properties of the Hamiltonian formulation of nonlinear PDEs were studied since the early times of integrable systems. In particular, B.A. Dubrovin and S.P. Novikov introduced homogeneous Hamiltonian operators (HHOs in this paper, for the sake of brevity) of first order [5] and higher order [4] as one of the essential ingredients of the Hamiltonian formalism for PDEs. Such operators are form-invariant with respect to transformations of the field variables and have interesting geometric properties (see the long review [23]) which continue to be discovered [1, 10, 9, 8].

It was understood long ago that the conditions under which a system of PDEs admits a Hamiltonian formulation by a given HHO are of differential-geometric nature. In particular, S. Tsarev proved [34] that a first-order quasi-linear system of PDEs, or hydrodynamic type system, in two independent variables:

$$u^i_t = V^i_j(u)u^j_x, \tag{1}$$

where $u^i = u^i(t, x)$ and $i = 1, \dots, n$, admits a Hamiltonian formulation

$$u^i_t = A^{ij}\left(\frac{\delta H(u)}{\delta u^j}\right) \tag{2}$$

through a first-order HHO

$$A^{ij} = g^{ij}(u)\partial_x + \Gamma^{ij}_k u^k_x \tag{3}$$

(in the non-degenerate case $\det(g^{ij}) \neq 0$) if and only if the following conditions are satisfied:

$$g^{ik}V^j_k = g^{jk}V^i_k, \qquad \nabla_i V^k_j = \nabla_j V^k_i, \tag{4}$$

where ∇_k is the covariant derivative with respect to the Levi-Civita connection determined by g_{ij} (the inverse matrix of g^{ij}).

However, the original proof made use of the *existence* of a Hamiltonian H. In general, it is not easy to predict the existence of a Hamiltonian in a certain class. This made a generalization of the above conditions to higher order HHOs quite difficult to achieve.

Recently, the geometric properties of third-order HHOs were studied [9, 8]. It was realized that a result like (4) for third-order HHOs was missing, while several examples of hydrodynamic type systems (1) admitting a Hamiltonian formulation by means of a third-order HHO were known at the time, mostly from Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations (see the discussion in [10]). Using the cotangent covering of the hydrodynamic type system and the equation for variational bivectors on third-order HHOs led to geometric conditions that reduced to a linear algebraic system in either the coefficients of the operator A or the velocity matrix (V^i_j). In particular, we found an interesting class of hydrodynamic type systems of conservation laws that are determined by the choice of a third-order local HHO (see [29] for the nonlocal case).

The aim of this paper is twofold.

First of all we would like to present the range of applicability of the method of tangent and cotangent coverings in the problem of determining the conditions on HHOs of a certain type to be variational bivectors for a system of PDEs. These are necessary conditions for the HHOs to be the Hamiltonian operators of the system of PDEs.

The key idea is that, since the coverings and the operator are both invariant with respect to point transformations of the field variables, we will get invariant conditions involving both the system of PDEs and the coefficients of the operator. We will call such conditions the *compatibility conditions* between HHOs and the system of PDEs.

More precisely, if the system of PDEs is form-invariant with respect to transformations of the dependent variables: $\bar{u}^i = \bar{u}^i(u^j)$ then the compatibility conditions will be tensor equations on the manifold which is parametrized by the dependent variables.

Since hydrodynamic type systems are form-invariant with respect to the above transformations of the dependent variables, our method allows to systematically derive geometrically invariant conditions of compatibility between HHOs of any order and hydrodynamic type systems.

In this paper we will use our method to reprove Tsarev's conditions of compatibility between local first-order HHOs and hydrodynamic type systems. We will show that the Hamiltonian is not needed at all in order to derive the conditions.

We will also consider non-local operators through the method of non-local odd variables, introduced by Kersten and Krasil'shchik [16], and obtain standard compatibility conditions between nonlocal HHOs of Ferapontov type (see [6] and Section 3.2) and hydrodynamic type systems.

Then, we will extend the method to derive compatibility conditions between second and third order HHOs and hydrodynamic type systems. While the third order case has already been successfully investigated in [29, 10], the second order case is new and surprisingly leads to a new class of systems which are highly likely to be integrable, and whose geometric properties will be studied in the future.

In a conclusive section we will indicate further directions where the above method seem to lead to potentially interesting results.

2 Coverings and Hamiltonian operators

In this Section we recall the method of cotangent covering to find Hamiltonian operators for (systems of) PDEs [14]. Consider an evolutionary system of PDEs in two independent variables t, x and n unknown functions $u^i = u^i(t,x)$, $i = 1, \ldots, n$, of the form

$$F^i = u^i_t - f^i(t,x,u^j,u^j_x,u^j_{xx},\ldots) = 0, \qquad i = 1,\ldots,n. \tag{5}$$

The above equation admits a Hamiltonian formulation if and only if there exists a linear differential operator $A = (A^{ij})$, $A^{ij} = a^{ij\sigma}\partial_\sigma$ and a linear functional $H = \int h\ dx$ such that the above system can be rewritten as

$$u^i_t = A^{ij}\left(\frac{\delta H}{\delta u^j}\right). \tag{6}$$

Here, both the coefficients $a^{ij\sigma}$ of the operator and the density h can depend on t, x, u^i, u^i_x, ... A further property is required to the operator A for the equation (6) to be Hamiltonian. Namely, the operation between densities:

$$\{F,G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\, dx \tag{7}$$

is required to be a Poisson bracket. This is equivalent to the following requirements on A: it must be skew-adjoint, $A^* = -A$, and its Schouten bracket must vanish, $[A,A] = 0$. An operator fulfilling the last two properties is said to be *Hamiltonian*. It can be proved that, for an Hamiltonian system of PDEs, the Poisson bracket of two conserved densities is a conserved density.

The presence of a Poisson bracket allows to reproduce the mathematical setting and some results that are more traditionally developed for Hamiltonian systems in mechanics. In particular, *integrability* of a system of PDEs holds if there exists an infinite sequence of conserved quantities in involution with respect to the above Poisson bracket. Magri's theorem yields conditions under which such sequences can be generated [21].

It is well-known that symmetries of (5) are vector functions $\varphi = (\varphi^i)$ such that $\ell_F(\varphi) = 0$ when $F = 0$, where ℓ_F is the linearization, or Fréchet derivative, of F. Conservation laws are equivalence classes of 1-forms $\omega = a\,dt + b\,dx$ that are closed modulo $F = 0$, up to total divergencies; they are uniquely represented by the generating function $\psi_i = \delta b/\delta u^i$. Such a vector function is a cosymmetry, *i.e.* $\ell^*_F(\psi) = 0$ when $F = 0$.

After the above definitions, it can be proved [18, 26] that a Hamiltonian operator A is a variational bivector for the equation $F = 0$, i.e. it intertwines the operators ℓ_F and ℓ^*_F, in the sense that

$$\ell_F \circ A = A^* \circ \ell^*_F. \tag{8}$$

This implies that a Hamiltonian operator maps conserved quantities into symmetries. The above property can be taken as a necessary condition for an operator A to be the Hamiltonian operator for a system of PDEs $F = 0$.

The equation (8) can be reformulated as follows. Introduce new variables p_i in such a way that $\partial_x\psi_i$ corresponds to $p_{i,x}$, $\partial_x^2\psi_i$ corresponds to $p_{i,xx}$ and so on. A bijective correspondence between operators and vector functions which are linear in p_i and their derivatives can be achieved just by evaluating the operator on the new variable(s) p_i. Let us introduce the *cotangent covering* [14] (also known as adjoint system):

$$\mathcal{T}^*\colon \begin{cases} F = 0, \\ \ell_F^*(\mathbf{p}) = 0. \end{cases} . \tag{9}$$

Working on the cotangent covering allows us to annihilate the right-hand sides of (8). This implies the following result.

Theorem 1 *[14] A linear differential operator A in total derivatives is a variational bivector* (5) *if and only if the following equation holds:*

$$\ell_F(A(\mathbf{p})) = 0 \tag{10}$$

on the cotangent covering (9)*.*

We stress that the equation $\ell_F(A(\mathbf{p})) = 0$ is a necessary condition for A to be the Hamiltonian operator of a system of PDEs $F = 0$: it does not imply that $A^* = -A$ or $[A, A] = 0$ in general. However, the equation $\ell_F(A(\mathbf{p})) = 0$ is linear and it does not contain any unknown Hamiltonian density, so it is easier to solve than finding a Hamiltonian formulation for the system of PDEs. Moreover, the condition is quite strong, so that in most cases it happens that all variational bivectors are also Hamiltonian operators. See [18] for a discussion.

We observe that the idea of representing differential operators with linear functions of new variables was also used in [11] to compute the Hamiltonian cohomology of a wide class of operators. In order to do that, one is led to assume that p_i and its derivatives are *anticommuting (odd) variables*.

In order to show the simplicity and the effectiveness of the method, let us consider the KdV equation $u_t = u_{xxx} + uu_x$. Its linearization is $\ell_F = \partial_t - \partial_{3x} - u_x - u\partial_x$, and its adjoint is $\ell_F^* = -\partial_t + \partial_{3x} + u\partial_x$. The cotangent covering is

$$\begin{cases} u_t = u_{xxx} + uu_x \\ p_t = p_{xxx} + up_x \end{cases} \tag{11}$$

Then the well-known Hamiltonian operators $A_1 = \partial_x$ and $A_2 = \frac{1}{3}(3\partial_{xxx} + 2u\partial_x + u_x)$ can be rewritten as functions which are linear in the new coordinate p:

$$A_1 = p_x, \quad A_2 = \frac{1}{3}(3p_{3x} + 2up_x + u_x p). \tag{12}$$

The above A_1, A_2 are the only linear functions A which fulfill $\ell_F(A) = 0$; this calculation is easily performed by pen and paper.

The invariance of the cotangent covering under coordinate changes of the type $\bar{u}^i = \bar{u}^i(u^j)$ is crucial in our subsequent results, and it is proved in [19].

Non-local operators also fit in the above scheme: an operator with a summand of the type $N^{ij}\psi_j = S^i(u^k, u^k_x, u^k_{xx}, \ldots)\partial_x^{-1}\big(S^j(u^k, u^k_x, u^k_{xx}, \ldots)\psi_j\big)$ (*weakly non-local operator* in the terminology of [22]) can be represented as $S^i r$, where r is a new (odd) variable such that $r_x = S^j p_j$. Such variables are potentials of conservation laws on the cotangent covering whose flux and density are linear with respect to odd variables.

3 First-order homogeneous Hamiltonian operators

In this section we use the necessary conditions of Theorem 1 to compute geometrically-invariant compatibility conditions between a hydrodynamic type system and a first-order local HHO. We will reprove well-known results by Tsarev [34] without using any Hamiltonian density.

The linearization and adjoint linearization of the system

$$F^i = u^i_t - V^i_j u^j_x \tag{13}$$

are:

$$\ell_F(\varphi) = \partial_t\varphi^i - V^i_{j,k}u^j_x\varphi^k - V^i_j\partial_x\varphi^j, \tag{14}$$

$$\ell^*_F(\varphi) = -\partial_t\psi_i - \psi_k V^k_{j,i}u^j_x + \partial_x(\psi_k V^k_i). \tag{15}$$

The cotangent covering (some Authors call it *the adjoint system*, see [13]) is determined by the following system of PDEs:

$$\mathcal{T}^*\colon \begin{cases} u^i_t = V^i_j u^j_x, \\ p_{i,t} = (V^k_{i,j}u^j_x - V^k_{j,i}u^j_x)p_k + V^k_i p_{k,x}. \end{cases} \tag{16}$$

Here and in what follows an index like $,i$ stands for the partial derivative of the indexed object with respect to u^i.

3.1 Local operators

A local first-order HHO can be identified with the linear vector function

$$A(\mathbf{p})^i = g^{ij}p_{j,x} + \Gamma^{ij}_k u^k_x p_j. \tag{17}$$

where g_{ij} is a nondegenerate matrix: $\det(g^{ij}) \neq 0$. Under a transformation $\bar{u}^i = \bar{u}^i(u^j)$ the coefficients g^{ij} transform as a symmetric contravariant 2-tensor, and $\Gamma^i_{jk} = -g_{js}\Gamma^{si}_k$ (where g_{ij} is the inverse matrix of g^{ij}) transform as the Christoffel symbols of a linear connection.

We assume that $A^* = -A$ and $[A, A] = 0$. This is equivalent to well-known conditions:

$$g^{ij} = g^{ji}, \tag{18a}$$
$$g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k, \tag{18b}$$
$$g^{ik}\Gamma^{jl}_k = g^{jk}\Gamma^{il}_k, \tag{18c}$$
$$R[g]^{ij}_{kl} = \Gamma^{ij}_{l,k} - \Gamma^{ij}_{k,l} + \Gamma^i_{ks}\Gamma^{sj}_l - \Gamma^j_{ks}\Gamma^{si}_l = 0, \tag{18d}$$

where $R[g]$ is the curvature of the metric g.

Proposition 2 *The equation $\ell_F(A(\mathbf{p})) = 0$, where A is a homogeneous operator and $F = 0$ is a hydrodynamic type system (1), is equivalent to the following system:*

$$V^i_k g^{kj} - V^j_k g^{ki} = 0, \tag{19}$$

$$\begin{aligned} &g^{ij}_k V^k_m + g^{ik}(V^j_{k,m} - V^j_{m,k}) + g^{ik}V^j_{k,m} + \Gamma^{ik}_m V^j_k \\ &\quad - V^i_{m,k}g^{kj} - V^i_k g^{kj}_m - V^i_k\Gamma^{kj}_m = 0, \end{aligned} \tag{20}$$

$$g^{ik}\left(V^j_{k,h} - V^j_{h,k}\right) + \Gamma^{ij}_k V^k_h - \Gamma^{kj}_h V^i_k = 0, \tag{21}$$

$$\begin{aligned} &g^{ik}\left(V^j_{k,ml} + V^j_{k,lm} - V^j_{m,kl} - V^j_{l,km}\right) \\ &\quad + \Gamma^{ij}_{m,k}V^k_l + \Gamma^{ij}_{l,k}V^k_m + \Gamma^{ij}_k V^k_{l,m} + \Gamma^{ij}_k V^k_{m,l} \\ &\quad + \Gamma^{ik}_l V^j_{k,m} + \Gamma^{ik}_m V^j_{k,l} - \Gamma^{ik}_l V^j_{m,k} - \Gamma^{ik}_m V^j_{l,k} \\ &\quad - \Gamma^{kj}_m V^i_{l,k} - \Gamma^{kj}_l V^i_{m,k} - \Gamma^{kj}_{m,l}V^i_k - \Gamma^{kj}_{l,m}V^i_k = 0. \end{aligned} \tag{22}$$

Proof. We have:

$$\begin{aligned} \ell_F(A(\mathbf{p})) =& \partial_t(g^{ij}p_{j,x} + \Gamma^{ij}_k u^k_x p_j) \\ & - V^i_{l,k}u^l_x(g^{kj}p_{j,x} + \Gamma^{kj}_h u^h_x p_j) \\ & - V^i_k\partial_x(g^{kj}p_{j,x} + \Gamma^{kj}_h u^h_x p_j) \end{aligned} \tag{23}$$

We must keep into account the system (16) and its differential consequences. So, u_{tx} should be replaced by $(V^i_j u^j_x)_x$ and similarly for $p_{i,tx}$. We obtain:

$$\begin{aligned}
\ell_F(A(\mathbf{p})) = {} & (-V^i_k g^{kj} + V^j_k g^{ki}) p_{j,xx} \\
& + \big(g^{ij}_k V^k_l u^l_x + g^{ik}(V^j_{k,m}u^m_x - V^j_{m,k}u^m_x) + g^{ik}V^j_{k,m}u^m_x + \Gamma^{ik}_h u^h_x V^j_k \\
& \quad - V^i_{l,k}u^l_x g^{kj} - V^i_k g^{kj}_h u^h_x - V^i_k \Gamma^{kj}_h u^h_x\big) p_{j,x} \\
& + \big(g^{ik}(V^j_{k,ml}u^l_x u^m_x + V^j_{k,m}u^m_{xx} - V^j_{m,kl}u^l_x u^m_x - V^j_{m,k}u^m_{xx}) \\
& \quad + \Gamma^{ij}_{k,h} V^h_l u^l_x u^k_x + \Gamma^{ij}_k V^k_{l,m} u^m_x u^l_x + \Gamma^{ij}_k V^k_l u^l_{xx} \\
& \quad + \Gamma^{ik}_l u^l_x (V^j_{k,h}u^h_x - V^j_{h,k}u^h_x) - V^i_{l,k}u^l_x \Gamma^{kj}_h u^h_x \\
& \quad - V^i_k(\Gamma^{kj}_{h,l}u^l_x u^h_x + \Gamma^{kj}_h u^h_{xx})\Big) p_j
\end{aligned} \tag{24}$$

Since the above expression is linear with respect to p_i, $p_{i,x}$, $p_{i,xx}$ and its coefficients are polynomial with respect to u^i_x, u^i_{xx}, the result follows. ∎

Lemma 3 *Equation (22) is a differential consequence of (21) and (19).*

Proof. Let us subtract a differential consequence of (21) from equation (22):

$$\begin{aligned}
& g^{ik}\left(V^j_{k,ml} + V^j_{k,lm} - V^j_{m,kl} - V^j_{l,km}\right) \\
& \quad + \Gamma^{ij}_{m,k}V^k_l + \Gamma^{ij}_{l,k}V^k_m + \Gamma^{ij}_k V^k_{l,m} + \Gamma^{ij}_k V^k_{m,l} \\
& \quad + \Gamma^{ik}_l V^j_{k,m} + \Gamma^{ik}_m V^j_{k,l} - \Gamma^{ik}_l V^j_{m,k} - \Gamma^{ik}_m V^j_{l,k} \\
& \quad - \Gamma^{kj}_m V^i_{l,k} - \Gamma^{kj}_l V^i_{m,k} - \Gamma^{kj}_{m,l} V^i_k - \Gamma^{kj}_{l,m} V^i_k \\
& \quad - \left(g^{ik}(V^j_{k,m} - V^j_{m,k}) + \Gamma^{ij}_k V^k_m - \Gamma^{kj}_m V^i_k\right)_l \\
& \quad - \Big(g^{ik}(V^j_{k,l} - V^j_{l,k}) + \Gamma^{ij}_k V^k_l - \Gamma^{kj}_l V^i_k\Big)_m \\
= {} & \left(\Gamma^{ij}_{m,k} - \Gamma^{ij}_{k,m}\right)V^k_l + \left(\Gamma^{ij}_{l,k} - \Gamma^{ij}_{k,l}\right)V^k_m \\
& \quad + \Gamma^{kj}_l\left(V^i_{k,m} - V^i_{m,k}\right) + \Gamma^{ki}_l\left(V^j_{m,k} - V^j_{k,m}\right) \\
& \quad + \Gamma^{kj}_m\left(V^i_{k,l} - V^i_{l,k}\right) + \Gamma^{ki}_m\left(V^j_{l,k} - V^i_{k,l}\right)
\end{aligned}$$

Using (21) again to replace all terms containing $V^i_{j,k}$ the above expression becomes

$$\begin{aligned}
& \left(\left(\Gamma^{ij}_{m,k} - \Gamma^{ij}_{k,m}\right) + \Gamma^j_{sm}\Gamma^{si}_k - \Gamma^i_{sm}\Gamma^{sj}_k\right)V^k_l + \\
& \qquad \left(\left(\Gamma^{ij}_{l,k} - \Gamma^{ij}_{k,l}\right) + \Gamma^j_{sl}\Gamma^{si}_k - \Gamma^i_{sl}\Gamma^{sj}_k\right)V^k_m + T^{ij}_{lm}
\end{aligned} \tag{25}$$

where

$$T^{ij}_{lm} = \Gamma^{aj}_{l} g_{as} V^s_k \Gamma^{ki}_{m} - \Gamma^{ki}_{l} g_{ks} V^s_a \Gamma^{aj}_{m} + \Gamma^{kj}_{m} g_{ks} V^s_a \Gamma^{ai}_{l} - \Gamma^{ki}_{m} g_{ks} V^s_a \Gamma^{aj}_{l} \qquad (26)$$

Using (19) it is easy to show that $T^{ij}_{lm} = 0$; then, (25) is equivalent to

$$R^{ij}_{km} V^k_l + R^{ij}_{kl} V^k_m \qquad (27)$$

(note that $\Gamma^{i}_{sk}\Gamma^{sj}_{l} = \Gamma^{j}_{sl}\Gamma^{si}_{k}$) which vanishes due to the Hamiltonian property of A. ∎

Lemma 4 *Equation* (20) *is a consequence of* (21) *and* (19)*.*

Proof. By (18b), equation (20) can be written in the following way:

$$\Gamma^{ij}_{k} V^k_h + \Gamma^{ji}_{k} V^k_h + g^{ik}(V^j_{k,h} - V^j_{h,k}) + g^{ik} V^j_{k,h} + \Gamma^{ik}_{h} V^j_k \\ - g^{kj} V^i_{h,k} - \Gamma^{kj}_{h} V^i_k - \Gamma^{jk}_{h} V^i_k - V^i_k \Gamma^{kj}_{h} = 0. \qquad (28)$$

By adding and subtracting $g^{kj}V^i_{k,h}$ and $\Gamma^{ki}_{h}V^j_k$ in the previous equation we realize that it is equivalent to the equation:

$$g^{ik}(V^j_{k,h} - V^j_{h,k}) + \Gamma^{ij}_{k} V^k_h - \Gamma^{kj}_{h} V^i_k + \\ g^{jk}(V^i_{k,h} - V^i_{h,k}) + \Gamma^{ji}_{k} V^k_h - \Gamma^{ki}_{h} V^j_k + \\ + \partial_h (g^{ik} V^j_k - g^{jk} V^i_k) = 0, \qquad (29)$$

thus proving the lemma. ∎

Theorem 5 *Let us consider a local first-order HHO A* (17) *and a hydrodynamic type system* (1)*. Then, the compatibility conditions $\ell_F(A(\mathbf{p})) = 0$ for the operator A to be a Hamiltonian operator for the hydrodynamic type system* (1) *are equivalent to the following system:*

1. $g^{ik}V^j_k = g^{jk}V^i_k$;
2. $\nabla^i V^j_k = \nabla^j V^i_k$.

Proof. Indeed, using $g^{ik}V^j_k = g^{jk}V^i_k$ and its differential consequences we observe that

$$g^{ik}\left(V^j_{k,h} - V^j_{h,k}\right) + \Gamma^{ij}_{k} V^k_h - \Gamma^{kj}_{h} V^i_k = g^{ik}(\nabla_h V^j_k - \nabla_k V^j_h) \qquad (30)$$

∎

Remark 6 *We stress that we proved Tsarev's Theorem without the need of a Hamiltonian density h such that $V^i_j = \nabla^i \nabla_j h$ [34], even if we are not aware of examples where there exists a Hamiltonian operator for a hydrodynamic type system but there is no Hamiltonian density.*

3.2 Non-local operators

In this section we would like to find conditions of compatibility between hydrodynamic type systems (1) and first order non-local HHOs of the type:

$$B = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + W^i_k u^k_x \partial_x^{-1} W^j_h u^h_x, \tag{31}$$

where $W^i_k = W^i_k(u^j)$. Such operators have been introduced and studied in full generality by Ferapontov; they have a beautiful geometric characterization, see [6] and references therein. The conditions for the above operator to be Hamiltonian can be found in the same reference.

The way to introduce non-local (odd) variables in order to rewrite operators as in the previous subsection makes use of new non-local variables on the cotangent covering. Non-local variables are usually introduced as conservation laws; it was proved in [15] that any symmetry of a system of PDEs yields a conservation law on the cotangent covering of the system that is linear with respect to odd variables.

Let us describe the construction of non-local variables in our case. Consider the equality that defines the adjoint linearization:

$$\langle \ell_F(\varphi), \psi\rangle - \langle \varphi, \ell^*_F(\psi)\rangle = \sum_{i=1}^{n} \partial_i(a^i). \tag{32}$$

By a restriction to $F = 0$, if φ is a symmetry the first summand at the left-hand side vanishes due to $\ell_F(\varphi) = 0$. If we lift the remaining identity on the cotangent covering $\ell^*_F(\mathbf{p}) = 0$, we have a conservation law on the right-hand side. Let us compute an explicit formula for the conservation law.

$$\begin{aligned}(\partial_t\varphi^i - V^i_{j,k}u^j_x\varphi^k - V^i_j\partial_x\varphi^j)\psi_i - \varphi^i(-\partial_t\psi_i + (V^k_{i,j} - V^k_{j,i})u^j_x\psi_k + V^k_i\partial_x\psi_k) \\ = \partial_t(\varphi^i\psi_i) - \partial_x(V^i_j\varphi^j\psi_i)\end{aligned}$$

So, the new non-local variable on the cotangent covering corresponding with each symmetry φ is denoted by r, where

$$r_t = V^i_j\varphi^j p_i, \qquad r_x = \varphi^i p_i. \tag{33}$$

The expression of B in odd variables becomes

$$B^i = g^{ij}p_{j,x} + \Gamma^{ij}_k u^k_x p_j + W^i_s u^s_x r. \tag{34}$$

Theorem 7 *Let us consider a non-local first order HHO B (31), whose non-local part is defined by a hydrodynamic type symmetry $\varphi^i = W^i_j u^j_x$, and the hydrodynamic type system (13). Then, the compatibility conditions $\ell_F(B(\mathbf{p})) = 0$ for the operator B to be a Hamiltonian operator for the hydrodynamic type system (13) are equivalent to the following system:*

1. $g^{ik}V^j_k = g^{jk}V^i_k$,
2. $\nabla^i V^j_k = \nabla^j V^i_k$.

Proof. We have

$$\ell_F(\tilde{A}^i) = \ell_F(A^i) + \ell_F(W^i_s u^s_x r), \tag{35}$$

where

$$\begin{aligned}
\ell_F(W^i_s u^s_x r) &= \partial_t(W^i_s u^s_x r) - V^i_{j,k} u^j_x W^k_s u^s_x r - V^i_j \partial_x(W^j_s u^s_x r) \\
&= W^i_{s,l} u^l_t u^s_x r + W^i_s u^s_{xt} r + W^i_s u^s_x r_t \\
&\quad - V^i_{j,k} u^j_x W^k_s u^s_x r \\
&\quad - V^i_j W^j_{s,l} u^l_x u^s_x r - V^i_j W^j_s u^s_{xx} r - V^i_j W^j_s u^s_x r_x \\
&= W^i_{s,l} V^l_k u^k_x u^s_x r + W^i_s \left(V^s_k u^k_x\right)_x r + W^i_s u^s_x V^k_l W^l_j u^j_x p_k \\
&\quad - V^i_{j,k} u^j_x W^k_s u^s_x r \\
&\quad - V^i_j W^j_{s,l} u^l_x u^s_x r - V^i_j W^j_s u^s_{xx} r - V^i_j W^j_s u^s_x W^k_l u^l_x p_k
\end{aligned}$$

The coefficient of r vanishes because when defining B we required that $\varphi^i = W^i_j u^j_x$ is a symmetry (or a commuting flow) of the hydrodynamic type system.

The only other change with respect to the local case are the coefficients of $u^l_x u^m_x p_j$. It is easy to calculate that, up to differential consequences of the conditions 1 and 2 in the statement of the Theorem, such coefficients are equal to

$$R^{ij}_{kl} V^k_m + R^{ij}_{km} V^k_l + W^i_l V^j_k W^k_m + W^i_m V^j_k W^k_l - V^i_k W^k_l W^j_m - V^i_k W^k_m W^j_l \tag{36}$$

hence they vanish due to the Hamiltonian property of B [6] and the condition $W^i_s V^s_j = W^s_j V^i_s$ from $\ell_F(\varphi) = 0$. ■

We observe that the conditions of compatibility between a non-local operator (34) and a hydrodynamic-type system are the same as the condition for a local operator (3) with the only additional requirement that the non-local part is constructed by symmetries of the system of PDEs.

Remark 8 *The above construction of non-local variables allows to avoid integrals ∂_x^{-1} which have no clean differential-geometric interpretation. However, the construction requires symmetries of the system (1); this implies that, unlike the local case, non-local operators are strongly linked to an underlying system of PDEs.*

Remark 9 *One might try to solve the systems of compatibility conditions in Theorem 5 or in Theorem 7 for a given operator A or B and unknown functions V^i_j. Usually, this approach does not work: there are too many systems that are Hamiltonian with respect to a single first-order local or non-local HHOs. We will see that the situation is completely different for higher order HHOs.*

4 Second-order homogeneous Hamiltonian operators

In this section we will derive compatibility conditions between the class of second order HHOs and systems of PDEs of hydrodynamic type. Higher order HHOs were introduced in [4] as a generalization of first-order HHOs; in particular, second order HHOs have the form $C = (C^{ij})$ where

$$C^{ij} = g^{ij}\partial_x^2 + b^{ij}_k u^k_x \partial_x + c^{ij}_k u^k_{xx} + c^{ij}_{kh} u^k_x u^h_x, \tag{37}$$

where the coefficients g^{ij}, b^{ij}_k, c^{ij}_k, c^{ij}_{kh} depend on field variables (u^j) only. The coefficients of the above operator C transform as differential-geometric objects; in the non degenerate case $\det(g^{ij}) \neq 0$, the symbols $\Gamma^i_{jk} = -g_{js}c^{si}_k$ (where (g_{ij}) is the inverse matrix of g^{ij}) transform as the Christoffel symbols of a linear connection. It was conjectured by S.P. Novikov (also for higher order HHOs) that if C is a Hamiltonian operator then Γ^i_{jk} is symmetric and flat. This statement was proved in [3, 30] (but see also [23, p. 76]). It can also be proved that in flat coordinates of that connection a Hamiltonian operator C takes the canonical form:

$$C^{ij} = \partial_x g^{ij} \partial_x, \tag{38}$$

where

$$g_{ij} = T_{ijk}u^k + g_{0ij}; \tag{39}$$

the residual conditions for C to be Hamiltonian are that T_{ijk}, g_{0ij} are constant and skew-symmetric with respect to any pair of indices.

Since a natural choice of Casimirs of the above operators is just the set of coordinates u^i, it is natural to assume that the hydrodynamic type system is conservative:

$$F^i = u^i_t - (V^i)_x = 0, \tag{40}$$

where $V^i = V^i(u^k)$.

Now, let us introduce potential coordinates $b^i_x = u^i$; then the equation can be rewritten as

$$F^i = b^i_t - V^i(b_x) = 0. \tag{41}$$

Using the standard formula for the coordinate change $\ell_b \circ C \circ \ell_b^*$ (see e.g. [25]) where the map $b = (b^i)$ is given by $b^i = \partial_x^{-1} u^i$ and $\ell_b = \partial_x^{-1}$ we have

$$C^{ij} = -g^{ij}(b^k_x), \tag{42}$$

so that the operator becomes of order 0.

The linearization of F is

$$\ell_F(\varphi)^i = \partial_t \varphi^i - V^i_{,j} \partial_x \varphi^j \tag{43}$$

and the cotangent covering is

$$\mathcal{T}^* : \begin{cases} b^i_t = V^i(b_x) \\ p_{i,t} = V^j_{,i} p_{j,x} + V^j_{,il} b^l_{xx} p_j \end{cases} \tag{44}$$

Theorem 10 *The compatibility conditions $\ell_F(C(\mathbf{p})) = 0$ for a second-order HHO C in canonical form (14) to be a Hamiltonian operator for the system (6) are*

$$g_{qj} V^j_{,p} + g_{pj} V^j_{,q} = 0, \tag{45a}$$

$$g_{qk} V^k_{,pl} + g_{pq,k} V^k_{,l} + g_{qk,l} V^k_{,p} = 0. \tag{45b}$$

Proof. The condition of compatibility $\ell_F(C) = 0$ of the Hamiltonian operator $C^i = -g^{ij} p_j$ with the system (41) is

$$\begin{aligned} \ell_F(C)^i &= (-g^{ij} p_j)_t - V^i_{,j}(-g^{jl} p_l)_x \\ &= -g^{ij}_k V^k_{,l} b^l_{xx} p_j - g^{ij} V^k_{,j} p_{k,x} - g^{ij} V^k_{,jl} b^l_{xx} p_k + V^i_{,j} g^{jl}_k b^k_{xx} p_l + V^i_{,j} g^{jl} p_{l,x} \\ &= 0. \end{aligned} \tag{46}$$

Then, $\ell_F(C) = 0$ if and only if the following two conditions hold:

$$-g^{il}V^j_{,l} + g^{lj}V^i_{,l} = 0, \tag{47}$$

$$-g^{ij}_k V^k_{,l} - g^{ik}V^j_{,kl} + g^{kj}_l V^i_{,k} = 0. \tag{48}$$

The result is obtained by lowering the indices and remembering that g_{ij} is skew-symmetric with respect to i, j. ■

At this stage, and having the previous experience with third-order HHO in mind [10], we might ask ourselves if it is possible to solve the system (45) for any given second-order HHO in order to find systems of hydrodynamic type that admit a Hamiltonian operator C as above. In the non degenerate case $\det(g^{ij}) \neq 0$, the answer is in the affirmative for low numbers n of depedent variables. Of course, there is no second-order HHO when $n = 1$.

The case $n = 2$. In this case, g_{ij} is a constant matrix. It can be easily realized that the only solution of (45) is $V^i_{,j}$ a constant for i, $j = 1, \ldots, n$, hence the resulting hydrodynamic type system is linear and not interesting to our purposes.

The case $n = 3$. In this case g_{ij}, being skew-symmetric, is always degenerate; the degenerate case will deserve a future investigation.

The case $n = 4$. In this case the space of 2-forms g_{ij} is 10-dimensional and subject to the single constraint $\det(g_{ij}) \neq 0$. Let us start by an example.

Example 11 *We consider the following* 2*-form:*

$$g_{ij} = \begin{pmatrix} 0 & b^3_x & -b^2_x & 0 \\ -b^3_x & 0 & b^1_x & 0 \\ b^2_x & -b^1_x & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{49}$$

where in (39) $T_{123} = 1$*,* $g_{034} = 1$*, other coefficients are either* 1 *or* -1 *if they are related to the above coefficients by an even or odd permutation, or they are defined to be* 0*. Then, solving the system* (45) *for the vector of fluxes* V^i

we obtain the following system (in potential coordinates)

$$\begin{cases} b^1_t = \dfrac{c_4\left(b^1_x\right)^2 + \left(c_1 b^2_x + c_2 b^3_x + c_8\right) b^1_x + c_{10} b^3_x - c_1 b^1_x - c_2}{b^3_x} \\ b^2_t = \dfrac{c_1\left(b^2_x\right)^2 + \left(c_3 b^3_x + c_4 b^1_x + c_8\right) b^2_x + c_9 b^3_x + c_4 b^4_x + c_6}{b^3_x} \\ b^3_x = c_1 b^2_x + c_3 b^3_x + c_4 b^1_x + c_7 \\ b^4_t = \dfrac{\left(c_1 b^2_x + c_3 b^3_x + c_4 b^1_x\right) b^4_x + c_2 b^2_x + c_5 b^3_x + c_6 b^1_x}{b^3_x} \end{cases} \tag{50}$$

where c_i are parameters, $i = 1, \ldots, 10$.

Indeed, we can provide a more general statement.

Theorem 12 *Let $n = 4$ and $\det(g_{ij}) \neq 0$. Then, for every second-order HHO the system (45) can be solved with respect to the unknown functions V^i, and the solution depends on at most 23 parameters.*

The resulting hydrodynamic type systems of conservation laws are linearly degenerate and diagonalizable, hence semi-Hamiltonian (according with [33, 35]). The eigenvalues of the velocity matrix of the systems are real and with algebraic multiplicity two.

Proof. The proof of the existence of solutions V^i is achieved by computer-solving of the system (45).

Then, we check that the condition

$$(\nabla f_1)(V^i_{,j})^4 + (\nabla f_2)(V^i_{,j})^3 + (\nabla f_3)(V^i_{,j})^2 + (\nabla f_4)(V^i_{,j}) = 0 \tag{51}$$

holds, where $\det(\lambda\delta^i_j - V^i_{,j}) = \lambda^4 + f_1\lambda^3 + f_2\lambda^2 + f_3\lambda + f_4$; that means that the hydrodynamic type system defined by V^i is linearly degenerate.

Finally, we check that the Haantijes tensor [12] of the tensor $V^i_{,j}$ vanishes identically; that ensures the diagonalizability of the hydrodynamic type system. ■

Remark 13 *Semi-Hamiltonian systems of hydrodynamic type are integrable by the generalized hodograph transform [35]. However, that method has been developed for hydrodynamic type systems with real distinct eigenvalues; so, strictly speaking, we cannot say that our systems are integrable. However, the fact that there are coinciding eigenvalues is not a strong restriction to the applicability of the generalized hodograph transform, as recent results show [38].*

We recall that the inverse $C_{ij} = -g_{ij}(b^k_x)$ of the Hamiltonian operator (42) is a symplectic operator of order 0 [23].

5 Third-order homogeneous Hamiltonian operators

In this section we will summarize the results of the papers [10, 29] in order to have a complete picture of the range of applicability of the method exposed in this paper.

5.1 Local operators

Local third-order homogeneous Hamiltonian operators $D = (D^{ij})$ can always be brought to the following canonical form by a transformation $\bar{u}^i = \bar{u}^i(u^j)$ (in the non-degenerate case $\det(g^{ij}) \neq 0$):

$$D^{ij} = \partial_x(g^{ij}\partial_x + c^{ij}_k u^k_x)\partial_x \tag{52}$$

[3, 32, 31]. Indeed, this is a transformation to flat coordinates of a linear symmetric connection, as in the case of second order operators. In this case the conditions for D^{ij} to be Hamiltonian are [8]:

$$g_{ij} = g_{ji}, \tag{53a}$$

$$c_{nkm} = \frac{1}{3}(g_{mn,k} - g_{kn,m}), \tag{53b}$$

$$g_{ij,k} + g_{jk,i} + g_{ki,j} = 0, \tag{53c}$$

$$c_{nml,k} + c^s_{ml}c_{snk} = 0. \tag{53d}$$

where $(g_{ij})^{-1} = (g^{ij})$ and $c_{ijk} = g_{iq}g_{jp}c^{pq}_k$. We recall that the pseudo-Riemannian metrics g_{ij} fulfilling (53c) represent quadratic line complexes in Monge form [8].

Theorem 14 ([10]) *The compatibility conditions $\ell_F(D(\mathbf{p})) = 0$ for a third-order HHO D (52) to be a Hamiltonian operator for the hydrodynamic type system of conservation laws (40) are equivalent to the following system:*

$$g_{im}V^m_{,j} = g_{jm}V^m_{,i}, \tag{54a}$$

$$c_{mkl}V^m_{,i} + c_{mik}V^m_{,l} + c_{mli}V^m_{,k} = 0, \tag{54b}$$

$$V^k_{,ij} = g^{ks}c_{smj}V^m_{,i} + g^{ks}c_{smi}V^m_{,j}, \tag{54c}$$

Proof. The proof goes exactly like in the case of first-order and second-order HHOs, being only considerably more complicated under the viewpoint of the calculations. See [10] for details. ■

It is also proved in [10] that, given a third-order HHO, there exists a multiparameter family of systems of conservation laws (40) solving (54). More precisely, the system (54) is reduced to a linear algebraic system. The solutions of the system admit the third-order HHO as its Hamiltonian operator; non-local Hamiltonian, momentum and Casimirs are provided.

It is proved that systems of conservation laws admitting a third-order HHO are *linearly degenerate* and **non-diagonalizable** (at difference with the second-order case). The invariance of the systems of conservation laws together with their third-order HHOs is up to projective reciprocal transformations of the type [8]

$$\begin{gathered} d\tilde{x} = \Delta dx, \qquad d\tilde{t} = dt, \\ \tilde{u}^i = \frac{a^i_j u^j + a^i_0}{\Delta}, \qquad \Delta = a^0_j u^j + a^0_0, \end{gathered} \tag{55}$$

where a^i_j, a^i_0, a^0_j, a^0_0 are constants.

This makes the above systems of conservation laws interesting objects of study. The integrability of such systems is still an open question, although in some cases it holds true by Lax pairs [10] or bi-Hamiltonian formalism by another HHO which is local, first-order and compatible with the third-order HHO in the case of WDVV equations (see [28] and references therein).

5.2 Non-local operators

Non-local third order HHOs have been considered in the literature; see [1] and references therein for a detailed study.

An interesting instance of such operators is the Hamiltonian operator of the Oriented Associativity Equation. Such a system can be written as the following hydrodynamic type system of conservation laws:

$$\begin{aligned} q^1_t &= q^2_x, \qquad q^2_t = \partial_x \frac{q^2 q^6 + q^1 q^4 - q^2 q^3}{q^5}, \\ q^3_t &= q^4_x, \qquad q^4_t = \partial_x \frac{q^2 + q^4 q^6}{q^5}, \end{aligned} \tag{56}$$

$$q^5_t = q^6_x, \qquad q^6_t = \partial_x \frac{(q^6)^2 - q^3 q^6 + q^4 q^5 - q^1}{q^5}.$$

The system was introduced in this form in [27], where a first-order local HHO was provided. In [29] the following ansatz was introduced for a non-local third-order HHO $E = (E^{ij})$:

$$E^{ij} = \partial_x (g^{ij}\partial_x + c^{ij}_k q^k_x + c^\alpha w^i_{\alpha k} q^k_x \partial_x^{-1} w^j_{\alpha h} q^h_x)\partial_x \tag{57}$$

and $w^i_{\alpha k} = w^i_{\alpha k}(q^j)$, with $c^\alpha \in \mathbb{R}$. The conditions on E to be Hamiltonian are [1]

$$w_{\alpha ij} + w_{\alpha ji} = 0, \tag{58a}$$

$$w_{\alpha ij,l} - c^s_{ij} w_{\alpha sl} = 0, \tag{58b}$$

$$c_{nml,k} + c^s_{ml} c_{snk} + c^\alpha w_{\alpha ml} w_{\alpha nk} = 0, \tag{58c}$$

in addition to (53a), (53b), (53c) (of course, (58c) is a modification of (53d)), where $w_{ij} = g_{is} w^s_j$. We remain with the problem of determining the tensors $w^i_{\alpha j}$. In this case, the condition $\ell_F(E(\mathbf{p})) = 0$ of compatibility between E and the Oriented Associativity equation (56) is equivalent to the system (54) supplemented by the equations

$$\begin{aligned} &- w^i_{\alpha h,k} V^k_m - w^i_{\alpha m,k} V^k_h - w^i_{\alpha k} V^k_{m,h} \\ &\quad - w^i_{\alpha k} V^k_{h,m} + V^i_k w^k_{\alpha m,h} + V^i_k w^k_{\alpha h,m} = 0 \end{aligned} \tag{59a}$$

$$- w^i_{\alpha k} V^k_h + V^i_k w^k_{\alpha h} = 0 \tag{59b}$$

The above conditions (59) are equivalent to the fact that $\varphi^i = w^i_{\alpha j}(\mathbf{b}_x) b^j_{xx}$ are symmetries of the system (41), which is the transformed system (40) after the potential substitution $u^i = b^i_x$. Indeed, it can be proved that any such symmetry yields the conservation law r_α on the cotangent covering that is determined by

$$r_{\alpha t} = V^i_j w^j_{\alpha k} b^k_{xx} p_i, \quad r_{\alpha x} = w^i_{\alpha k} b^k_{xx} p_i. \tag{60}$$

The non-local variables r_α allow us to represent the operator E as

$$E^i(\mathbf{p}) = -g^{ij} p_{j,x} - c^{ij}_k b^k_{xx} p_j - c^\alpha w^i_{\alpha k} b^k_{xx} r_\alpha. \tag{61}$$

The solution g_{ij} of the system (54) for the Oriented Associativity Equation (56) is unique; indeed, g_{ij} turns out to be a Monge metric of a quadratic line complex, a fact already observed for WDVV equations. The non-local part of the operator has three summands generated by two symmetries of (56). See [29] for the detailed expression of the operator E.

6 Conclusions

In this paper we showed how the cotangent covering of a system of PDEs can help to find geometrically invariant conditions of compatibility with homogeneous Hamiltonian operators. This is not the only domain of applicability of the method: indeed, the following areas might benefit of a similar approach.

- Multidimensional HHOs are considered in the literature, and are present in a number of examples (see, for example, [23, 7]). The cotangent covering might be used to relate the operators to hydrodynamic type systems in more than two independent variables.

- Dually, homogeneous symplectic operators could be considered [2]; in this case the tangent covering should be employed. See [18] for more details.

- Non-homogeneous cases might be considered, splitting them into homogeneous components with different scaling. That could be a general framework for operators of KdV-type [20].

As a by-product of the systematic presentation of results for HHOs in this paper, we obtained a new family of hydrodynamic type systems associated with second-order HHOs. It is highly likely that the systems are integrable, being semi-Hamiltonian and endowed with a second-order HHO. An important task would be the integration of the systems by the generalized hodograph transform. By analogy with [29], a non-local ansatz for a second order HHO can be easily figured out.

Along the lines of [10], we *conjecture* that both the second-order HHOs and the associated hydrodynamic type systems could be invariant with respect to projective reciprocal transformations (55). If this turned out to be true, then that might be another sign (together with similar results for third-order operators, see Subsection 5.1) that projective geometry underlies the deformation theory as developed by B.A. Dubrovin and co-workers.

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