Hamiltonian structures for general PDEs

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Abstract

We sketch out a new geometric framework to construct Hamiltonian operators for generic, non-evolutionary partial differential equations. Examples on how the formalism works are provided for the KdV equation, Camassa–Holm equation, and Kupershmidt's deformation of a bi-Hamiltonian system.

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1 Introduction

In this short paper we will discuss the following question: What happens to a Hamiltonian operator of an evolution system if we change coordinates so that the system becomes non-evolution?

Using the traditional definition of a Hamiltonian structure one cannot answer this question, since the definition is tied to evolution form of the system at hand. However, first, not all equations have a natural evolution form, and, second, an evolution form of a system of equations is not unique. Let us consider some examples.

1 Example (KdV)**.** It is well known that the KdV equation $u_t = u_{xxx}+6uu_x$ has two compatible Hamiltonian operators:

$$A_1 = D_x, \qquad A_2 = D_{xxx} + 4uD_x + 2u_x, \tag{1}$$

so that the equation can be written in the following ways:

$$\begin{aligned} u_t = u_{xxx} + 6uu_x &= D_x\frac{\delta}{\delta u}(u^3 - u_x^2/2) \\ &= (D_{xxx} + 4uD_x + 2u_x)\frac{\delta}{\delta u}(u^2/2), \end{aligned}$$

where $\delta/\delta u$ denotes the Euler operator (the variational derivative) and is applied to the two Hamiltonian densities.

Let us introduce new dependent variables v and w and rewrite the KdV equation in the form

$$u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv. \tag{2}$$

In the new coordinates, the KdV still has an evolutionary form, but with respect to another independent variable (x instead of t). A natural question arises then: is the KdV equation in the form (2) Hamiltonian? An affirmative answer to this question was obtained by Tsarev in [6]. He proved that transformations of the type (2) preserve the Hamiltonian property of all evolution systems for which the Cauchy problem is solvable. Our approach is very different from Tsarev's one. Below we explain why this fact holds true for all transformations of variables and without the assumption on the Cauchy problem. We will also show how to compute the Hamiltonian structure in

new coordinates. For the above example the answer is the following:

(3)
$$\begin{aligned}\begin{pmatrix}u\\v\\w\end{pmatrix}_x &= \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -6u\\ 0 & 6u & D_t\end{pmatrix}\frac{\delta}{\delta u}(uw - v^2/2 + 2u^3)\\ &= \begin{pmatrix}0 & -2u & -D_t - 2v\\ 2u & D_t & -12u^2 - 2w\\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t\end{pmatrix}\frac{\delta}{\delta u}(-3u^2/2 - w/2).\end{aligned}$$

2 Example (Camassa–Holm equation)**.** Camassa and Holm have written their equation $u_t - u_{txx} - uu_{xxx} - 2u_xu_{xx} + 3uu_x = 0$ in a bi-Hamiltonian form by introducing the new variable $m = u - u_{xx}$. The equation now takes the form

$$m_t = -um_x - 2u_xm = B_1\frac{\delta\mathscr{H}_1}{\delta m} = B_2\frac{\delta\mathscr{H}_2}{\delta m} \tag{4}$$

with

$$\begin{aligned}B_1 &= -(mD_x + D_xm), \quad \mathscr{H}_1 = \frac{1}{2}\int mu\,dx,\\ B_2 &= D_x^3 - D_x, \quad \mathscr{H}_2 = \frac{1}{2}\int (u^3 + uu_x^2))\,dx.\end{aligned}$$

Note that $\mathscr{H}_1$ and $\mathscr{H}_2$ are viewed as functionals in m and u, but not in u solely. To get rid of m, one is forced to assume that $u = (1 - D_x^2)^{-1}m$ in the Hamiltonian densities. The use of the inverse of the operator $1 - D_x^2$ is not elegant from mathematical viewpoint. We will find a bi-Hamiltonian structure for the Camassa–Holm equation written in the initial non-evolution form and thus get rid of the term $(1 - D_x^2)^{-1}$.

3 Example (Kupershmidt deformation)**.** Consider a bi-Hamiltonian evolution system of equations $u_t = f(t, x, u, u_x, u_{xx}, \dots)$, u and f being vector functions, with compatible Hamiltonian operators A_1 and A_2 and a Magri hierarchy of conserved densities H_1, H_2, ...

$$D_t(H_i) = 0, \qquad A_1\frac{\delta H_i}{\delta u} = A_2\frac{\delta H_{i+1}}{\delta u}.$$

In [5], Kupershmidt defined what he called the *nonholonomic deformation* of the above system:

$$u_t = f - A_1(w), \qquad A_2(w) = 0. \tag{5}$$

We call system (5) the *Kupershmidt deformation* of the system $u_t = f$. The motivating example of this construction is the so-called KdV6 equation (see [2])

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0 \tag{6}$$

which is the Kupershmidt deformation of the KdV equation. The authors of [2] have shown that the KdV6 passes the Painlevé test and conjectured that the system is integrable. Kupershmidt, in [5], found an hierarchy of conservation laws of the KdV6 as a particular case of the following general fact.

Theorem (Kupershmidt)**.** *Let $u_t = f$ be an evolution bi-Hamiltonian system, with A_1, A_2 being the corresponding Hamiltonian operators. If this equation has a Magri hierarchy of conserved densities $\frac{dH_i}{dt} = 0$, $A_1\frac{\delta H_i}{\delta u} = A_2\frac{\delta H_{i+1}}{\delta u}$ then H_1, H_2, ... are conserved densities for* (5)*.*

Proof.

$$\frac{dH_i}{dt} = \left\langle \frac{\delta H_i}{\delta u}, f + A_1(w) \right\rangle = \left\langle -A_1\frac{\delta H_i}{\delta u}, w \right\rangle$$
$$= \left\langle -A_2\frac{\delta H_{i+1}}{\delta u}, w \right\rangle = \left\langle \frac{\delta H_{i+1}}{\delta u}, A_2(w) \right\rangle = 0. \qquad \square$$

$\square$

Kupershmidt also conjectured that H_1, H_2, ... commute in some sense so that the KdV6 is indeed integrable. Below we will see that this is true and, moreover, system (5) is bi-Hamiltonian.

Our framework to study Hamiltonian structures for general PDEs is the geometry of jet spaces and differential equations. We assume the reader to be familiar with the geometric approach to differential equations and hence we include only the notation and the coordinate descriptions in the next section. We refer the reader to the book [1] for further information.

2 Notation: infinite jets and differential equations

In what follows everything is supposed to be smooth.

We denote an infinite jet space by J^∞. This can be the space of jets of submanifolds, maps, sections of a bundle, and so on, and it is not important

to us here. Coordinates on J^∞ are x_i (independent variables, $i = 1, \dots, n$) and u^j_σ (dependent variables, $j = 1, \dots, m$, σ being multi-indices).

The formulas

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}$$

provide expressions for the total derivatives. The vector fields D_i span the Cartan distribution on J^∞. To every vector function on J^∞, there corresponds the evolutionary field

$$E_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}.$$

The matrix differential operator

$$\ell_f = \left\| \sum_\sigma \frac{\partial f^i}{\partial u^j_\sigma} D_\sigma \right\|.$$

is the linearization of a vector function f. It is defined by the formula $\ell_f(\varphi) = E_\varphi(f)$. The linearization is a differential operator in total derivatives; we shall call such operators *$\mathscr{C}$-differential operators*.

The coordinate expression for the adjoint $\mathscr{C}$-differential operator is

$$\Delta^* = \left\| \sum_\sigma (-1)^{|\sigma|} D_\sigma a^{ji}_\sigma \right\|$$

if $\Delta = \|\sum_\sigma a^{ij}_\sigma D_\sigma\|$.

Let $F_k(x_i, u^j_\sigma) = 0$, $k = 1, \dots\ l$, be a system of differential equations. Then the relations $F = (F_1, \dots, F_l) = 0$ together with $D_\sigma(F) = 0$ define its infinite prolongation $\mathscr{E} \subset J^\infty$. For the sake of brevity we shall call the infinite prolongation of a system of differential equations the equation. The operator $\ell_\mathscr{E} = \ell_F|_\mathscr{E}$ is the linearization of the equation $\mathscr{E}$.

In this paper, we consider *normal* equations only, i.e., such that the compatibility operators for both $\ell_\mathscr{E}$ and $\ell^*_\mathscr{E}$ are trivial. In other words, if there exists a $\mathscr{C}$-differential operator Δ such that $\Delta \circ \ell_\mathscr{E} = 0$ on $\mathscr{E}$ then $\Delta = 0$ on $\mathscr{E}$ as well, and the same holds true with $\ell^*_\mathscr{E}$ instead of $\ell_\mathscr{E}$.

An evolutionary field E_φ is a symmetry of the equation $\mathscr{E}$ if $E_\varphi(F)|_\mathscr{E} = \ell_\mathscr{E}(\varphi) = 0$. If E_φ is a symmetry then φ is said to be its generating function. We often identify symmetries with their generating functions.

A vector function $S = (S^1, \dots, S^n)$ on $\mathscr{E}$ is a conserved current if $\sum_i D_i(S^i) = 0$ on $\mathscr{E}$. A conserved current is trivial if there exist functions T_{ij} on $\mathscr{E}$ such that $S^i = \sum_{j<i} D_j(T^{ji}) - \sum_{i<j} D_j(T^{ij})$.

Conservation laws of $\mathscr{E}$ are classes of conserved currents modulo trivial ones. To every conservation law, there correspond its generating function, which is computed in the following way. If $S = (S^1, \dots, S^n)$ is a conserved current, so that $\sum_i D_i(S^i) = 0$ on $\mathscr{E}$ then there exists a $\mathscr{C}$-differential operator Δ such that $\sum_i D_i(S^i) = \Delta(F)$ on J^∞. The generating function of the conservation law is defined by $\psi = (\psi_1, \dots, \psi_m) = \Delta^*(1)$. Note that $\psi = 0$ if and only if the conserved current S is trivial. One can prove that every generating function ψ satisfies the equation $\ell^*_{\mathscr{E}}(\psi) = 0$, so that the set $\mathrm{CL}(\mathscr{E})$ of conservation laws of $\mathscr{E}$ is a subset in the kernel of $\ell^*_{\mathscr{E}}$, $\mathrm{CL}(\mathscr{E}) \subset \ker \ell^*_{\mathscr{E}}$.

3 Cotangent bundle to an equation

Let us introduce our main hero. For every differential equation $\mathscr{E}$ we define a canonical covering $\tau^*\colon \mathscr{L}^*(\mathscr{E}) \to \mathscr{E}$, called the *$\ell^*$-covering*. The equation $\mathscr{L}^*(\mathscr{E})$ is given by the system

$$\ell^*_F(p) = 0, \quad F = 0,$$

if $\mathscr{E}$ is given by $F = 0$. Here $p = (p^1, \dots, p^l)$ are new dependent variables, l being the number of equations $F = (F_1, \dots, F_l)$. We endow $\mathscr{L}^*(\mathscr{E})$ with the structure of a supermanifold by choosing the variables p^k to be odd. The covering τ^* is the natural projection $\tau^*\colon (u^j_\sigma, p^k_\sigma) \mapsto (u^j_\sigma)$.

Note that

$$\langle F, p\rangle = \sum_{i=1}^{l} F_i p^i \tag{7}$$

is the Lagrangian for the equation $\mathscr{L}^*(\mathscr{E})$.

From the above definition it is not seen why we said that ℓ^*-covering is *canonical*. Indeed, the definition uses the embedding $\mathscr{E} \to J^\infty$, but later we will show that $\mathscr{L}^*(\mathscr{E})$ is independent of the choice of this embedding.

4 Remark. For an arbitrary $\mathscr{C}$-differential operator Δ one can define the Δ-covering in the same way as the ℓ^*-covering is associated with the operator $\ell^*_{\mathscr{E}}$.

The most interesting for us property of the ℓ^*-covering is given by the following theorem.

5 Theorem. *There is a natural* 1-1 *correspondence between the symmetries of $\mathscr{E}$ and the conservation laws of $\mathscr{L}^*(\mathscr{E})$ linear along the fibers of τ^*.*

The expression "linear conservation law" means that the corresponding conserved current is linear along the fibers of τ^* (i.e., linear in variables p^k). Here and below we skip the proofs that can be found in [3]. Let us nevertheless describe the correspondence stated in the theorem in terms of generating functions. If φ is a symmetry of equation $\mathscr{E}$ then there exists a $\mathscr{C}$-differential operator Δ such that $\ell_F(\varphi) = \Delta(F)$. Consider the adjoint operator Δ^*. It can be naturally identified with a fiberwise linear vector function φ_Δ on $\mathscr{L}^*(\mathscr{E})$. Then the vector function $(\varphi, \varphi_\Delta)$ is the generating function of the conservation law that corresponds to the symmetry φ.

In the geometry of differential equation it is very useful to construct an analogy with geometry of finite dimensional manifolds. We shall now use this approach to clarify the meaning of the above theorem. Let us start building our analogy with the following two rather standard correspondences (cf. [7] and references therein):

Manifold M		**PDE** $\mathscr{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries

Now, using Theorem 5, we can say that the analog of the ℓ^*-covering is a vector bundle such that vector fields on the base are in 1-1 correspondence with fiberwise linear functions on the total space of the bundle. Obviously, such a bundle is the cotangent bundle. So, the ℓ^*-covering is the cotangent bundle to an equation, and we can continue our manifold-equation dictionary:

Manifold M		**PDE** $\mathscr{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries
$T^*(M)$	$\longleftrightarrow$	$\mathscr{L}^*(\mathscr{E})$

6 Remark. This dictionary can be easily extended:

Manifold M		**PDE** $\mathscr{E}$	
functions	$\longleftrightarrow$	conservation laws	
vector fields	$\longleftrightarrow$	symmetries	
$T^*(M)$	$\longleftrightarrow$	$\mathscr{L}^*(\mathscr{E})$	q
$T(M)$	$\longleftrightarrow$	$\mathscr{L}(\mathscr{E})$	
De Rham complex	$\longleftrightarrow$	$E_1^{0,n-1} \to E_1^{1,n-1} \to E_1^{2,n-1} \to \cdots$	

q Here $\mathscr{L}(\mathscr{E})$ is the ℓ-covering (see Remark 4). The complex $E_1^{0,n-1} \to E_1^{1,n-1} \to E_1^{2,n-1} \to \cdots$ is $(n-1)$st line of the Vinogradov $\mathscr{C}$-spectral sequence

(see [7] and references therein). In this paper we use only the first three entries of the dictionary.

7 Remark. In [4], Kupershmidt defined the cotangent bundle to a bundle. This construction can be identified with the ℓ^*-covering of the system

$$u^1_t = 0, \quad u^2_t = 0, \quad \dots \quad u^m_t = 0.$$

At this point, a natural quaestion may arise: what is the analog of the Poisson bracket on the cotangent bundle? The answer is that the ℓ^*-covering is endowed with a canonical Poisson bracket. More precisely, since we changed the parity of fibers in the ℓ^*-covering, this bracket is a super-bracket and is the analog of Schouten bracket. We shall call it the *variational Schouten bracket*.

To define the bracket, recall that $\mathscr{L}^*(\mathscr{E})$ has the Lagrangian structure (7). Hence, by the Noether theorem there is a 1-1 correspondence between conservation laws on $\mathscr{L}^*(\mathscr{E})$ and Noether symmetries of $\mathscr{L}^*(\mathscr{E})$. If ψ is the generating function of a conservation law, then E_ψ is the corresponding Noether symmetry. The set of Noether symmetries is a Lie superalgebra with respect to the commutator, so we obtain a structure of Lie superalgebra on conservation laws on $\mathscr{L}^*(\mathscr{E})$ uniquely determined by the equality

$$E_{[\![\psi_1,\psi_2]\!]} = [E_{\psi_1}, E_{\psi_2}]. \tag{8}$$

It is easy to show that $[\![\psi_1, \psi_2]\!] = E_{\psi_1}(\psi_2) + \ell^*_{\psi_1}(\psi_1)$.

According to our manifold-equation dictionary, conservation laws on $\mathscr{L}^*(\mathscr{E})$ correspond to functions on $T^*(M)$. The latter are skew multivectors on M (this is why we have changed the parity of fibers of the ℓ^*-covering—to get skew-symmetric multivectors). So, we shall call conservation laws on $\mathscr{L}^*(\mathscr{E})$ the *variational multivectors*. Linear conservation laws, as we saw, are vectors, biliner ones are bivectors and so on.

The generating function of a variational k-vector is a vector function on $\mathscr{L}^*(\mathscr{E})$ which is $(k-1)$-linear along τ^*-fibers. Such a function can be identified with a $(k-1)$-linear $\mathscr{C}$-differential operator on $\mathscr{E}$. In coordinates, this correspondence boils down to the change $p_\sigma \mapsto D_\sigma$. Thus, we can (and will) identify variational multivectors to multilinear $\mathscr{C}$-differential operators.

More precisely, in the above identification we will use not operators but equivalence classes of $\mathscr{C}$-differential operators modulo operators divisible by $\ell^*_{\mathscr{E}}$. This is being done, because operators of the form $\square \circ \ell^*_{\mathscr{E}}$ correspond to trivial functions on $\mathscr{L}^*(\mathscr{E})$. But we will not change terminology, we say operator instead of the equivalence class.

For the sake of brevity and because we are interested in the Hamiltonian formalism, let us restrict ourselves to bivectors, which are identified

with linear $\mathscr{C}$-differential operators. Formulas presented below for bivectors (= linear operators) can be easily generalised to multivectors (= multilinear operators).

8 Theorem. *An operator A is a variational bivector on equation $\mathscr{E}$ if and only if it satisfies the condition*

$$\ell_{\mathscr{E}} A = A^* \ell^*_{\mathscr{E}}.$$

9 Remark. If $\mathscr{E}$ is written in evolution form then the above condition implies that $A^* = -A$.

From this theorem it follows that a Hamiltonian operator A takes a conservation law ψ to a symmetry $A(\psi)$.

This is the formula for the variational Schouten bracket of two bivectors:

$$\begin{aligned}
&[\![A_1, A_2]\!](\psi_1, \psi_2) \\
&\quad = \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\quad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\qquad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}$$

where $\ell_{A,\psi} = \ell_{A(\psi)} - A\ell_\psi$ and the operators B_i^* are defined by the equalities:

$$\begin{gathered}
\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot) \quad \text{on } J^\infty, \\
B_i^*(\psi_1, \psi_2) = B_i^{*_1}(\psi_1, \psi_2)|_{\mathscr{E}}.
\end{gathered}$$

Here ^{*_1} denotes that the adjoint operator is computed with respect to the first argument. The operators B_i^* are skew-symmetric and skew-adjoint in each argument. Note that if $\mathscr{E}$ is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell^*_{A_i,\psi_2}(\psi_1)$.

Now we are in position to give a definition of a Hamiltonian structure for a general PDE.

10 Definition. A variational bivector A is called *Hamiltonian* if $[\![A, A]\!] = 0$.

A Hamiltonian bivector A gives rise to a Poisson bracket

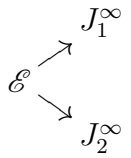
$$\{\psi_1, \psi_2\}_A = E_{A(\psi_1)}(\psi_2) + \ell^*_{A(\psi_1)}(\psi_2), \tag{9}$$

where ψ_1 and ψ_2 are conservation laws of $\mathscr{E}$.

As in the evolution case, we call an equation *bi-Hamiltonian* if it possesses two Hamiltonian structures A_1 and A_2 such that $[\![A_1, A_2]\!] = 0$.

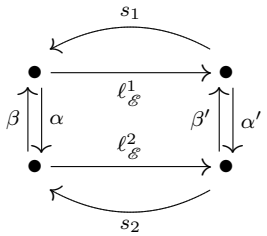
An infinite series of conservation laws ψ_1, ψ_2, ... is called a *Magri hierarchy* if for all i we have $A_1(\psi_i) = A_2(\psi_{i+1})$. In the standard way one can show that $\{\psi_i, \psi_j\}_{A_1} = \{\psi_i, \psi_j\}_{A_2} = 0$ for all i and j.

Now let us return to the question of invariance of the ℓ^*-covering. Suppose the equation $\mathscr{E}$ under consideration is embedded in two different jet spaces



We encountered an example of this situation when discussed the KdV equation, with J_1^∞ being jets with coordinates x, t and u, while J_2^∞ being jets with coordinates x, t, u, v, and w. Now, we have two linearization operators, $\ell^1_{\mathscr{E}}$ and $\ell^2_{\mathscr{E}}$, the former computed using the embedding $\mathscr{E} \to J_1^\infty$ and the latter is obtained using the embedding $\mathscr{E} \to J_2^\infty$. It is not difficult to show that these two linearization operators are related by the following diagram:

(10)



where all arrows are $\mathscr{C}$-differential operators on $\mathscr{E}$ satisfying the following relations:

$$\ell^1_{\mathscr{E}}\,\beta = \beta'\,\ell^2_{\mathscr{E}}, \quad \ell^2_{\mathscr{E}}\,\alpha = \alpha'\,\ell^1_{\mathscr{E}}, \qquad \beta\,\alpha = \mathrm{id} + s_1\,\ell^1_{\mathscr{E}}, \quad \alpha\,\beta = \mathrm{id} + s_2\,\ell^2_{\mathscr{E}}. \tag{11}$$

We use the dots $\bullet$ to avoid introducing new notations for the corresponding spaces of sections of vector bundles.

11 Definition. Two $\mathscr{C}$-differential operators Δ_1 and Δ_2 on $\mathscr{E}$ are called *equivalent* if there exist $\mathscr{C}$-differential operators α, β, α', β', s_1, and s_2 such that

$$\Delta_1\,\beta = \beta'\,\Delta_2, \quad \Delta_2\,\alpha = \alpha'\,\Delta_1, \qquad \beta\,\alpha = \mathrm{id} + s_1\,\Delta_1, \quad \alpha\,\beta = \mathrm{id} + s_2\,\Delta_2.$$

Thus, we can say that the linearization operators $\ell^1_{\mathscr{E}}$ and $\ell^2_{\mathscr{E}}$ are equivalent.

The following simple Lemma explains why this notion is really important.

12 Lemma. *$\mathscr{C}$-differential operators Δ_1 and Δ_2 are equivalent if and only if the Δ_1- and Δ_2-coverings are isomorphic.*

So, to prove that ℓ^*-covering is invariant we have to establish that the operators $\ell^{1*}_{\mathscr{E}}$ and $\ell^{2*}_{\mathscr{E}}$ are equivalent. This is implied by the following result.

13 Theorem. *If Δ_1 is equivalent to Δ_2 then Δ_1^* is equivalent to Δ_2^*.*

14 Corollary. *The equation $\mathscr{L}^*(\mathscr{E})$ does not depend on the embedding $\mathscr{E} \to J^\infty$.*

Now, recall that bivectors were defined as conservation laws on $\mathscr{L}^*(\mathscr{E})$, while operators that corresponds to them are essentially generating functions of these conservation laws. Thus, the operators depend on using an embedding $\mathscr{E} \to J^\infty$. Assume that we have two different embeddings as above, so that they give rise to two operators A_1 and A_2 that correspond to the same bivector. Here are the formulas that relate these two operators:

$$\begin{aligned} A^2 &= \alpha\, A^1\, \alpha'^*, \\ A^1 &= \beta\, A^2\, \beta'^*. \end{aligned} \tag{12}$$

4 Examples

Let us revise the three examples from the Introduction.

15 Example (KdV)**.** We considered two different embeddings of the KdV equations to jets:

$$u_t - u_{xxx} - 6uu_x = 0$$

$$\begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0$$

Here are all operators of diagram (10):

$$\ell^1_{\mathscr{E}} = D_t - D_{xxx} - 6uD_x - 6u_x, \qquad \ell^2_{\mathscr{E}} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix}, \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{aligned} \beta &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\ \beta' &= \begin{pmatrix} -D_{xx} - 6u & -D_x & -1 \end{pmatrix}, \end{aligned}$$

$$s_1 = 0, \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}.$$

Formulas (12) relate Hamiltonian operators (1) and (3).

16 Remark. If we take (1) for A_1 and compute A_2 via (12) we will not get (3), but an operator equivalent to (3).

17 Example (Camassa–Holm equation)**.** The Camassa–Holm equation written in the usual form $u_t - u_{txx} - uu_{xxx} - 2u_xu_{xx} + 3uu_x = 0$ has a bi-Hamiltonian structure:

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

If we rewrite the equation in the form

$$\begin{aligned} &m_t + um_x + 2u_xm = 0,\\ &m - u + u_{xx} = 0 \end{aligned}$$

then the bi-Hamiltonian structure takes the form

$$A_1' = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

Note that the operators B_1 and B_2 from Example 2 are entries (up to sign) of the matrix A_1' and A_2'. Thus we see that studying bi-Hamiltonian structure of the Camassa–Holm equation does not require the use of the $(1 - D_x^2)^{-1}$ "operator".

18 Example (Kupershmidt deformation)**.** Let $\mathscr{E}$ be a bi-Hamiltonian equation given by $F = 0$ and A_1 and A_2 be the Hamiltonian operators.

19 Definition. The Kupershmidt deformation $\tilde{\mathscr{E}}$ of $\mathscr{E}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables.

20 Theorem. *The Kupershmidt deformation $\tilde{\mathscr{E}}$ is a bi-Hamiltonian system.*

The proof of this theorem consists of checking that the following two bivectors define a bi-Hamiltonian structures:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

The generalisation of Kupershmidt's theorem from the Introduction is the following.

21 Theorem. *If ψ_1, ψ_2, ... is a Magri hierarchy for $\mathscr{E}$ then $(\psi_i, -\psi_{i+1})$, $i = 1, 2, \dots$, is a Magri hierarchy for the Kupershmidt deformation $\tilde{\mathscr{E}}$.*

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