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Lie–Remarkable PDEs

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Abstract

Within the context of the inverse Lie problem the question whether there exist PDEs that are characterized by their Lie point symmetries may be addressed. In a recent paper the authors called these equations *Lie remarkable*. In this paper we exhibit various examples of Lie remarkable equations, including some multidimensional Monge-Ampère type equations.
Keywords: Lie symmetries of differential equations, jet spaces

Dedicated to Antonio Greco
on the occasion of his 65th birthday

1 Introduction

One of the most powerful tools for studying differential equations (DEs), either ordinary or partial, is provided by the theory of symmetries (see Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9]). Symmetries of DEs are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former, with the further property of sending solutions into solutions. Among symmetries, there is a distinguished class, that of symmetries coming from a transformation of the independent and dependent variables, namely point symmetries.

[1]Speaker at the Conference

Given a (system of) DE(s) the *direct* Lie problem consists in finding the admitted algebra of point symmetries. This task is accomplished by means of the Lie's algorithm, requiring the straightforward though tedious solution of an overdetermined system of DEs, possibly by using some computer algebra packages like Dimsym[10], MathLie[9], or Relie[11].

The problem of finding the symmetries of a DE has associated a natural *inverse* problem, namely, the problem of finding the most general form of a DE admitting a given (abstract) Lie algebra as subalgebra of point symmetries. This problem was considered, for instance, in Refs. [12, 13].

An aspect of this problem has been considered in Ref. [14], where the authors, starting from a given DE, found necessary and sufficient conditions for it to be uniquely determined by its point symmetries. By following the terminology already used in Refs. [15, 16, 14, 17], we call such a DE *Lie remarkable.* A similar problem was also considered in Refs. [18, 19].

In this paper we review some of the results obtained in Ref. [14], and give examples of Lie remarkable equations. Here we also treat the case of a multidimensional Monge-Ampère equation, *i.e.*, with more than two independent variables.

The plan of the paper is the following. In section 2, we introduce a DE of order r as a submanifold of a suitable jet space (of order r). Then we distinguish two types of Lie remarkable equations: *strongly* and *weakly* Lie remarkable equations. Strongly Lie remarkable equations are uniquely determined by their point symmetries; weakly Lie remarkable equations are equations which do not intersect other equations admitting the same symmetries. Then we report [14, 16] necessary as well as sufficient conditions for an equation to be strongly or weakly Lie remarkable.

In section 3, we give various examples of either strongly or weakly Lie remarkable equations: they include equations of Monge-Ampère type, with two and three independent variables, and minimal surface equations.

2 Theoretical framework

Here we recall some basic facts regarding jet spaces (for more details, see Refs. [8, 5, 20]) and the basic theory on DEs determined by their Lie point symmetries[14].

All manifolds and maps are supposed to be C^∞. If E is a manifold then we denote by $\chi(E)$ the Lie algebra of vector fields on E. Also, for the sake of simplicity, all submanifolds of E are *embedded* submanifolds.

Let E be an $(n+m)$-dimensional smooth manifold and L an n-dimensional submanifold of E. Let (V, y^A) be a local chart on E. The coordinates (y^A) can be divided in two sets, $(y^A) = (x^\lambda, u^i)$, $\lambda = 1 \dots n$ and $i = 1 \dots m$, such that the submanifold L is locally described as the graph of a vector function $u^i = f^i(x^1, \dots, x^n)$. In what follows, Greek indices run from 1 to n and Latin indices run from 1 to m unless otherwise specified.

The set of equivalence classes $[L]^r_p$ of submanifolds L having at $p \in E$ a contact of order r is said to be the *r-jet of n-dimensional submanifolds of E* (also known as extended bundles [5]), and is denoted by $J^r(E, n)$. If E is endowed with a bundle

$\pi : E \to M$ where $\dim M = n$, then the r-th order jet $J^r\pi$ of local sections of π is an open dense subset of $J^r(E,n)$. We have the natural maps $j_rL\colon L \to J^r(E,n)$, $p \mapsto [L]^r_p$, and $\pi_{k,h}\colon J^k(E,n) \to J^h(E,n)$, $[L]^k_p \mapsto [L]^h_p$, $k \geq h$.

The set $J^r(E,n)$ is a smooth manifold whose dimension is

$$\dim J^r(E,n) = n + m \sum_{h=0}^{r} \binom{n+h-1}{n-1} = n + m \binom{n+r}{r}, \tag{1}$$

whose charts are $(x^\lambda, u^i_{\boldsymbol{\sigma}})$, where $u^i_{\boldsymbol{\sigma}} \circ j_rL = \partial^{|\boldsymbol{\sigma}|} f^i / \partial x^{\boldsymbol{\sigma}}$, where $0 \leq |\boldsymbol{\sigma}| \leq r$. On $J^r(E,n)$ there is a distribution, the contact distribution, which is generated by the vectors

$$D_\lambda \overset{\text{def}}{=} \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma}\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \quad \text{and} \quad \frac{\partial}{\partial u^j_{\boldsymbol{\tau}}},$$

where $0 \leq |\boldsymbol{\sigma}| \leq r-1$, $|\boldsymbol{\tau}| = r$ and $\boldsymbol{\sigma}\lambda$ denotes the multi-index $(\sigma_1, \dots, \sigma_{r-1}, \lambda)$. Any vector field $\Xi \in \chi(E)$ can be lifted to a vector field $\Xi^{(k)} \in \chi(J^k(E,n))$ which preserves the contact distribution. In coordinates, if $\Xi = \Xi^\lambda \partial/\partial x^\lambda + \Xi^i \partial/\partial u^i$ is a vector field on E, then its k-lift $\Xi^{(k)}$ has the coordinate expression

$$\Xi^{(k)} = \Xi^\lambda \frac{\partial}{\partial x^\lambda} + \Xi^i_{\boldsymbol{\sigma}} \frac{\partial}{\partial u^i_{\boldsymbol{\sigma}}}, \tag{2}$$

where $\Xi^j_{\boldsymbol{\tau},\lambda} = D_\lambda(\Xi^j_{\boldsymbol{\tau}}) - u^j_{\boldsymbol{\tau},\beta} D_\lambda(\Xi^\beta)$ with $|\boldsymbol{\tau}| < k$.

A *differential equation $\mathcal{E}$ of order r on n-dimensional submanifolds of a manifold E* is a submanifold of $J^r(E,n)$. The manifold $J^r(E,n)$ is called the *trivial equation*. An *infinitesimal point symmetry* of $\mathcal{E}$ is a vector field of the type $\Xi^{(r)}$ which is tangent to $\mathcal{E}$.

Let $\mathcal{E}$ be locally described by $\{F^i = 0\}$, $i = 1 \dots k$ with $k < \dim J^r(E,n)$. Then finding point symmetries amounts to solve the system

$$\Xi^{(r)}\left(F^i\right) = 0 \quad \text{whenever} \quad F^i = 0$$

for some $\Xi \in \chi(E)$.

We denote by $\operatorname{sym}(\mathcal{E})$ the Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$.

By an *r-th order differential invariant* of a Lie subalgebra $\mathfrak{s}$ of $\chi(E)$ we mean a smooth function $I\colon J^r(E,n) \to \mathbb{R}$ such that for all $\Xi \in \mathfrak{s}$ we have $\Xi^{(r)}(I) = 0$.

The problem of determining the Lie algebra $\operatorname{sym}(\mathcal{E})$ is said to be the *direct Lie problem.* Conversely, given a Lie subalgebra $\mathfrak{s} \subset \chi(E)$, we consider the *inverse Lie problem*, *i.e.*, the problem of characterizing the equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathfrak{s} \subseteq \operatorname{sym}(\mathcal{E})$ [1, 21].

1 Definition. Let E be a manifold, $\dim E = n + m$, and let $r \in \mathbb{N}$, $r > 0$. An l-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be

1. *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ passing at any $\theta \in \mathcal{E}$ admitting $\operatorname{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries;

2. *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) l-dimensional equation in $J^r(E,n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of the algebra of its infinitesimal point symmetries.

Of course, a strongly Lie remarkable equation is also weakly Lie remarkable. Some direct consequences of our definitions are due. For each $\theta \in J^r(E,n)$ denote by $S_\theta(\mathcal{E}) \subset T_\theta J^r(E,n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at θ. Let us set $S(\mathcal{E}) \stackrel{\text{def}}{=} \bigcup_{\theta \in J^r(E,n)} S_\theta(\mathcal{E})$. In general, $\dim S_\theta(\mathcal{E})$ may change with $\theta \in J^r(E,n)$. The following inequality holds:

$$\dim \mathrm{sym}(\mathcal{E}) \geq \dim S_\theta(\mathcal{E}), \ \forall\, \theta \in J^r(E,n), \tag{3}$$

where $\dim \mathrm{sym}(\mathcal{E})$ is the dimension, as real vector space, of the Lie algebra of infinitesimal point symmetries $\mathrm{sym}(\mathcal{E})$ of $\mathcal{E}$. If the rank of $S(\mathcal{E})$ at each $\theta \in J^r(E,n)$ is the same, then $S(\mathcal{E})$ is an involutive (smooth) distribution.

A submanifold N of $J^r(E,n)$ is an *integral submanifold* of $S(\mathcal{E})$ if $T_\theta N = S_\theta(\mathcal{E})$ for each $\theta \in N$. Of course, an integral submanifold of $S(\mathcal{E})$ is an equation in $J^r(E,n)$ which admits all elements in $\mathrm{sym}(\mathcal{E})$ as infinitesimal point symmetries. The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$ [14]. It follows that $\mathcal{E}$ can not coincide with the set of points of maximal rank of $S(\mathcal{E})$. The following theorems [14] can be proved.

2 Theorem.

1. *A necessary condition for the differential equation $\mathcal{E}$ to be strongly Lie remarkable is that*
$$\dim \mathrm{sym}(\mathcal{E}) > \dim \mathcal{E}.$$

2. *A necessary condition for the differential equation $\mathcal{E}$ to be weakly Lie remarkable is that*
$$\dim \mathrm{sym}(\mathcal{E}) \geq \dim\, \mathcal{E}.$$

In Ref. [14] also sufficient conditions have been established, that reveal useful when computing examples and applications.

3 Theorem.

1. *If $S(\mathcal{E})|_{\mathcal{E}}$ is an l-dimensional distribution on $\mathcal{E} \subset J^r(E,n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.*

2. *Let $S(\mathcal{E})$ be such that for any $\theta \notin \mathcal{E}$ we have $\dim S_\theta(\mathcal{E}) > l$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.*

The next theorem [14] gives the relationship between Lie remarkability and differential invariants.

4 Theorem. *Let $\mathfrak{s}$ be a Lie subalgebra of $\chi(J^r(E,n))$. Let us suppose that the r-prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E,n)$ and that the set of r-th order functionally independent differential invariants of $\mathfrak{s}$ reduces to a unique element $I \in C^\infty(J^r(E,n))$. Then the submanifold of $J^r(E,n)$ described by $\Delta(I) = 0$ (in particular $I = k$ for any $k \in \mathbb{R}$), with Δ an arbitrary smooth function, is a weakly Lie remarkable equation.*

To prove that a PDE is strongly or weakly Lie remarkable the following steps are required:

1. determine its Lie point symmetries;
2. determine the rank k of the distribution generated by its r–order prolongations and compare it with the dimension of the equation;
3. determine the submanifolds where the rank of the distribution decreases.

3 Examples

In what follows we give some examples of Monge-Ampère equations (of various order) and minimal surface equations which are Lie remarkable. Since we deal with (infinitesimal) point symmetries of these equations, as we are interested to local aspects, we will interpret them as submanifolds of jets of a trivial bundle $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, with $n = 2, 3$, which we denote, following our notation, by $J^r(\mathbb{R}^{n+1}, n)$.

3.1 Second order Monge-Ampère equation

The 2nd order Monge-Ampère equation in 2 independent variables has been introduced by Ampère in 1815; in 1968, Boillat[22] discovered that it is the only second order equation being completely exceptional in the sense of Lax. The requirement of complete exceptionality has been used to derive Monge-Ampère equations involving more than 2 independent variables[23, 24, 25].

5 Proposition (Boillat, 1991)**.** *Given an unknown field*

$$u(x_0, x_1, \ldots, x_n), \qquad (x_0 \text{ denoting the time}),$$

and its associated Hessian matrix $H = \left\| \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} \right\|$, the most general 2nd order PDE being completely exceptional (and called Monge-Ampère equation) is provided by a linear combination of all minors extracted from H, with coefficients depending at most on x_α, u and first order derivatives of u.

The classical Monge-Ampère equation written in the form

$$\kappa_1(u_{tt}u_{xx} - u_{tx}^2) + \kappa_2 u_{tt} + \kappa_3 u_{tx} + \kappa_4 u_{xx} + \kappa_5 = 0,$$

where the coefficients κ_i ($\kappa_1 \neq 0$) are constant, through the substitution

$$u \to u - \frac{\kappa_4}{2\kappa_1}x^2 + \frac{\kappa_3}{2\kappa_1}xt - \frac{\kappa_2}{2\kappa_1}t^2$$

is mapped to

$$u_{tt}u_{xx} - u_{tx}^2 = \kappa, \qquad \kappa = \frac{4\kappa_2\kappa_4 - 4\kappa_1\kappa_5 - \kappa_3^2}{4\kappa_1^2}.$$

If $\kappa = 0$ we have the homogeneous Monge-Ampère equation for the surface $u(t,x)$ with zero Gaussian curvature.

6 Theorem. *Equation*

$$u_{tt}u_{xx} - u_{tx}^2 = \kappa \tag{4}$$

is weakly Lie remarkable if $\kappa \neq 0$, whereas it is strongly Lie remarkable if $\kappa = 0$.

Proof. Equation (4) is a hypersurface of $J^2(\mathbb{R}^3, 2)$. If $\kappa \neq 0$, equation (4) admits a 9-parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\begin{array}{lll} \Xi_1 = \frac{\partial}{\partial x}, & \Xi_2 = \frac{\partial}{\partial t}, & \Xi_3 = \frac{\partial}{\partial u}, \\ \Xi_4 = x\frac{\partial}{\partial t}, & \Xi_5 = t\frac{\partial}{\partial x}, & \Xi_6 = x\frac{\partial}{\partial u}, \\ \Xi_7 = t\frac{\partial}{\partial u}, & \Xi_8 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, & \Xi_9 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \end{array}$$

The 2nd order prolonged vector fields give rise to a distribution of rank 7 (equation (4) is a 7–dimensional submanifold!) on the whole jet space provided we exclude the 5–dimensional submanifolds locally described by $u_{xx} = u_{tx} = u_{tt} = 0$ where the rank reduces to 5. Thus, non-homogeneous equation (4) is weakly Lie remarkable.

On the contrary, if $\kappa = 0$, equation (4) admits a 15–dimensional Lie algebra of point symmetries spanned by

$$\frac{\partial}{\partial a}, \quad a\frac{\partial}{\partial b}, \quad a\frac{\partial}{\partial b}, \quad a\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial u}\right), \quad \forall a, b \in \{x, t, u\}.$$

In this case, the 2nd order prolonged vector fields give rise to a distribution of rank 8 on $J^2(\mathbb{R}^3, 2)$ (which has dimension 8), provided we exclude the submanifold characterized by the equation itself (where the rank is at most 7; in fact on the submanifold $u_{xx} = u_{tx} = u_{tt} = 0$ the rank reduces to 5). Hence, the homogeneous equation for a surface with vanishing Gaussian curvature is strongly Lie remarkable. $\square$

3.2 Monge-Ampère equation in 3 independent variables

Consider the 2nd order Monge-Ampère equation in 3 independent variables[23]:

$$\begin{aligned} &\kappa_1[u_{tt}(u_{xx}u_{yy} - u_{xy}^2) + u_{tx}(u_{ty}u_{xy} - u_{tx}u_{yy} + u_{ty}(u_{tx}u_{xy} - u_{ty}u_{xx})] \\ &+\kappa_2(u_{xx}u_{yy} - u_{xy}^2) + \kappa_3(u_{ty}u_{xy} - u_{tx}u_{yy}) + \kappa_4(u_{tx}u_{xy} - u_{ty}u_{xx}) \\ &+\kappa_5(u_{tt}u_{yy} - u_{ty}^2) + \kappa_6(u_{tx}u_{ty} - u_{tt}u_{xy}) + \kappa_7(u_{tt}u_{xx} - u_{tx}^2) \\ &+\kappa_8 u_{tt} + \kappa_9 u_{tx} + \kappa_{10}u_{ty} + \kappa_{11}u_{xx} + \kappa_{12}u_{xy} + \kappa_{13}u_{yy} + \kappa_{14} = 0, \end{aligned}$$

where κ_i $(i = 1, \dots, 14)$ are taken constant.

The explicit determination of the infinitesimal generators of the admitted Lie group results quite complicated and the use of Computer Algebra packages reveals extremely memory consuming since the expression of the infinitesimals involves thousands of terms. Nevertheless, without loss of generality, it is possible to introduce the substitution

$$u \to u + \alpha_1 t^2 + \alpha_2 tx + \alpha_3 ty + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2,$$

α_i being suitable constants, so reducing the equation to the equivalent form

$$\begin{aligned}
&\kappa_1[u_{tt}(u_{xx}u_{yy} - u_{xy}^2) + u_{tx}(u_{ty}u_{xy} - u_{tx}u_{yy} + u_{ty}(u_{tx}u_{xy} - u_{ty}u_{xx})]\\
&+\kappa_2(u_{xx}u_{yy} - u_{xy}^2) + \kappa_3(u_{ty}u_{xy} - u_{tx}u_{yy}) + \kappa_4(u_{tx}u_{xy} - u_{ty}u_{xx})\\
&+\kappa_5(u_{tt}u_{yy} - u_{ty}^2) + \kappa_6(u_{tx}u_{ty} - u_{tt}u_{xy}) + \kappa_7(u_{tt}u_{xx} - u_{tx}^2) = \kappa,
\end{aligned} \tag{5}$$

In general this equation admits an 11–dimensional Lie algebra of point symmetries. Since the equation represents a 12–dimensional submanifold of $J^2(\mathbb{R}^4, 3)$ (which is a 13–dimensional manifold), it can not be neither strongly nor weakly Lie remarkable. However, the following theorem holds.

7 Theorem. *The 2nd order Monge-Ampère equation* (5)*, when the coefficients are such that*

$$\kappa_1 = 1, \qquad \kappa_2\kappa_6^2 - \kappa_3\kappa_4\kappa_6 + \kappa_4^2\kappa_5 - (4\kappa_2\kappa_5 - \kappa_3^2)\kappa_7 = 0,$$

is weakly Lie remarkable.

Proof. In fact:

1. the dimension of the submanifold described by (5) is 12;
2. the Lie algebra of point symmetries is 13–dimensional;
3. the 2nd order prolongations of the admitted vector fields give rise to a distribution of rank 12 provided that we exclude some submanifolds not contained in the equation itself.

Hence, due to theorem 3, equation (5) is weakly Lie remarkable. □

8 Remark. It may be verified that the unique 2nd order differential invariant of the Lie symmetries of equation (5) is

$$\begin{aligned}
I = {}&(u_{tt}u_{xx}u_{yy} - u_{tt}u_{xy}^2 - u_{tx}^2u_{yy} + 2u_{tx}u_{ty}u_{xy} - u_{ty}^2u_{xx})\\
&+\kappa_2(u_{xx}u_{yy} - u_{xy}^2) - \kappa_3(u_{tx}u_{yy} - u_{ty}u_{xy}) + \kappa_4(u_{tx}u_{xy} - u_{ty}u_{xx})\\
&+\kappa_5(u_{tt}u_{yy} - u_{ty}^2) - \kappa_6(u_{tt}u_{xy} - u_{tx}u_{ty}) + \kappa_7(u_{tt}u_{xx} - u_{tx}^2),
\end{aligned}$$

whereupon if follows that they characterize the equation

$$\Delta(I) = 0 \quad \Rightarrow \quad I = \kappa, \quad \kappa \text{ constant}$$

More generally, some other 2nd order Monge-Ampère equations, involving more than 3 independent variables, are weakly Lie remarkable.

3.3 Higher order Monge-Ampère equations

The property of complete exceptionality has been used by Boillat[26] to determine higher order Monge-Ampère equations for the unknown $u(t, x)$.

By considering an equation of order $N > 2$, we need to consider the Hankel matrix

$$H = \begin{bmatrix} X_0 & X_1 & X_2 & \dots & X_{M-1} & X_M \\ X_1 & X_2 & X_3 & \dots & X_M & X_{M+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{M-1} & X_M & X_{M+1} & \dots & X_{2M-2} & X_{2M-1} \\ X_M & X_{M+1} & X_{M+2} & \dots & X_{2M-1} & X_{2M} \end{bmatrix},$$

where $X_i = \dfrac{\partial^N u}{\partial t^i \partial x^{N-i}}$.

9 Theorem (Boillat, 1992)**.** *The most general nonlinear completely exceptional equation is given, if $N = 2M$, by a linear combination of all minors, including the determinant, of the Hankel matrix, whereas in the case where $N = 2M-1$, we have to consider the linear combination of all minors extracted from the Hankel matrix where the last row has been removed.*

In both cases the coefficients of the linear combination are functions of t, x, u and its derivatives up to the order $N-1$. Let us limit ourselves to the case where these coefficients are constant.

Consider the 3rd order Monge-Ampère equation

$$\begin{aligned} &\widetilde{\kappa}_1(u_{ttx}u_{xxx} - u_{txx}^2) + \widetilde{\kappa}_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \widetilde{\kappa}_3(u_{ttt}u_{txx} - u_{ttx}^2) \\ &\quad + \widetilde{\kappa}_4 u_{ttt} + \widetilde{\kappa}_5 u_{ttx} + \widetilde{\kappa}_6 u_{txx} + \widetilde{\kappa}_7 u_{xxx} + \widetilde{\kappa}_8 = 0. \end{aligned}$$

The substitution

$$u \to u + \alpha_1 t^3 + \alpha_2 t^2 x + \alpha_3 t x^2 + \alpha_4 x^3$$

provides the equation

$$\kappa_1(u_{ttx}u_{xxx} - u_{txx}^2) + \kappa_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \kappa_3(u_{ttt}u_{txx} - u_{ttx}^2) = \kappa. \tag{6}$$

Equation (6) describes an 11–dimensional submanifold of $J^3(\mathbb{R}^3, 2)$ (which is a 12–dimensional manifold); since the Lie algebra of its point symmetries is 10–dimensional, it can not be in general Lie remarkable.

Nevertheless, the following theorem may be proved.

10 Theorem. *The equation*

$$(u_{ttx}u_{xxx} - u_{txx}^2) + \lambda(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \lambda^2(u_{ttt}u_{txx} - u_{ttx}^2) = \mu,$$

where

$$\lambda = \frac{\kappa_3}{\kappa_2}, \qquad \mu = \frac{\kappa\kappa_3}{\kappa_2^2},$$

obtained from (6) *by choosing* $\kappa_1 = \dfrac{\kappa_2^2}{\kappa_3}$, *is weakly Lie remarkable.*

Proof. In fact, the Lie algebra of point symmetries admitted is spanned by

$$\begin{array}{lll}
\Xi_1 = \frac{\partial}{\partial t}, & \Xi_2 = \frac{\partial}{\partial x}, & \Xi_3 = \frac{\partial}{\partial u}, \\
\Xi_4 = (2t - 3\lambda x)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, & \Xi_5 = \lambda^2 x\frac{\partial}{\partial t} + (2t - \lambda x)\frac{\partial}{\partial x}, & \\
\Xi_6 = \lambda x\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, & \Xi_7 = t\frac{\partial}{\partial u}, & \Xi_8 = x\frac{\partial}{\partial u}, \\
\Xi_9 = t^2\frac{\partial}{\partial u}, & \Xi_{10} = tx\frac{\partial}{\partial u}, & \Xi_{11} = x^2\frac{\partial}{\partial u}, \\
\Xi_{12} = F(t - \lambda x)\frac{\partial}{\partial u}, & &
\end{array}$$

where F is an arbitrary function of $(t - \lambda x)$, and their 3rd order prolongations give rise, provided $F''' \neq 0$, to a distribution of rank 11, provided that we exclude some singular subsets. □

11 Remark. It may be verified that the unique third order differential invariant is

$$I = (u_{ttx}u_{xxx} - u_{txx}^2) + \lambda(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \lambda^2(u_{ttt}u_{txx} - u_{ttx}^2),$$

whereupon if follows that the operators characterize the equation

$$\Delta(I) = 0 \quad \Rightarrow \quad I = \mu.$$

12 Theorem. *The fourth order Monge-Ampère equation*

$$u_{tttt}(u_{ttxx}u_{xxxx} - u_{txxx}^2) + 2u_{tttx}u_{ttxx}u_{txxx} - u_{ttxx}^3 - u_{tttx}^2 u_{xxxx} = 0$$

is weakly Lie remarkable.

Proof.
1. The previous equation describes a 16–dimensional submanifold of $J^4(\mathbb{R}^3, 2)$, which is a 17–dimensional manifold;
2. The Lie algebra of point symmetries is 19–dimensional;
3. The rank of 4th order prolongations give rise to a distribution of rank 16 provided we exclude a singular subset.

□

3.4 Equation of minimal surface in $\mathbb{R}^{m+2}$

13 Theorem. *The equation of minimal surface in* $\mathbb{R}^{m+2}$

$$(1 + |\mathbf{u}_y|^2)\mathbf{u}_{xx} - 2|\mathbf{u}_x| \cdot |\mathbf{u}_y|\mathbf{u}_{xy} + (1 + |\mathbf{u}_x|^2)\mathbf{u}_{yy} = \mathbf{0}, \quad \mathbf{u} \in \mathbb{R}^m, \tag{7}$$

is nor strongly neither weakly Lie remarkable when $m = 2$ or $m = 3$, whereas is weakly Lie remarkable when $m = 1$ or $m = 4$.

Proof. If $m = 2$ or $m = 3$, the theorem follows immediately by dimensional reasons in view of theorem 2.

Then let us discuss the case $m = 1$.

In this case, equation (7) admits the following Lie algebra of point symmetries (which is formed by isometries and scaling):

$$\begin{array}{lll} \Xi_1 = \frac{\partial}{\partial x}, & \Xi_2 = \frac{\partial}{\partial y}, & \Xi_3 = \frac{\partial}{\partial u}, \\ \Xi_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, & \Xi_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, & \Xi_6 = u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u}, \\ \Xi_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, & & \end{array} \tag{8}$$

The 2nd order prolongations of (8) give rise to a distribution of rank 7 on the equation, provided we exclude the differential equation of planes. Since equation (7) describes a 7–dimensional submanifold in $J^2(\mathbb{R}^3, 2)$ (which is an 8–dimensional manifold), from theorem 3 the result follows.

The case $m = 4$ is analogous to the case $m = 1$, then we omit computations. □

14 Remark. The unique 2nd order differential invariant of (8) is:

$$I = \frac{\left((1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}^2\right)^2}{(1 + u_x^2 + u_y^2)(u_{xx}u_{yy} - u_{xy}^2)} = \frac{4H^2}{G}, \tag{9}$$

where H is the scalar mean curvature and G the Gaussian curvature.

Then most general equation admitting the point symmetries of minimal surface equation in $\mathbb{R}^3$ is given by

$$\Delta(I) = 0, \quad \Rightarrow \quad I = \kappa, \quad \kappa \text{ constant}.$$

It is worth of noticing (see Ref. [27]) that the equation

$$\frac{H^2}{G} = 1$$

is a strongly Lie remarkable equation characterized by the conformal algebra of $\mathbb{R}^3$.

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