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Symmetries in covariant classical mechanics

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Abstract

In the framework of covariant classical mechanics (i.e. general relativistic classical mechanics on a spacetime with absolute time), developed by Jadczyk and Modugno, we analyse systematically the relationship between symmetries of geometric objects. We show that the (holonomic) infinitesimal symmetries of the cosymplectic structure on spacetime and of its horizontal potentials are also symmetries of spacelike metric, gravitational and electromagnetic fields, Euler-Lagrange morphism and Lagrangians. Then, we provide a definition for a covariant momentum map associated with a group of cosymplectic symmetries by means of a covariant lift of functions of phase space. In the case of holonomic symmetries, we see that the any covariant momentum map takes values in the quantizable functions in the sense of Jadczyk and Modugno, i.e. functions quadratic in velocities with leading coefficient proportional to the spacelike metric. Finally, we illustrate the results by some examples.

Key words: covariant classical mechanics, cosymplectic manifolds, Lie groups and symmetries, jets, contact structure.

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Introduction

At the beginning of the 90's, M. Modugno and A. Jadczyk proposed a new geometric framework for covariant classical and quantum mechanics on a curved spacetime with absolute time [1, 2, 3], based on jets, connections and cosymplectic forms. This approach was later developed in [4, 5, 6, 7, 8, 9, 10, 11]. In the following, this theory will be referred to as 'covariant classical Galilei' (*CCG*) and 'covariant quantum Galilei' (*CQG*) theory.

The goal of this paper is to analyse systematically the symmetries of the structures involved in the *CCG* theory and to introduce an associated momentum map for conserved quantities.

The *CCG*–*CQG* theory was partially inspired by the wide literature on geometric formulations of classical and quantum mechanics. Main sources were symplectic and cosymplectic classical mechanics [12, 13, 14, 15, 16, 17, 18, 19], Newton or Galilei classical mechanics as general relativistic theories [20, 21, 22, 23, 24, 25, 26, 27, 28], and quantum theories of mechanics within a symplectic or cosymplectic framework (such as geometric quantization) [29, 23, 15, 30, 31, 32, 33, 28, 34].

The *CCQ*–*CQG* theory shares nice ideas with the above literature, trying at the same time to avoid some typical problems [8]. For instance, the theory is explicitly covariant with respect to changes of coordinates, even time–dependent ones. This feature partially comes from the cosymplectic structure of the classical phase space, and overcomes the problem of explicit time independence of symplectic mechanics. Additionally, the theory is covariant with respect to the choice of units of measurement, due to the use of 'unit spaces'. Regarding the application, the theory is supported by non trivial physically relevant examples, such as the quantized rigid body [7]. Another important feature is that *CCG* theory is a one–dimensional multisymplectic theory providing a 'working' quantum theory, i.e. *CQG*. Such a problem has not been solved, yet, in field theory (see [35, 36] and references therein). See [8] for a more complete comparison of the *CCG*–*CQG* theory with standard literature.

In this paper, we provide two main results.

Firstly, we analyze systematically the *symmetries* of the structures involved in *CCG* theory. We find the set of natural bijections between geometric objects that rule the dynamics of particle in *CCG* theory. By these bijections we deduce analogous correspondences for symmetries and infinitesimal symmetries. Note that some similar bijection is known in literature about non degenerate mechanical systems (see [19], for instance), but neither to the same extent nor in the same framework as us.

Secondly, in order to study conserved quantities, we define a *momentum map* associated to the action of a group of symmetries of *CCG* theory. We show that all conserved quantities coming from holonomic symmetries are of a special kind, i.e. *quantizable functions*, according to Jadczyk and Modugno. Such functions are quadratic polynomials with respect to velocities, and their second derivative is proportional to the spacelike metric on spacetime. This result essentially comes from covariance, and seems to be a completely new feature of *CCG* mechanics with respect to symplectic and cosymplectic classical mechanics. In a sense, it is a way by which the geometry of spacetime selects a subspace of the space of functions on phase space, namely the space of quantizable functions. Moreover, this result seems to be a covariant and classical analogue of the Groenwald–van Hove theorem [37].

In a subsequent paper we shall apply these results to the covariant quantum Galilei theory.

The paper has the following structure.

In the second section we recall the basic geometric objects of the *CCG* theory [2, 1, 3, 6], emphasising natural bijective correspondences between these objects.

Classical spacetime in *CCG* theory is an $(n+1)$–dimensional manifold fibred over a 1-dimensional affine space. The geometric objects by which *CCG* mechanics is built are a scaled vertical metric over spacetime, a linear connection of the tangent bundle of spacetime which is compatible with the time form, called spacetime connection (*gravitational field*), and a scaled 2–form f of spacetime (*electromagnetic field*). The phase space is taken to be the first jet space with respect to the spacetime fibring. When a mass and a charge are chosen, the fields can be incorporated naturally into a distinguished 2–form on spacetime, called 'total' dynamical phase 2–form. The dynamical equations for the fields are assumed by requiring the total form to be closed. In this case, phase space endowed with this form and the time form constitutes a *cosymplectic manifold*. The corresponding Reeb vector field γ [12, 13, 14, 16, 17] yields the dynamics on spacetime through its flow.

We shall see that there is a natural bijective correspondence between spacetime connections, affine connections of the phase space (called phase connections), and homogenous connections of the phase space (called dynamical connections). Reeb vector fields of dynamical phase 2–forms are dynamical connections, so any dynamical phase 2–form yields a dynamical connection encoding gravitational and electromagnetic structures.

Moreover, there is a natural bijective correspondence between the pairs of a phase connection and a spacelike metric and the dynamical phase 2–forms. On the other hand, there is a natural bijective correspondence between the pairs of a dynamical connection and a spacelike metric and distinguished 2–forms of the second jet space, called the 'horizontal phase 2–forms'. These forms can also be obtained by dynamical phase 2–forms through horizontalization with respect to the contact structure of the second jet space.

Finally, we introduce a special class of potentials of closed phase 2–forms Ω, namely dynamical phase 1–forms Θ. Their horizontal part $\mathcal{L}$ is a Lagrangian density on the phase space. We prove that dynamical phase 1–forms are in bijection with Lagrangian densities obtained in this way. Moreover, it turns out that dynamical phase 1–forms are the Poincaré–Cartan forms for their corresponding Lagrangians. On the other hand, if we evaluate the Euler–Lagrange operator $\epsilon(\mathcal{L})$ of these Lagrangians, we discover that it is exactly the horizontal phase 2–form corresponding to the differential of the dynamical phase 1–form. Summarising, we have the following diagram

$$\begin{tikzcd} \Theta \arrow[r, "d"] \arrow[d, "i_{\mathcal{D}_1}"'] & \Omega \arrow[d, "i_{\vartheta_2} i_{\mathcal{D}_2}"] \\ \mathcal{L} \arrow[r, "\epsilon"] & \mathcal{E} \end{tikzcd}$$

We also recall the notion of τ-Hamiltonian lift [2, 1, 3], which is a covariant lift of functions of the phase space to vector fields of phase space, motivated from the *CQG* theory. We will see that this lift plays an important role in the definition of momentum map.

In the third section, we analyse systematically the infinitesimal symmetries of the geometric objects that we have introduced so far. Hereby, we are mainly interested in holonomic symmetries.

As one could expect, the bijective correspondences in our model imply correspondences also in symmetries. We prove, indeed, that any holonomic infinitesimal symmetry of a spacetime connection is a symmetry of the corresponding phase connection and of the corresponding dynamical connection, and vice-versa.

Moreover, we prove that any holonomic infinitesimal symmetry of a dynamical phase 2–form form is also a symmetry of the corresponding pair of a spacelike metric and a phase connection, and vice–versa.Similarly, we prove that any holonomic infinitesimal symmetry of an Euler–Lagrange morphism is also a symmetry of the corresponding pair of a spacelike metric and a dynamical connection, and vice–versa.Eventually, we prove that the dynamical connection corresponding to a closed dynamical phase 2–form is the only second order connection which is a (nonholonomic) symmetry of the cosymplectic structure.

Then, we prove that any holonomic infinitesimal symmetry of a Poincaré–Cartan form is a symmetry of the corresponding Lagrangian, and vice–versa.Consequently, we are able to give two equivalent versions of Noether's theorem, first for symmetries of Poincaré–Cartan forms and then, for symmetries of Lagrangians.

In subsection II.0.7, we provide a definition of covariant momentum map for an action of a group of symmetries of the cosymplectic structure. This definition is similar to momentum map in presymplectic and cosymplectic literature [12, 13, 14, 16, 17]. However, the covariance of our theory requires the concept of τ-Hamiltonian lifts. The momentum map has its values in the pairs of a conserved quantity and an element of the space of time units.This yields a Lie algebra morphism between such pairs with respect to the (extended) Poisson bracket, and τ-Hamiltonian lifts of these pairs, with respect to the standard Lie bracket.In the particular case of a group of symmetries which projects on spacetime, the momentum map turns out to take values into the quantizable functions, i.e. functions which are polynomials of second degree in the velocities, and whose second derivative (with respect to velocities) is proportional to the metric.Quantizable functions are of fundamental importance in *CQG* theory [2, 1, 3]. This feature is not present in general symplectic or cosymplectic momentum maps. Then, we consider an action of a group of symmetries that, additionally, is a group of symmetries of a Poincaré–Cartan form. Here, a natural momentum map is determined by the Poincaré–Cartan form.

Finally, three very simple examples are provided in order to show the machinery at work. These examples show also that, in standard time–independent situations of particle mechanics, we get the same results as standard symplectic mechanics [18]. However, we are able to treat time–dependent cases, as well.

I Classical theory

We recall the main geometric objects of *CCG* theory [4, 1, 2, 3], and give natural bijective correspondences between them.

Unit spaces

Now, we are going to assume the fundamental spaces of units of measurement and constants.

The theory of *unit spaces* has been developed in [2, 1] to make the independence of classical and quantum mechanics from scales explicit. Unit spaces are defined similarly to vector spaces, but using $I\!R^+$ instead of $I\!R$. We will use 1–dimensional (over $I\!R^+$) unit spaces. It is possible to define n–th tensorial powers and n–th roots of unit spaces. Moreover, if $\mathbb{P}$ is a positive unit space and $p \in \mathbb{P}$, then we denote by $1/p \in \mathbb{P}^*$ the dual element. Hence, we can set $\mathbb{P}^{-1} := \mathbb{P}^*$. In this way, we can introduce rational powers of unit spaces.

We assume the following unit spaces.

–$\mathbb{T}$, the oriented one–dimensional semi-vector space of *time intervals*;

–$\mathbb{L}$, the positive unit space of *length units*;

–$\mathbb{M}$, the positive unit space of *mass units*;

We denote by $\overline{\mathbb{P}} := \mathbb{P} \otimes I\!R$ the associated vector space to the unit space $\mathbb{P}$. An element $u_0 \in \mathbb{T}$ (or $u^0 \in \mathbb{T}^{-1}$) represents a *time unit of measurement*, a *charge* is represented by an element $q \in \mathbb{Q} := \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$, and a *particle* is represented by a pair (m, q), where m is a mass and q is a charge. A tensor field with values into mixed rational powers of $\mathbb{T}$, $\mathbb{L}$, $\mathbb{M}$ is said to be *scaled*. We assume the *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We remark that Lie derivative of scaled tensor fields commutes with the scaling.

In the following, we assume all manifolds and maps to be C^∞.

Spacetime and phase space

Assumption 0.1. We assume *spacetime* to be a $(n+1)$-dimensional oriented fibred manifold

$$t : \boldsymbol{E} \to \boldsymbol{T}$$

over a 1-dimensional oriented affine space $\boldsymbol{T}$ (*time*) associated with the vector space $\bar{\mathbb{T}}$, where $n \in \mathbb{N}$ with $n \geq 2$. $\square$

We shall refer to *spacetime charts* (x^0, x^i), which are adapted to the fibring, to a time unit of measurement $u_0 \in \mathbb{T}$ and to the chosen orientation of $\boldsymbol{E}$. The index 0 will refer to the base space, Latin indices $i, j, \dots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both to the base space and the fibres.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$. The coordinate expression of a motion s is of the type $s^i := x^i \circ s : \boldsymbol{T} \to I\!R$.

We shall be involved with the *tangent bundle* $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$ and the *vertical tangent subspace* $V\boldsymbol{E} := \ker Tt \subset T\boldsymbol{E}$. We denote the charts induced on $T\boldsymbol{E}$ by $(x^\lambda, \dot{x}^\lambda)$; moreover, we denote the induced local bases of vector fields, of forms and of vertical forms of $\boldsymbol{E}$, respectively, by (∂_λ), (d^λ) and $(\check{d}^i)$.

The *phase space* is defined to be the first jet space of motions $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ (see [38, 39] for the theory of jet spaces). We denote the charts induced on $J_1\boldsymbol{E}$ by (x^0, x^i, x^i_0). We will be involved with the second jet space of motions $t^2_0 : J_2\boldsymbol{E} \to \boldsymbol{E}$. We denote the charts induced on $J_2\boldsymbol{E}$ by $(x^0, x^i, x^i_0, x^i_{00})$.

The velocity of a motion s is the section $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$, with coordinate expression $x^i_0 \circ j_1 s = \partial_0 s^i$.

The phase space is equipped with the natural maps $\mathcal{D}_1 : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\vartheta_1 : J_1\boldsymbol{E} \to T^*\mathbb{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$, with coordinate expressions $\mathcal{D}_1 = u^0 \otimes \mathcal{D}_{1,0} = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\vartheta_1 = \vartheta^i_1 \otimes \partial_i = (d^i - x^i_0\, d^0) \otimes \partial_i$. Analogously, the jet space $J_2\boldsymbol{E}$ is equipped with the natural maps $\mathcal{D}_2 : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ and $\vartheta_2 : J_2\boldsymbol{E} \to T^*\mathbb{E} \underset{J_1\boldsymbol{E}}{\otimes} VJ_1\boldsymbol{E}$, with coordinate expressions $\mathcal{D}_2 = u^0 \otimes \mathcal{D}_{2,0} = u^0 \otimes (\partial_0 + x^i_0\, \partial_i + x^i_{00}\, \partial^0_i)$ and $\vartheta_2 = \vartheta^i_2 \otimes \partial_i = (d^i_0 - x^i_{00}\, d^0) \otimes \partial^0_i + (d^i - x^i_0\, d^0) \otimes \partial_i$.

An *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. Its coordinate expression is of the type $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$, where $o^i_0 : \boldsymbol{E} \to I\!R$.

An observer o can be regarded as a scaled vector field of $\boldsymbol{E}$. The *integral motions* of an observer o are defined to be the motions s such that $j_1 s = o \circ s$. An observer o yields locally a fibred splitting $\boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{P}$, where $\boldsymbol{P}$ is the manifold of integral motions of o. An observer is said to be *complete* if it yields a global splitting of $\boldsymbol{E}$. A spacetime chart is said to be *adapted* to o if it is adapted to the local splitting of $\boldsymbol{E}$ induced by o, i.e. if $o^i_0 = 0$.

An observer o can be regarded as a connection of the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$. Accordingly, it yields the translation fibred isomorphism $\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$, given by $\nabla[o](e_1) := e_1 - o(t^1_0(e_1))$. We have the coordinate expression $\nabla[o] = (x^i_0 - o^i_0)\, d^0 \otimes \partial_i$.

Natural bijective correspondences

In the following we define distinguished objects living on spacetime or phase space and we investigate their relations. In a concrete model of a classical system these objects will be determined by further assumptions (see subsection II.0.7).

Definition I.1. A scaled vertical Riemannian metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}) \tag{I.1}$$

is said to be a *spacelike metric.* □

The coordinate expression of a spacelike metric g is $g = g_{ij}\, \check{d}^i \otimes \check{d}^j$, where $g_{ij} : \boldsymbol{E} \to \mathbb{L}^2$.

Given a mass $m \in \mathbb{M}$, it is convenient to introduce a "normalized" metric $G \equiv \frac{m}{\hbar} g$, with coordinate expression $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, where $G^0_{ij} : \boldsymbol{E} \to I\!\!R$.

Next, we analyse distinguished connections that can be defined on spacetime or phase space.

Definition I.2. A *spacetime connection* is defined to be a dt–preserving torsion free linear connection of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E}\,. \tag{I.2}$$

□

The coordinate expression of any spacetime connection K is of the type $K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\nu\, \dot{x}^\nu\, \dot{\partial}_i)$, where $K_\nu{}^i{}_\lambda = K_\lambda{}^i{}_\nu : \boldsymbol{E} \to I\!\!R$. The compatibility with dt, i.e. the condition $\nabla[K]dt = 0$, is expressed by $K_\mu{}^0{}_\nu = 0$.

The restriction of a spacetime connection K to the vertical tangent bundle is a linear connection $K' : V\boldsymbol{E} \to T^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} TV\boldsymbol{E}$.

A spacetime connection K is said to be *metric* if $\nabla[K']\,G = 0$. In this case, K is partially determined by the metric according to the local formulas $K_{ihj} = -\frac{1}{2}(\partial_i G_{hj} + \partial_j G_{hi} - \partial_h G_{ij})$ and $K_{0ij} + K_{0ji} = -\partial_0 G_{ij}$, where indices have been raised or lowered by the metric G.

Definition I.3. A *phase connection* is defined to be a torsion–free affine connection of the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$

$$\Gamma : \boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E}\,. \tag{I.3}$$

Here the torsion is defined through the vertical valued form ϑ_1 [41]. □

The coordinate expression of Γ is of the type $\Gamma = d^\lambda \otimes \big(\partial_\lambda + \Gamma_{\lambda}{}^{i}_{0}\, \partial^0_i\big)$, where $\Gamma_{\lambda}{}^{i}_{0} \equiv \Gamma_{\lambda}{}^{i\,0}_{0h}\, x^h_0 + \Gamma_{\lambda}{}^{i\,0}_{00}$ and $\Gamma_{\lambda}{}^{i\,0}_{0\mu} : \boldsymbol{E} \to I\!\!R$.

It can be easily seen that there is a natural bijective correspondence [1, 5, 3]

$$K \leftrightarrow \Gamma[K]$$

between spacetime connections and phase connections, namely such that the coordinate expression are given by $\Gamma_{\lambda}{}^{i\,0}_{0\mu} = K_\lambda{}^i{}_\mu$.

A *second order connection* of spacetime is defined to be a (nonlinear) connection of the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$

$$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}\,, \tag{I.4}$$

which is projectable on the contact map $\mathcal{D}_1$. The coordinate expression of γ is of the type $\gamma = u^0 \otimes (\partial_0 + x^i_0\partial_i + \gamma_{00}{}^i_0)$, where $\gamma_{00}{}^i_0 : J_1\boldsymbol{E} \to I\!R$.

Definition I.4. A second order connection is said to be a *dynamical connection* if it is "homogeneous" in the sense of [40], i.e. if its coordinate expression is of the type $\gamma_{00}{}^i_0 \equiv \gamma_{0}{}_{0}{}^{i\,0\,0}_{hk}\, x^h_0 x^k_0 + 2\,\gamma_{0}{}_{0}{}^{i\,0}_{h}\, x^h_0 + \gamma_{0}{}_{00}{}^{i\,0}_{0}$ and $\gamma_{0}{}_{0}{}^{i\,0\,0}_{hk},\ \gamma_{0}{}_{0}{}^{i\,0}_{\mu} : \boldsymbol{E} \to I\!R$. □

We will be involved with the covariant derivative of a dynamical connection, i.e. the morphism $\nabla[\gamma] : J_2\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E}$ with coordinate expression $\nabla[\gamma] = u^0 \otimes (x^i_{00} - \gamma_{00}{}^i_0)d^0 \otimes \partial_i$.

We can easily see that the map $\Gamma \mapsto \gamma[\Gamma] := \mathcal{D}_1 \lrcorner\, \Gamma$ is a natural bijective correspondence [6]

$$\Gamma \leftrightarrow \gamma[\Gamma]$$

between the phase connections and dynamical connections, namely such that $\gamma_{0}{}_{0}{}^{i\,0\,0}_{hk} = \Gamma_{h}{}^{i\,0}_{0k}$, $\gamma_{0}{}_{0}{}^{i\,0}_{\lambda} = \Gamma_{\lambda}{}^{i\,0}_{00}$.

Next, we define distinguished forms of $J_1\boldsymbol{E}$ and $J_2\boldsymbol{E}$ through the above objects.

Definition I.5. A 2–form Ω of $J_1\boldsymbol{E}$ of the type

$$\Omega[G,\Gamma] = \nu[\Gamma]\,\bar{\wedge}\,\vartheta_1 : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}\,, \tag{I.5}$$

where $\nu[\Gamma]$ is the vertical projection associated with a phase connection Γ and where the contracted wedge product is taken with respect to a spacelike metric G a *dynamical phase 2–form.* □

We have the coordinate expression $\Omega[G,\Gamma] = G^0_{ij}\,(d^i_0 - \Gamma_{\lambda}{}^i_0\, d^\lambda) \wedge \vartheta^j_1 = G^0_{ij}\,(d^i_0 - \gamma_{00}{}^i_0 d^0 - \Gamma_{h}{}^i_0\, \vartheta^h_1) \wedge \vartheta^j_1$.

We observe that the above form is the only natural 2–form which can be obtained from Γ and G [5, 1]. Moreover, the form $\Omega[G,\Gamma]$ is non degenerate in the sense that $dt \wedge \Omega[G,\Gamma]^n$ is a volume form of $J_1\boldsymbol{E}$. We can easily prove that there is a unique scaled vector field $X : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ such that $i_X\, dt = 1$ and $i_X\,\Omega[G,\Gamma] = 0$, namely, $X = \gamma[\Gamma]$, the dynamical connection. This yields a natural bijective correspondence

$$(G,\Gamma) \leftrightarrow \Omega[G,\Gamma]$$

between pairs of a spacelike metric and a phase connection and dynamical phase 2–forms.

Definition I.6. A 2–form $\mathcal{E}$ of $J_2\boldsymbol{E}$ of the type

$$\mathcal{E}[G,\gamma] = \nabla[\gamma]\,\bar{\wedge}\,\vartheta_1 : J_2\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}\,, \tag{I.6}$$

where $\nabla[\gamma]$ is the covariant differential associated with a dynamical connection γ and where the contracted wedge product is taken with respect to a spacelike metric G is called a *horizontal phase 2–form.* □

We have the coordinate expression $\mathcal{E}[G,\gamma] = G^0_{ij}(x^i_{00} - \gamma^i_{00})d^0 \wedge (d^j - x^j_0 d^0)$.

Analogously to the case of a dynamical phase 2–form, it turns out that there is a natural bijective correspondence

$$(G,\gamma) \leftrightarrow \mathcal{E}[G,\gamma]$$

between pairs of a spacelike metric and a dynamical connection and horizontal phase 2–forms. Consequently, we obtain the natural bijective correspondence

$$\Omega[\Gamma, G] \leftrightarrow \mathcal{E}[\nabla\gamma[\Gamma], G]$$

between dynamical phase 2–forms and horizontal phase 2–forms. The horizontal phase 2–form $\mathcal{E}$ corresponding to a dynamical phase 2–form Ω coincides with the horizontal part of Ω according to the contact splitting of forms of $J_1\boldsymbol{E}$ induced by $J_2\boldsymbol{E}$ [6].

Let us consider the case of a closed dynamical phase 2–form Ω. We can prove [2, 1, 41, 3] that $d\Omega = 0$ is equivalent to the conditions that $\nabla[K']G = 0$ and that, in coordinates, the curvature $R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}$ fulfills $R_{\lambda}{}^{i}{}_{\mu}{}^{j} = R_{\mu}{}^{j}{}_{\lambda}{}^{i}$. Given an observer o we can define the 2–form of $\boldsymbol{E}$ $\Phi[o] := 2o^*\Omega : \boldsymbol{E} \to \overset{2}{\wedge} T^*\boldsymbol{E}$. It turns out that $d\Omega = 0$ is also equivalent to the conditions $\nabla[K']G = 0$ and $d\Phi[o] = 0$.

Remark I.1. If the phase space $J_1\boldsymbol{E}$ is equipped with a closed dynamical phase 2–form Ω, the triple $(J_1\boldsymbol{E}, \Omega, dt)$ turns out to be a (scaled) cosymplectic manifold in the sense of [12, 13] (see also references therein). The associated dynamical connection turns out to be the (scaled) Reeb vector field for this cosymplectic structure.

Moreover, if Ω is a closed dynamical phase 2–form, it admits potential 1–forms of $J_1\boldsymbol{E}$. In the following we introduce a special kind of such potential forms.

Definition I.7. A horizontal 1–form $\Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ such that $d\Theta = \Omega$ where Ω is a closed dynamical phase 2–form is said to be a *dynamical phase 1–form* associated with Ω. □

For any observer o, the expression of a dynamical phase 1–form associated with Ω in adapted coordinates is given by $\Theta = -(\frac{1}{2}G^0_{ij}x^i_0 x^j_0 - A_0)d^0 + (G^0_{ij} + A_i)d^i$ where $A_\lambda d^\lambda$ is a potential of the closed 2–form $\Phi[o] = 2o^*\Omega$. Clearly, a dynamical phase 1–form associated with Ω is determined up to a closed 1–form of $\boldsymbol{E}$.

Remark I.2. The triple $(J_1\boldsymbol{E}, \Omega, \Theta)$ *is not a contact structure* in the sense of [16]: the form $\Theta \wedge \Omega^n$ is not a volume form on $J_1\boldsymbol{E}$.

According to the contact splitting of Θ induced by $J_1\boldsymbol{E}$ we define the following object

Definition I.8. The *Lagrangian* $\mathcal{L}$ associated with a phase 1–form Θ is defined to be the horizontal 1–form

$$\mathcal{L}[\Theta] := \mathcal{D}_1 \lrcorner \Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{T}\,. \tag{I.7}$$

The *momentum* $\mathcal{P}[\mathcal{L}]$ of the Lagrangian $\mathcal{L}$ is defined to be the vertical restriction of $\mathcal{L}$ with respect to the fibring $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$, i.e. $\mathcal{P}[\mathcal{L}] := V_{\boldsymbol{E}}\mathcal{L}$. □

We have the coordinate expressions $\mathcal{L}[\Theta] = (\frac{1}{2}G^0_{ij}x^i_0x^j_0 + A_i x^i_0 + A_0)d^0$ and $\mathcal{P}[\mathcal{L}] = (G^0_{ij}x^j_0 + A_i)(d^i - x^i_0 d^0)$.

It turns out that Θ splits into $\Theta = \mathcal{L}[\Theta] + \mathcal{P}[\mathcal{L}[\Theta]]$. This splitting coincides with the contact splitting of Θ induced by $J_1\boldsymbol{E}$.

This yields directly the natural bijective correspondence

$$\Theta \leftrightarrow \mathcal{L}[\Theta]$$

between dynamical phase 1–forms and Lagrangians.

Moreover, $\Theta[\mathcal{L}]$ turns out to be the Poincaré–Cartan form associated with the Lagrangian $\mathcal{L}$. Hence, in the following, we say any dynamical phase 1–form to be a *Poincaré–Cartan form.*

Given a closed dynamical phase 2–form Ω it can be proved that the corresponding horizontal phase 2–form $\mathcal{E}[\Omega]$ coincides with the Euler-Lagrange morphism associated with any corresponding Lagrangian $\mathcal{L}$ of a Poncaré-Cartan form Θ of Ω. Hence, the above results concerning Θ, Ω, $\mathcal{L}$ and $\mathcal{E}$ are described by the following commutative diagram

$$\begin{tikzcd}
\Theta \arrow[r, "d"] \arrow[d, "i_{\mathcal{D}_1}"'] & \Omega \arrow[d, "i_{\vartheta_2} i_{\mathcal{D}_2}"] \\
\mathcal{L} \arrow[r, "\epsilon"] & \mathcal{E}
\end{tikzcd}$$

This diagram can be regarded as a piece of a more comprehensive natural bicomplex, which accounts for the Lagrangian formalism via a cohomological scheme [42, 6, 43].

We stress the fact that the objects Θ, $\mathcal{L}$ and $\mathcal{P}$ do not depend on an observer but on a chosen local gauge. However, it is convenient that we have given their coordinate expressions with respect to an observer, since there is another splitting of a Poincaré–Cartan form that is observer dependent. Namely, given an observer o, each Poincaré–Cartan form splits, according to the splitting of $T^*\boldsymbol{E}$ induced by o, as

$$\begin{aligned}
\Theta &= -\mathcal{H}[o] + \mathcal{P}[o]\,, \\
-\mathcal{H}[o] &:= -o \lrcorner \Theta : J_1\boldsymbol{E} \to T^*\boldsymbol{T}\,, \\
\mathcal{P}[o] &:= \vartheta_1 \lrcorner(\nu[o] \lrcorner \Theta) : J_1\boldsymbol{E} \to T^*\boldsymbol{E}\,,
\end{aligned}$$

where $\mathcal{H}[o]$ and $\mathcal{P}[o]$ are said to be the observed *Hamiltonian* and *momentum.*

Remark I.3. The above splitting shows that *CCG* theory fulfills the axioms for *multisymplectic first order field theories* (see [35, 36] and references therein for definitions and properties), in the special case of one–dimensional base space. In this hypothesis we are able to provide a 'working' quantum theory, namely *CQG* theory. This problem is still open in field theory. □

We summarize the above results in the following theorem.

Theorem I.1. *The following natural bijective correspondences hold*

$$K \leftrightarrow \Gamma \leftrightarrow \gamma \tag{I.8}$$

$$\Omega \leftrightarrow (\Gamma, G) \leftrightarrow (\gamma, G) \leftrightarrow \mathcal{E} \tag{I.9}$$

$$\Theta \leftrightarrow \mathcal{L} \tag{I.10}$$

Classical dynamics

We are involved with two different approaches to the classical dynamics in our context. The first consists in a direct definition of the law of particle motion, namely, the (generalized) Newton law.

Definition I.9. Let γ be a dynamical connection and s a motion. Then, the condition on s,

$$\nabla[\gamma] j_1 s := j_2 s - \gamma \circ j_1 s = 0\,. \qquad \square$$

is said to be the *law of motion* for the dynamical connection γ [4, 1, 2, 3].

The law of motion has the coordinate expression $\partial_0\partial_0 s^i - \gamma_{00}^{i} \circ s = 0$.

A dynamical connection γ can be regarded (up to a time scale) as a vector field of $J_1\boldsymbol{E}$, hence, a motion fulfilling the above equation is just an integral curve of γ.

It is easy to see that a motion s fulfills the above law if and only if, for any $f : J_1\boldsymbol{E} \to I\!R$, we have

$$d(f \circ j_1 s) = (\gamma.f) \circ j_1 s\,. \tag{I.11}$$

where $\gamma.f := df(\gamma)$. In the particular case when $\gamma.f = 0$ we call f a *conserved quantity*.

On the other hand, if the metric G is given by a concrete model, the natural bijective correspondence $\mathcal{E} \leftrightarrow (G, \gamma)$ leads to an equivalent approach to the equations of motion, namely, the equation $\mathcal{E}(\mathcal{L}) j_2 s = 0$ for any Lagrangian $\mathcal{L}$.

Hamiltonian lift and quantizable functions

A dynamical phase 2–form Ω yields in a natural way the Hamiltonian lift of functions $f : J_1\boldsymbol{E} \to I\!R$ to vertical vector fields $H[f] : J_1\boldsymbol{E} \to VJ_1\boldsymbol{E}$. The musical morphism $\Omega^\flat : VJ_1\boldsymbol{E} \to T^*_\gamma J_1\boldsymbol{E}$ turns out to be an isomorphism of vector spaces between vertical vector fields of $J_1\boldsymbol{E}$ and forms of $J_1\boldsymbol{E}$ that annihilate the corresponding γ. Clearly, in the case when $\Omega = \Omega[G, \Gamma]$, then $\gamma = \gamma[\Gamma]$.

More generally, taking into account the independence of units, a time scale $\tau : J_1\boldsymbol{E} \to T\boldsymbol{T}$ and a cosymplectic form Ω yield, in a natural (covariant) way, the τ-Hamiltonian lift of functions $f : J_1\boldsymbol{E} \to I\!R$ to vector fields

$$H_\tau[f] :=< \tau, \gamma > + (\Omega^\flat)^{-1}(df - < \gamma.f, dt >) : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E} \tag{I.12}$$

whose time component is τ. Its coordinate expression is

$$H_\tau[f] = \tau^0(\partial_0 + x_0^h\partial_h + \gamma_{00}^{\ i}\partial_h^0) + G_{hk}^0(-\partial_k^0 f\partial_h + (\partial_k f + (\Gamma_{k0}^{\ l} - G_{kr}^0 G_0^{ls}\Gamma_{s0}^{\ r})\,\partial_l^0 f)\,\partial_h^0)$$

where $\tau^0 := < \tau, u^0 >$.

In view of later developments in the quantum theory, it can be proved [1, 2] that $H_\tau[f]$ is projectable on a vector field $X[f] : \boldsymbol{E} \to T\boldsymbol{E}$ if and only if the following conditions hold: i) the function f is quadratic with respect to the affine fibres of

$J_1\boldsymbol{E} \to \boldsymbol{E}$ with second fibre derivative $f'' \otimes G$, where $f'' : \boldsymbol{E} \to T\boldsymbol{T}$ and ii) $\tau = f''$. A function of this kind is called *special quadratic* and is of the type

$$f = \frac{1}{2} f^0 G^0_{ij} x^i_0 x^j_0 + f^0_i x^i_0 + \overset{o}{f}, ,$$

with $f^0, f^0_i, \overset{o}{f} : \boldsymbol{E} \to I\!R$.

The τ-Hamiltonian lift yields a Poisson bracket for functions of $J_1\boldsymbol{E}$, namely, the bracket $\{f, g\} := = i_{H_0[f]}\, i_{H_0[g]}\, \Omega$. We observe that if τ, σ are time scales, then $\{f, g\} = i_{H_\tau[f]}\, i_{H_\sigma[g]}\, \Omega$. The vector space of special functions is not closed under the Poisson bracket, but it turns out to be an $I\!R$-Lie algebra through the natural *special bracket* $[f, g] = \{f, g\} + \gamma(f'').g - \gamma(g'').f$, where $\{f, g\}$ is the Poisson bracket of the functions f and g. Of particular interest are such special functions whose time component is a constant. They are called *quantizable functions with constant time component*, and play a fundamental role in the covariant quantum Galilei theory.

II Symmetries in covariant classical mechanics

In this section we introduce the notion of symmetry of the objects which we have defined on spacetime and phase space. The bijections that we found so far provide connections between symmetries of these objects. Moreover, we give a definition for a covariant momentum map which directly leads us to conserved quantities which are quantizable functions. Finally, we provide simple examples showing our theory at work.

Symmetries and infinitesimal symmetries

First, we want to recall the basic facts about symmetries, groups of symmetries and infinitesimal symmetries of manifolds, fibred manifolds and tensors.

Let $\boldsymbol{M}$ be a manifold. Then, we define a *symmetry of the manifold* $\boldsymbol{M}$ to be a diffeomorphism $f : \boldsymbol{M} \to \boldsymbol{M}$. The group of diffeomorphisms $\mathcal{D}iff(\boldsymbol{M})$ operates on the manifold $\boldsymbol{M}$ via a natural left action In practice, we are interested in finite dimensional subgroups of $\mathcal{D}iff(\boldsymbol{M})$ with the structure of a Lie group.

By taking the tangent prolongation of the action Φ with respect to $\boldsymbol{G}$, at the unit element $e \in \boldsymbol{G}$, we obtain the map

$$\partial\Phi : \mathfrak{g} \to Sec(T\boldsymbol{M}) : \xi \mapsto X[\xi] := \partial\Phi(\xi)\,,$$

(where $\mathfrak{g}$ is the Lie algebra of $\boldsymbol{G}$) which turns out to be an antihomomorphism of Lie algebras. We say $\partial\Phi$ to be an *infinitesimal left action* of $\mathfrak{g}$ on $\boldsymbol{M}$ and the vector field $X[\xi]$ of $\boldsymbol{M}$ to be the *infinitesimal generator* of the infinitesimal action corresponding to ξ. Let us say a vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$ to be an *infinitesimal symmetry of the manifold* $\boldsymbol{M}$ (since its local flow is a local group of diffeomorphisms). Now, by considering a left action Φ of a Lie group $\boldsymbol{G}$ on $\boldsymbol{M}$, the set of infinitesimal generators $\{X[\xi] : \boldsymbol{M} \to T\boldsymbol{M}, \forall\, \xi \in \mathfrak{g}\}$ turns out to be a subalgebra of the Lie algebra of infinitesimal symmetries of $\boldsymbol{M}$.

Now we extend the definition of symmetries to manifolds that are equipped with further structure.

Let $p : \boldsymbol{E} \to \boldsymbol{B}$ be a fibred manifold. Then, we define a *symmetry of the fibred manifold* p to be a fibred diffeomorphism f of $\boldsymbol{E}$ over $\boldsymbol{B}$. We say any projectable vector field X of $\boldsymbol{E}$ to be an *infinitesimal symmetry of the fibred manifold* p.

Let $f : \boldsymbol{M} \to \boldsymbol{M}$ be a symmetry of the manifold $\boldsymbol{M}$. Then, the tangent prolongation $Tf : T\boldsymbol{M} \to T\boldsymbol{M}$ turns out to be a symmetry of the fibred manifold $T\boldsymbol{M} \to \boldsymbol{M}$. Moreover, for each left action $\Phi : \boldsymbol{G} \times \boldsymbol{M} \to \boldsymbol{M}$, the tangent prolongation $\overline{T}\Phi : \boldsymbol{G} \times T\boldsymbol{M} \to T\boldsymbol{M} : (g, y) \mapsto T(\Phi_g)(y)$ turns out to be a left action.

Let ν be a tensor field of $\boldsymbol{M}$, which is contravariant of order s and covariant of order r. Then, we define a *symmetry of the tensor field* ν to be a diffeomorphism $f : \boldsymbol{M} \to \boldsymbol{M}$ such that $\nu \circ \overset{s}{\otimes} Tf = \overset{r}{\otimes} Tf \circ \nu$. We say any vector field X of $\boldsymbol{M}$ such that $L_X \nu = 0$ to be an *infinitesimal symmetry of the tensor field* ν.

Now, let $f : \boldsymbol{E} \to \boldsymbol{E}$ be a symmetry of the fibred manifold $p : \boldsymbol{E} \to \boldsymbol{B}$. Then, for each $1 \leq k$, the k-jet prolongation $J_k f : J_k\boldsymbol{E} \to J_k\boldsymbol{E}$ turns out to be a symmetry of the fibred manifolds $p^k_h : J_k\boldsymbol{E} \to J_h\boldsymbol{E}$ and $p^k : J_k\boldsymbol{E} \to \boldsymbol{B}$, for each $1 \leq h < k$. Moreover, for each left action of symmetries $\Phi : \boldsymbol{G} \times \boldsymbol{E} \to \boldsymbol{E}$, the k-jet prolongation $\overline{J}_k\Phi : \boldsymbol{G} \times J_k\boldsymbol{E} \to J_k\boldsymbol{E} : (g, e_k) \mapsto J_k(\Phi_g)(e_k)$ turns out to be a left action, called the *k-jet prolongation* of Φ.

Let us recall the natural involution $s : TT\boldsymbol{M} \to TT\boldsymbol{M}$ [45]. This map yields the natural prolongation of each vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$ to the vector field $X_{(T)} := s \circ TX : T\boldsymbol{M} \to TT\boldsymbol{M}$. If $X = X^\lambda \partial_\lambda$, then $X_{(T)} = X^\lambda \partial_\lambda + \partial_\mu X^\lambda \dot{x}^\mu \dot{\partial}_\lambda$. The map $X \mapsto X_{(T)}$ turns out to be a morphism of Lie algebras. Let $\Phi : \boldsymbol{G} \times \boldsymbol{M} \to \boldsymbol{M}$ be a left action of $\boldsymbol{G}$ on $\boldsymbol{M}$. Then, we obtain $s \circ \overline{T}\partial\Phi = \partial\overline{T}\Phi : \mathfrak{g} \times T\boldsymbol{M} \to TT\boldsymbol{M}$, hence, $s \circ \overline{T}\partial\Phi$ turns out to coincide with the infinitesimal left action of the Lie algebra $\mathfrak{g}$ on the manifold $T\boldsymbol{M}$.

We recall the natural map $r^k : J_kT\boldsymbol{E} \to TJ_k\boldsymbol{E}$ [38]. This map yields the natural prolongation of each vector field $X : \boldsymbol{E} \to T\boldsymbol{E}$ to the vector field $X_{(k)} := r^k \circ J_kX : J_k\boldsymbol{E} \to TJ_k\boldsymbol{E}$. In particular, if $\boldsymbol{B} = \boldsymbol{T}$, then $X = X^\lambda \partial_\lambda$ and $X_{(1)} = X^\lambda \partial_\lambda + (\partial_0 X^i + \partial_k X^i x^k_0 - \partial_0 X^0 x^i_0)\partial^\mu_i$. The map $X \mapsto X_{(k)}$ turns out to be a morphism of Lie algebras. Let $\Phi : \boldsymbol{G} \times \boldsymbol{E} \to \boldsymbol{E}$ be a left action of a group $\boldsymbol{G}$ such that $\{\Phi_g, g \in \boldsymbol{G}\}$ is a group of symmetries of the fibred manifold $p : \boldsymbol{E} \to \boldsymbol{B}$. Then, we obtain $r^k \circ \overline{J}_k\partial\Phi = \partial\overline{J}_k\Phi : \mathfrak{g} \times J_k\boldsymbol{E} \to TJ_k\boldsymbol{E}$, hence, $r^k \circ \overline{J}_k\partial\Phi$ turns out to coincide with the infinitesimal left action of the Lie algebra $\mathfrak{g}$ on the manifold $J_k\boldsymbol{E}$.

In general, we say all above natural prolongation of symmetries, infinitesimal symmetries and actions to be *holonomic*. By an abuse of language we often call the group $\boldsymbol{G}$ a *group of symmetries* if the left action Φ of $\boldsymbol{G}$ is given such that Φ_g is a symmetry for all $g \in \boldsymbol{G}$.

Infinitesimal symmetries of spacetime structures

Now, we apply the above general definitions of infinitesimal symmetries to covariant classical mechanics. However, some of the morphisms (contact maps, spacelike metric) cannot be regarded as tensors, naturally. Consequently, a (direct) definition of their symmetries would require naturality techniques [44]. Instead, we show that it is possible in our case to give a meaning to their infinitesimal symmetries by keeping the standard Lie derivative.

II.0.1 Symmetries of spacetime

An infinitesimal symmetry of spacetime is an infinitesimal symmetry of the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ which, additionally, preserves the affine structure of $\boldsymbol{T}$. More precisely, we define an *infinitesimal symmetry of spacetime* as a vector field $X : \boldsymbol{E} \to T\boldsymbol{E}$ which is projectable on a vector field $\underline{X} : \boldsymbol{T} \to T\boldsymbol{T}$ and such that $\underline{X}$ is constant.

An easy calculation shows that

Proposition II.1. *A vector field $X : \boldsymbol{E} \to T\boldsymbol{E}$ is an infinitesimal symmetry of spacetime if and only if $L_X\, dt = 0$.*

If $X = X^0\partial_0 + X^i\partial_i$ is the coordinate expression, then the conditions are equivalent to $X^0 \in I\!R$.

II.0.2 Symmetries of the contact maps

On any fibred manifold $p : \boldsymbol{E} \to \boldsymbol{B}$, the contact maps ϑ_1 and $\mathcal{D}$ are not tensors. Hence, their symmetries require naturality techniques. However, it is possible to use a standard Lie derivative for both objects by using the 1–dimensional affine structure of $\boldsymbol{T}$ in the following way. The affine structure of $p^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ allows us to regard ϑ_1 as a (scaled) tensor $\vartheta_1 : J_1\boldsymbol{E} \to \mathbb{T} \otimes (T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E})$. Easy calculations yield the following results.

Proposition II.2. *Let X be an infinitesimal spacetime symmetry. Then, $X_{(1)}$ is a holonomic infinitesimal symmetry of ϑ_1, i.e. $L_{X_{(1)}}\vartheta_1 = 0$.*

Lemma II.1. *Let $p : \boldsymbol{F} \to \boldsymbol{B}$ a fibred manifold. Let $X : \boldsymbol{F} \to T\boldsymbol{F}$ be a vector field which projects on a vector field $\underline{X} : \boldsymbol{B} \to T\boldsymbol{B}$, and let $Y : \boldsymbol{F} \to T\boldsymbol{B}$ be a fibred morphism (over the identity on $\boldsymbol{B}$). Let $\tilde{Y} : \boldsymbol{F} \to T\boldsymbol{F}$ be any extension of Y, i.e. a vector field projectable on Y. Then, $L_X Y := Tp \circ (L_X \tilde{Y})$ is well defined, i.e. it does not depend on the extension $\tilde{Y}$ of Y.*

We obtain the following coordinate expression

$$L_X Y = (X^\mu\partial_\mu Y^\lambda - Y^\mu\partial_\mu X^\lambda + X^i\partial_i Y^\lambda)\partial_\lambda\,.$$

Proposition II.3. *Let $X : \boldsymbol{E} \to T\boldsymbol{E}$ an infinitesimal spacetime symmetry. Then, $X_{(1)}$ is a holonomic infinitesimal symmetry of $\mathcal{D}$, i.e. $L_{X_{(1)}}\mathcal{D}_1 = 0$.*

II.0.3 Symmetries of spacelike metrics

A spacelike metric is not a tensor of $\boldsymbol{E}$. Hence, in order to define its infinitesimal symmetries, we need the following lemma [3].

Lemma II.2. *Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, a vector field X of $\boldsymbol{F}$ which is projectable on a vector field $\underline{X}$ of $\boldsymbol{B}$ and a vertical covariant tensor $\alpha : \boldsymbol{F} \to \overset{r}{\otimes} V^*\boldsymbol{F}$. Then, the vertical restriction $(L[X]\tilde{\alpha})\check{} : \boldsymbol{F} \to \overset{r}{\otimes} V^*\boldsymbol{F}$ of the Lie derivative $L[X]\tilde{\alpha}$, where $\tilde{\alpha} : \boldsymbol{F} \to T^*\boldsymbol{F}$ is an extension of α, does not depend on the choice of the extension $\tilde{\alpha}$.*

Hence, the Lie derivative $L[X]\alpha := (L[X]\tilde{\alpha})\check{} : \boldsymbol{F} \to \overset{r}{\otimes} V^*\boldsymbol{F}$ is well defined.

Its coordinate expression is

$$L_X\, \alpha = (X^\lambda \partial_\lambda \alpha_{j_1 \dots j_r} + X^i \partial_i \alpha_{j_1 \dots j_r} + \alpha_{i j_2 \dots j_r} \partial_{j_1} X^i + \dots \\ + \alpha_{j_1 \dots j_{r-1} i} \partial_{j_r} X^i) d^{j_1} \otimes \dots \otimes d^{j_r}$$

where $(j_1, \dots j_r)$ is any permutation of the fibre indices, and where we have used greek indices for the coordinates of the base space and latin indices for the fibres.

Thus, for each spacelike metric G, we call a projectable vector field X of $\boldsymbol{E}$ such that $L_X\, G = 0$ an *infinitesimal symmetry of the spacelike metric* G. An easy calculation shows

Proposition II.4. *Let X be an infinitesimal spacetime symmetry and G a spacelike metric. Then, we have the coordinate expression*

$$L_X\, G = \{X^\lambda (\partial_\lambda G^0_{ij}) + G^0_{kj} (\partial_i X^k) + G^0_{ih} (\partial_j X^k)\} d^i \otimes d^j \,.$$

II.0.4 Symmetries of connections

Let K be a spacetime connection, Γ a phase connection and γ a dynamical connection. An easy calculation yields the following coordinate expressions.

Proposition II.5. *Let X be an infinitesimal spacetime symmetry. Then,*

$$\begin{aligned} L_{X_{(1)}}\, \Gamma \;&=\; \{\partial_\mu X^i_0 - (\Gamma_\lambda{}^i_{00} + \Gamma_\lambda{}^i_{0k} x^k_0)(\partial_\mu X^\lambda) - \partial_\lambda (\Gamma_\mu{}^i_{00} + \Gamma_\mu{}^i_{0k} x^k_0) X^\lambda \\ &\quad - \Gamma_\mu{}^i_{0k} X^k_0 + (\Gamma_\mu{}^j_{00} + \Gamma_\mu{}^j_{0k} x^k_0)(\partial_j X^i)\}\, d^\mu \otimes \partial^0_i \,, \end{aligned}$$

$$\begin{aligned} L_{X_{(1)}}\, \gamma \;&=\; u^0 \{X^0 (\partial_0 \gamma^i_{00}) + X^i (\partial_j \gamma^j_{00}) - (\partial_0 X^i_0) - (\partial_j X^i_0) x^j_0 \\ &\quad + X^j_0 (\partial^0_j \gamma^i_{00}) - \gamma^j_{00} (\partial^0_j X^i_0)\}\, \partial^0_i \,, \end{aligned}$$

$$\begin{aligned} L_{X_{(T)}}\, K \;&=\; \{(\partial_\lambda X^i) \Gamma_i{}^k_{0\mu} \dot{x}^\nu - \partial_\lambda \partial_\nu X^k \dot{x}^\nu + X^\alpha \partial_\alpha \Gamma_\lambda{}^k_{0\mu} \dot{x}^\mu \\ &\quad + (\partial_\alpha X^i) \Gamma_\lambda{}^k_{0\mu} \dot{x}^\alpha - (\partial_j X^k) \Gamma_\lambda{}^i_{0\mu} \dot{x}^\mu\}\, \dot{\partial}_k \otimes d^\lambda \,. \end{aligned}$$

Now let us consider the particular case when Γ and γ are corresponding to K. The natural bijective correspondences I.8 suggest the following theorem.

Theorem II.1. *Let X be an infinitesimal spacetime symmetry. The following equivalences hold*

$$1)\; L_{X_{(T)}}\, K = 0 \qquad \Leftrightarrow \qquad 2)\; L_{X_{(1)}}\, \Gamma = 0 \qquad \Leftrightarrow \qquad 3)\; L_{X_{(1)}}\, \gamma = 0$$

PROOF. The proof of 1) $\Rightarrow$ 2) $\Rightarrow$ 3) follows easily in virtue of the Leibnitz rule and the fact that $X_{(1)}$ is a symmetry of the contact maps.

The proof of 3) $\Rightarrow$ 2) $\Rightarrow$ 1) can be obtained easily by considering the expressions of Lemma II.5 which are polynomial in the coordinates x^i_0 or $\dot{x}^\mu$. QED

II.0.5 Symmetries of phase 2–forms

Let us consider a spacelike metric G and a phase connection Γ. Moreover, let the dynamical phase 2–form Ω, the Euler-Lagrange morphism $\mathcal{E}$ and the dynamical connection γ be the corresponding objects. The natural correspondences I.9 suggest the following theorem

Theorem II.2. *Let X be an infinitesimal symmetry of spacetime. Then, the following equivalence holds*

$$1)\ L_{X_{(1)}}\Omega = 0 \qquad \Leftrightarrow \qquad 2)\ L_{X_{(1)}}\Gamma = 0\,,\ L_X G = 0\,.$$

PROOF. The definition (I.5) yields, $L_{X_{(1)}}\Omega = L_{X_{(1)}}\nu[\Gamma]\bar{\wedge}\vartheta_1 + \nu[\Gamma]\bar{\wedge}L_{X_{(1)}}\vartheta_1 + \nu[\Gamma]\tilde{\wedge}\vartheta_1$. The proof of 2) $\Rightarrow$ 1) can be seen directly because of the fact that $X_{(1)}$ is a symmetry of ϑ_1.

For the proof of 1) $\Rightarrow$ 2) we consider the coordinate expressions $L_{X_{(1)}}\nu[\Gamma]\bar{\wedge}\vartheta_1 = G^0_{ij}\alpha^i_{\mu 0}(d^\mu \wedge d^j - x^j_0 d^\mu \wedge d^0)$ and $\nu[\Gamma]\tilde{\wedge}\vartheta_1 = \beta^0_{ij}(d^i_0 \wedge d^j - x^j_0 d^i_0 \wedge d^0 - \Gamma^i_{\lambda 0} d^\lambda \wedge d^j + x^j_0 \Gamma^i_{\lambda 0} d^\lambda \wedge d^0)$, where we have set $\alpha^i_{\mu 0}$ to be the coefficient of $L_{X_{(1)}}\Gamma$ in Proposition II.5 and β^0_{ij} the coefficient of $L_X G$ in Proposition II.4. It can be easily seen that, if the sum of these expressions is zero, then, all $\alpha^i_{\mu 0}$ and all β^0_{ij} have to be zero. QED

Moreover, the correspondences I.9 suggest the following

Theorem II.3. *Let X be an infinitesimal spacetime symmetry. Then, the following equivalence holds.*

$$1)\ L_{X_{(1)}}\Omega = 0 \qquad \Leftrightarrow \qquad 2)\ L_{X_{(2)}}\mathcal{E} = 0$$

PROOF. Expression I.6 yields $L_{X_{(2)}}\mathcal{E} = L_{X_{(2)}}\nabla[\gamma]\bar{\wedge}\vartheta_1 + \nabla[\gamma]\bar{\wedge}L_{X_{(1)}}\vartheta_1 + \nabla[\gamma]\tilde{\wedge}\vartheta_1$. In analogy to the proof of Theorem II.2 this yields the equivalence between the condition $L_{X_{(2)}}\mathcal{E} = 0$ and the two conditions $L_X G = 0$, $L_{X_{(1)}}\nabla[\gamma] = 0$. Clearly, $L_{X_{(1)}}\nabla[\gamma] = 0$ is equivalent to $L_{X_{(1)}}\gamma = 0$. Thus, Theorem II.2 and Theorem II.1 yield the result. QED

Let us additionally consider that Ω is closed. Eventually, we add an example of a distinguished nonholonomic infinitesimal symmetry of Ω.

Proposition II.6. *The dynamical connection γ is a nonholonomic (scaled) infinitesimal symmetry of the cosymplectic manifold $(J_1\boldsymbol{E}, \Omega, dt)$, i.e. $L_\gamma \Omega = 0$ and $L_\gamma dt = 0$.*

PROOF. This follows directly from Cartan's formula using $i_\gamma \Omega = 0, i_\gamma dt = 1$ and the closure of Ω. QED

We observe that this proposition is essentially the standard result for the Reeb vector field of any cosymplectic structure applied to our particular case.

But we can even say more about γ.

Theorem II.4. *There is exactly one second order connection $\tilde{\gamma}$ such that $L_{\tilde{\gamma}}\,\Omega = 0$. Namely, $\tilde{\gamma} = \gamma$.*

Proof. Let $\tilde{\gamma}$ be a second order connection such that $L_{\tilde{\gamma}}\Omega = 0$. This implies that there exists a local function $f : J_1\boldsymbol{E} \to I\!R$ such that $i_{\tilde{\gamma}}\Omega = df$. Using $i_\gamma\,\Omega = 0$, we get $i_{\tilde{\gamma}-\gamma}\,\Omega = df$. Let us set $c := \tilde{\gamma} - \gamma$; c is valued into $\mathbb{T}^* \otimes \mathbb{T}^* \otimes V\boldsymbol{E}$, and has coordinate expression $c = c_{00}^{\,i} u^0 \otimes \partial_i^0$, with $c_{00}^{\,i} := \tilde{\gamma}_{00}^{\,i} - \gamma_{00}^{\,i} : J_1\boldsymbol{E} \to I\!R$. We have to show that a local function f only exists if $c = 0$. Calculating in coordinates using $c_{0i} := G^0_{ij} c_{00}^{\,i}$ one gets the following system of equalities $\partial_0 f = -c_{0i} x_0^i$, $\partial_i f = c_{0i}$, $\partial_i^0 f = 0$. This systems implies that $c = 0$. Thus, $\tilde{\gamma} = \gamma$. QED

Noether Symmetries

Let us consider a closed dynamical phase 2–form Ω. The next proposition relates infinitesimal symmetries of Ω to conserved quantities.

Lemma II.3. *Let $Y : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ be an infinitesimal symmetry of Ω. Then, the 1–form $i_Y\,\Omega$ is closed, and any local potential function f of $i_Y\,\Omega$ is a conserved quantity.*

Proof. $L_Y\,\Omega = 0$ is locally equivalent to the closure of $i_Y\,\Omega$. Hence, there is a local function f such that $df = i_Y\,\Omega$. Therefore, $\gamma.f = df(\gamma) = i_Y\,\Omega(\gamma) = i_Y\,i_\gamma\,\Omega = 0$. QED

II.0.6 Symmetries of Poincaré–Cartan forms

Let us consider a local Poincaré–Cartan form Θ. Clearly, any infinitesimal symmetry $Y : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ of Θ is an infinitesimal symmetry of $\Omega = d\Theta$.

Now, we can formulate the following (Noether) theorem which relates holonomic infinitesimal symmetries of Θ to conserved quantities.

Theorem II.5. *Let X be an infinitesimal spacetime symmetry which is a holonomic infinitesimal symmetry of a Poincaré–Cartan form Θ. Then, on the domain of Θ, $i_{X_{(1)}}\,\Omega$ is exact and $f := -i_X\,\Theta$ is a potential, hence, a conserved quantity.*

Remark II.1. If an observer o is a (scaled) infinitesimal symmetry of Θ, then the Hamiltonian $H[o]$ turns out to be the associated conserved quantity.

II.0.7 Symmetries of Lagrangians

Let $\mathcal{L}$ be the Lagrangian corresponding to a Poincaré-Cartan form Θ and let $\mathcal{P}$ be the corresponding momentum. The natural correspondence I.10 indicates the following theorem

Theorem II.6. *Let X be an infinitesimal spacetime symmetry. Then, the following equivalence holds*

$$1)\ L_{X_{(1)}}\,\Theta = 0 \qquad \Leftrightarrow \qquad 2)\ L_{X_{(1)}}\,\mathcal{L} = 0$$

PROOF. Both directions can be proved in analogy to the proof of the equivalence $L_{X_{(1)}}\Gamma = 0 \;\Leftrightarrow\; L_{X_{(1)}}\gamma = 0$. QED

Theorem II.6 yields immediately another formulation of the (Noether) Theorem II.5. This version may be more popular to the physicist.

Corollary II.1. *Let X be an infinitesimal spacetime symmetry which, additionally, is a holonomic infinitesimal symmetry of $\boldsymbol{\mathcal{L}}$. Then, on the domain of Θ, a conserved quantity is given by*

$$f := -(X \lrcorner \mathcal{P} + \underline{X} \lrcorner \mathcal{L})\,.$$

Momentum map in $\boldsymbol{CCG}$ theory

Let us suppose a closed dynamical phase 2–form Ω and a left action $\hat{\Phi} : \boldsymbol{G} \times J_1\boldsymbol{E} \to J_1\boldsymbol{E}$ of a group $\boldsymbol{G}$ of symmetries of the cosymplectic structure $(J_1\boldsymbol{E}, \Omega, dt)$. That is, $\hat{\Phi}^*_g\Omega = \Omega$ and $\hat{\Phi}^*_g dt = dt$ for all $g \in \boldsymbol{G}$. Let $\mathfrak{g}$ be the associated Lie algebra. Hence, $L_{\partial\hat{\Phi}(\xi)}\,\Omega = 0$ and $L_{\partial\hat{\Phi}(\xi)}\,dt = 0$ for all $\xi \in \mathfrak{g}$.

We would like to define a *momentum map* in our setting by analogy with the standard symplectic and cosymplectic literature [12, 14, 17, 18] and ref. therein. However, the scaling of the time form dt requires the incorporation of a time unit in a momentum map.

Lemma II.3 shows that any vector field Y of $J_1\boldsymbol{E}$ is an infinitesimal symmetry of Ω if and only if there exists a (local) function f such that $i_Y\,\Omega = d\,f$. Clearly, f is not unique; each f of this type is a conserved quantity.

Analogously, we can easily see that the following lemma holds

Lemma II.4. *Let Y be any vector field of $J_1\boldsymbol{E}$. Then, Y is an infinitesimal symmetry of dt if and only if $i_Y\,dt$ is a constant $c \in \bar{\mathbb{T}}$.*

Hence, by Lemma II.3 and Lemma II.4, we can associate to any infinitesimal symmetry $\partial\hat{\Phi}(\xi)$ of Ω and of dt the (unique) constant $dt(\partial\hat{\Phi}(\xi))$ and a conserved quantity f_ξ, which is determined up to an additive constant $c \in I\!R$. In the following we denote by $Co(J_1\boldsymbol{E})$ the vector space of conserved quantities.

Definition II.1. A (local) map J

$$J : \mathfrak{g} \to Co(J_1\boldsymbol{E}) \times \bar{\mathbb{T}} :\; \xi \mapsto (J_\xi, \tau_\xi)\,,$$

where J_ξ is a potential of $i_{\partial\hat{\Phi}(\xi)}\,\Omega$ and $\tau_\xi := i_{\partial\hat{\Phi}(\xi)}\,dt$ for all $\xi \in \mathfrak{g}$, is said to be a *momentum map* for the action $\hat{\Phi}$. □

Clearly, for an action of symmetries for which $i_{\partial\hat{\Phi}(\xi)}\,dt = 0$, we get a standard momentum map for canonical actions in the sense of [17].

Remark II.2. The momentum map J is locally defined, in general. But if we assume further hypotheses on spacetime or on the Lie algebra $\mathfrak{g}$, then there exists a global momentum map. Of course, a global J always exists if $H^1(\boldsymbol{E}) = \{0\}$. A detailed list of other hypotheses under which J is globally defined is given in [18]; they are the same as in our case.

On the other hand, given a time scale τ, it is possible to associate to any function f of $J_1\boldsymbol{E}$ a distinguished vector field of $J_1\boldsymbol{E}$, namely, the τ-Hamiltonian lift of f. The following theorem holds.

Theorem II.7. *Let J be a momentum map for the action $\hat{\Phi}$ and let $H_\tau[J_\xi]$ be the τ-Hamiltonian lift of J_ξ with respect to an arbitrary time scale τ.*

Then, the following equivalence holds

$$\partial\hat{\Phi}(\xi) = H_\tau[J_\xi] \;\Leftrightarrow\; \tau = \tau_\xi\,.$$

Proof. By recalling that $\gamma.J_\xi = 0$ we obtain that $i_{H_\tau[J_\xi]}\,\Omega = dJ_\xi - \langle\gamma.J_\xi, dt\rangle = dJ_\xi$. Hence, by observing that two vector fields X, $Y : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ are equal if and only if $i_X\,dt = i_Y\,dt$ and $i_X\,\Omega = i_Y\,\Omega$, we obtain the result. QED

This theorem shows, why we have included the time scale τ_ξ in the definition of momentum map.

It is obvious that we want to know if the map that associates to a pair (J_ξ, τ_ξ) its τ-Hamiltonian lift $H(J_\xi, \tau_\xi) := H_{\tau_\xi}[J_\xi]$ is a homomorphism of Lie algebras. Therefore, we define the following bracket for pairs in $Co(J_1\boldsymbol{E}) \times \bar{\mathbb{T}}$.

Definition II.2. The bracket $\{(f,\tau), (g,\sigma)\} := (0, \{f, g\})$ is said to be the *Poisson bracket of pairs.* □

Then we can prove the following theorem

Theorem II.8. *The map H is a homomorphism of Lie algebras between pairs $(f,\tau) \in Co(J_1\boldsymbol{E}) \times \bar{\mathbb{T}}$, with respect to the Poisson bracket of pairs, and vector fields $H_\tau[f]$ of $J_1\boldsymbol{E}$, with respect to the standard Lie bracket.*

Proof. We have to show that $[H(f,\tau), H(g,\sigma)] = H(0, \{f,g\})$. We use the equalities $i_{[H_\tau[f],H_\sigma[g]]}\,dt = \big[L_{H_\tau[f]}, i_{H_\sigma[g]}\big]\,dt = L_{H_\tau[f]}\,\sigma - i_{H_\sigma[g]}\,d\tau = 0$ and $i_{[H_\tau[f],H_\sigma[g]]}\,\Omega = \big[L_{H_\tau[f]}, i_{H_\sigma[g]}\big]\,\Omega = d\,i_{H_\tau[f]}\,i_{H_\sigma[g]}\,\Omega + i_{H_\tau[f]}\,L_{H_\sigma[g]}\,\Omega - i_{H_\sigma[g]}\,L_{H_\tau[f]}\,\Omega = d\{f,g\}$.

On the other hand, the definition of $H_0[\{f,g\}]$ yields $i_{H_0[\{f,g\}]}\,dt = 0$ and $i_{H_0[\{f,g\}]}\,\Omega = d\{f,g\}$. QED

Let us recall that the τ-Hamiltonian lift of a function $f : J_1\boldsymbol{E} \to I\!\!R$ is projectable on a vector field X of $\boldsymbol{E}$ if and only if f is a special quadratic function and the second fibre derivative of f (with respect to the velocities) is equal to the time scale τ. Thus, Theorem II.7 yields the following theorem that relates the function J_ξ to the time scale τ_ξ.

Corollary II.2. *Let $\hat{\Phi}$ be projectable on a left action $\Phi : \boldsymbol{G} \times \boldsymbol{E} \to \boldsymbol{E}$. Then, any momentum map for $\hat{\Phi}$ takes values into the space of quantizable function and the second fibre derivative of J_ξ is equal to the constant time scale $\tau_\xi = dt(\hat{\Phi}(\xi))$.*

Thus, in this case, each function J_ξ encodes all information of the pair (J_ξ, τ_ξ). Hence, in this case we call a map $J : \mathfrak{g} \to Co(J_1\boldsymbol{E}) : \xi \to J(\xi) := J_\xi$ momentum map, denoted by the same symbol J.

Now let us consider a Poincaré–Cartan form Θ. Furthermore, let us suppose that $\hat{\Phi}$ is holonomic, i.e. $\hat{\Phi} = \Phi_{(1)}$ where Φ is a left action of $\boldsymbol{G}$ on $\boldsymbol{E}$ and, additionally, we suppose that Φ preserves Θ. Then the following holds

Theorem II.9. *There exists a momentum map on the domain of* Θ. *Namely, the map*

$$J_\xi = \partial\Phi(\xi) \lrcorner \mathcal{P} + \underline{\partial\Phi}(\xi) \lrcorner L \,. \tag{II.13}$$

Let (e_p) *be a basis of* $\mathfrak{g}$, *and* $\xi = \xi^p e_p$. *Then, the coordinate expression is*

$$J_\xi = \xi^p \big((\partial_p \phi^i - x^i_0 \partial_p \phi^0)\, \partial^0_i L + \partial_p \phi^0\, L \big) \,,$$

Given an observer o, *the momentum map can be expressed in terms of the observed Hamiltonian* $\mathcal{H}[o]$ *and the observed momentum* $\mathcal{P}[o]$ *by*

$$J_\xi = \partial\Phi(\xi) \lrcorner \mathcal{P}[o] + \underline{\partial\Phi}(\xi) \lrcorner \mathcal{H}[o] \,. \tag{II.14}$$

Proof. The first expression follows simply from the contact splitting of Θ and Theorem II.5. The observer dependent expression of J follows simply from the splitting of Θ through the observer.

The coordinate expression $\partial\phi(\xi) = \xi^p \partial_p \phi^0 \partial_0 + \xi^p \partial_p \phi^i \partial_i$ with respect to a basis (e_p) yields the second expression. QED

Remark II.3. There is a connection between the momentum of a Lagrangian and the momentum map. In fact, let $\boldsymbol{G}$ be a group of vertical holonomic symmetries of Θ, i.e. $i_{\partial\Phi(\xi)}\, dt = 0$. Then we have the expression

$$J_\xi = \partial\phi(\xi) \lrcorner \mathcal{P} \equiv \mathcal{P}(\partial\phi(\xi)) \,,$$

so the momentum map coincides with the momentum of the Lagrangian.

Examples

Before we give some examples of applications of the above results, we show how the geometric objects incorporate the physical information in the general model for the covariant classical mechanics.

Namely, one chooses a spacelike metric G. The existence of such a metric is assured by the possibility of a length measurement at any instant of time. Moreover, a gravitational field is assumed to be a spacetime connection $K^\natural$ and an electromagnetic field is assumed to be a scaled 2–form $f : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ of spacetime. Given a charge $q \in \mathbb{Q}$, it is convenient to introduce the normalized form $F := \frac{q}{m} f : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$.

The gravitational connection $K^\natural$ yields the gravitational objects $\Gamma^\natural$, $\gamma^\natural$ and $\Omega^\natural$ through the correspondences discussed in the above section: $K \leftrightarrow \Gamma[K] \leftrightarrow \gamma[\Gamma]$ and $\Gamma \leftrightarrow \Omega[G, \Gamma]$.

The first field equations for the gravitational and the electromagnetic field are assumed to be $d\Omega^\natural = 0$ and $dF = 0$.

There is a natural way to incorporate the electromagnetic field in the gravitational objects. We can start by considering a natural "minimal coupling" [26, 23]

$$\Omega := \Omega^\natural + \frac{1}{2}F\,, \tag{II.15}$$

where the factor $\frac{1}{2}$ has been chosen just in order to obtain standard normalisation. Then the other total objects are obtained by means of the correspondences. $K \leftrightarrow \Gamma[K] \leftrightarrow \gamma[\Gamma]$ and $\Gamma \leftrightarrow \Omega[G,\Gamma]$. The coefficients of K turn out to be $K_h{}^i{}_k = K^\natural{}_h{}^i{}_k$, $K_0{}^i{}_k = K^\natural{}_0{}^i{}_k + \frac{q}{2m}F^i{}_k$ and $K_0{}^i{}_0 = K^\natural{}_0{}^i{}_0 + \frac{q}{m}F^i{}_0$.

This yields that $dt \wedge \Omega^n = dt \wedge (\Omega^\natural)^n : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^n T^* J_1\boldsymbol{E}$ is a (scaled) volume form of $J_1\boldsymbol{E}$ and that $d\Omega = 0$. Therefore, the phase space $J_1\boldsymbol{E}$ together with the total dynamical phase 2–form Ω and the time form dt turns out to be a (scaled) cosymplectic manifold [12, 13, 14]. The total dynamical connection γ turns out to be the (scaled) Reeb vector field for this cosymplectic structure since $i_\gamma\,\Omega = 0$ and $i_\gamma\,dt = 1$.

It is interesting to interpret the form Ω through an observer o. We will suppose the observer to be complete, for the sake of simplicity. We recall that G yields the family $\varkappa$ of Riemannian connections of the fibres of $\boldsymbol{E} \to \boldsymbol{T}$, with coordinate expression $\varkappa = d^k \otimes (\partial_k + \varkappa_k{}^i{}_h\,\dot{x}^h\,\dot{\partial}_i)$, where $\varkappa_k{}^i{}_h$ are the usual Christoffel symbols related to G. Let us introduce the 2–form $\Omega[\kappa] := \nu[\kappa] \mathbin{\bar\wedge} \mathrm{id}_{V\boldsymbol{E}}$. Then, it can be seen that the splitting $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$ induces the following splittings

$$\Omega^\natural = \Omega[\kappa] + \frac{1}{2}\,\Phi^\natural\,, \qquad F = -2dt \wedge E + B$$

where $\Phi^\natural := 2o^*\Omega^\natural$, $E : \boldsymbol{T} \times \boldsymbol{P} \to \mathbb{T}^* \otimes T^*\boldsymbol{P}$ and $B : \boldsymbol{T} \times \boldsymbol{P} \to \overset{2}{\wedge} T^*\boldsymbol{P}$.

We say E to be the *electric field* and B to be the *magnetic field*.

With respect to a complete observer, the form $\Omega[\kappa]$ is just the standard time–dependent symplectic form on $T\boldsymbol{P}$ induced by the natural symplectic form on $T^*\boldsymbol{P}$ via the metric isomorphism. Hence, the form $\Omega[\kappa] + B$ is just the *charged* (or *deformed*) (time–dependent) *symplectic structure* [18]. In our setting, we have something more: there is the electric field E and the term $\Phi^\natural$ which is a 2–form on spacetime (as F) but is induced by the gravitational connection.

Remark II.4. The above considerations show that in *CCG* theory the dynamics is incorporated into the cosymplectic form. In symplectic mechanics one has to postulate the Hamiltonian H *besides* the symplectic form ω_0 on the phase space $\boldsymbol{M}$. On the other hand, such a structure yields naturally the cosymplectic manifold $(I\!R \times \boldsymbol{M}, \omega_0 - dH \wedge dt, dt)$ [12]. $\square$

Now, we apply the machinery developed in the above subsection to analyse three groups of symmetries acting in simple cases.

Example II.1. We suppose the spacetime $\boldsymbol{E}$ to be an affine space with affine projection t. In this case $V\boldsymbol{E} \simeq \boldsymbol{E} \times \boldsymbol{S}$, where $\boldsymbol{S} := \ker Dt$. So, we assume an Euclidean scaled metric g on $\boldsymbol{S}$.

Let us consider the vertical action

$$\boldsymbol{S} \times \boldsymbol{E} \to \boldsymbol{E} : (v, e_0) \mapsto (e_0 + v)\,;$$

Let $K^\natural$ be the natural flat connection on $\boldsymbol{E}$ and $F = 0$. Then, any Poincaré–Cartan form exists globally and $\boldsymbol{S}_0$ is a group of symmetries of a Θ. The momentum map J is just the standard linear momentum.

In fact, Θ is invariant with respect to spacelike translations. Of course, the Lie algebra of $\boldsymbol{S}_0$ is $\boldsymbol{S}_0$ and we have the momentum map

$$J : \boldsymbol{S}_0 \to C^\infty(J_1\boldsymbol{E}, I\!R) : v \mapsto J(v) \equiv \mathcal{P}(v)\,,$$

where, by definition, $\mathcal{P} = V_{\boldsymbol{E}}L$, with coordinate expression $\mathcal{P}_0(v) = v^i\, G_{ij} x^j_0$ (see remark II.3). □

Example II.2. Assume the same spacetime and fields as in the above example, and assume additionally that $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}$, i.e. , assume a complete observer o. Then, we can consider the natural action

$$\overline{\mathbb{T}} \times (\boldsymbol{T} \times \boldsymbol{P}) \to \boldsymbol{T} \times \boldsymbol{P} : (v, (\tau, \boldsymbol{p})) \mapsto (v + \tau, \boldsymbol{p})\,.$$

It turns out that $\overline{\mathbb{T}}$ is a group of symmetries of Θ, i.e. o is a (scaled) infinitesimal symmetry of Θ, and the momentum map J is just the (observed) kinetic energy $\mathcal{H}[o]$.

In fact, Θ is as in the above example, hence it is invariant with respect to time translations because the metric does not depend on time. Of course, the Lie algebra of $\overline{\mathbb{T}}$ is $\overline{\mathbb{T}}$ and we have the momentum map

$$J : \overline{\mathbb{T}} \to Co(J_1\boldsymbol{E}) : \xi \mapsto J(\xi) \equiv \xi \lrcorner (o \lrcorner \Theta)\,.$$

Obviously, $J = \mathcal{H}[o]$. □

Example II.3. Now, we suppose our spacetime to be $\boldsymbol{T} \times SO(g)$, where g is the metric of the above spacetime. The manifold $\boldsymbol{T} \times SO(g)$ is interpreted as the configuration space for the relative configurations of a rigid body with respect to the center of mass (see [18, 7] for a more detailed account).

We assume the *inertia tensor* I as the scaled vertical metric. Consider the action

$$SO(g) \times (\mathbb{T} \times SO(g)) \to \mathbb{T} \times SO(g) : (A, (\tau, B)) \mapsto (\tau, AB)\,.$$

Let $K^\natural$ be the natural flat connection on $\boldsymbol{T} \times SO(g)$ and $F = 0$. Then, $SO(g)$ is a group of symmetries of Θ and a momentum map J is just the angular momentum.

In fact, as in the previous examples, Θ reduces to the kinetic energy of particles with respect to the center of mass. This is obviously invariant with respect orthogonal transformations [7]. We have the momentum map

$$J : so(g_a) \to Co(\boldsymbol{T} \times \mathbb{T} \otimes TSO(g)) : \omega \mapsto J_\omega \equiv \omega^* \lrcorner \mathcal{P}\,,$$

where, by definition, $\mathcal{P} = V_{\boldsymbol{E}} L$, with the coordinate expression $\mathcal{P} = I_{ij} x_0^j \check{d}^i$. A simple computation shows that

– $\omega^* : SO(g) \to TSO(g) : r \mapsto \omega(r)$;

– $\omega^* \lrcorner \mathcal{P}(v) = I(\omega(r), v) = \omega(r \times v)$.

The Lie algebra of $SO(g_a)$ is $so(g_a)$, but the Hodge star isomorphism yields a natural Lie algebra isomorphism $so(g_a) \simeq \mathbb{L}^{-1} \otimes \boldsymbol{S}_a$. The isomorphism carries the Lie bracket of $so(g_a)$ into the cross product. In this way, if $\omega \in so(g_a)$ and $\bar{\omega} \in \mathbb{L}^{-1} \otimes \boldsymbol{S}_a$ is the corresponding element, we can equivalently write

$$J : \mathbb{L}^{-1} \otimes \boldsymbol{S}_a \to Co(\boldsymbol{T} \times \mathbb{T} \otimes TSO(g)) : \bar{\omega} \mapsto J_{\bar{\omega}} \equiv I(r \times v, \omega)\,,$$

where $v \in \mathbb{T}^* \otimes T\boldsymbol{R}_a \equiv J_1(\boldsymbol{T} \times \boldsymbol{R}_a)$. This proves the last part of the statement. □

III Conclusions

The use of natural bijective correspondences between distinguished objects in *CCG* theory provided us relations between symmetries of these objects. In particular, we saw that the holonomic infinitesimal symmetries of the cosymplectic structure yield the holonomic infinitesimal symmetries of all physical objects and of the dynamics, and conversely.

The momentum map of *CCG* theory has a surprising feature, due to covariance. Namely, its values turn out to be ‘quantizable functions’. This feature is not present in general symplectic or cosymplectic momentum maps. It seems that covariance naturally implements a classical analogue of Groenwald–van Hove theorem [37].

Besides their importance in covariant classical mechanics, the above results constitute the first fundamental step towards a research about symmetries in covariant quantum mechanics. This will be our next goal.

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