Quantum connection and Poincaré–Cartan form

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This paper is devoted to A. Lichnerowicz with admiration for his deep work interrelating geometry and physics

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Abstract

The present paper relates the covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time proposed by Jadczyk and Modugno to the finite order Lagrangian bicomplex due to Krupka. Namely, the classical mechanics is formulated in terms of the Lagrangian bicomplex and the cohomological condition for existence of quantum connection is recovered in terms of the classical Poincaré–Cartan form.

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Introduction

The present paper is aimed at analysing some aspects of the general relativistic classical and quantum mechanics on a curved spacetime with absolute time, in terms of recent formulations of a finite order Lagrangian bicomplex [Kru90, Kru95] and of a covariant approach to classical and quantum mechanics [JaMo93a, JaMo93b].

There is a large literature on geometric formulations of Lagrangian theories on jets. The approaches based on infinite jets (see, for example, [AnDu80, Bau82, DeTu80, Tak79, Tul77, Tul80, Vin77, Vin78, Vin84]) are attractive for their formal elegance arising from a natural splitting of the sheaves involved. However, the finite order Lagrangian theory based on a bicomplex according to [Kru90] seems to be more appropriate to our present aims, because we are involved with a strictly first order setting and, even more, are interested in restrictions of sheaves. More specifically, we use the formalism developed in [Vit95, Vit96]. In a few words, the differential structure and the fibring of the manifold produce naturally a bicomplex whose vertices and arrows recover the main objects of Lagrangian theory and organise them in a rational scheme.

On the other hand, there are several covariant formulations of classical and quantum mechanics on a curved spacetime with absolute time (see, for example, [DBKP85, DuKü84, Kuc80, Kün84, Tra63, Tra66, Tra96, Tul85]). In this paper, we refer to the formulation of Galilei classical and quantum mechanics based on jets, connections and cosymplectic forms according to [JaMo93a, JaMo93b, CJM95]. This approach presents important analogies with geometric quantisation but novelties as well. In a few words, spacetime is a fibred manifold equipped with a vertical metric, a gravitational connection and an electromagnetic field; these structures produce naturally a cosymplectic form. On the other hand, quantum mechanics is formulated on a line bundle over spacetime equipped with a connection whose curvature is proportional to the above form.

The goal of the present paper is twofold.

Firstly, we formulate the Lagrangian approach to Galilei classical mechanics in terms of the above bicomplex. Namely, we show that the geometric structure of spacetime exhibits naturally the global cosymplectic form and places it in a certain vertex of the bicomplex associated with spacetime fibring. Then, we derive further objects such as, the global Euler–Lagrange operator and the local Poincaré–Cartan form, Lagrangian, momentum, etc. and place them in the appropriate location of the bicomplex. We stress that the primitive object of our theory is the cosymplectic form and not the Lagrangian. This inverse approach affects essentially the perspective of Lagrangian theory. For instance, the problem of definition, uniqueness and globality of the Poincaré–Cartan form changes essentially with respect to the standard direct approach. Actually, we claim that a covariant formulation of classical mechanics on a curved spacetime with absolute time cannot be achieved starting directly from a global distinguished Lagrangian. On the other hand we show that the local and gauge dependent Lagrangian, Poincaré–Cartan form and momentum are observer independent.

Secondly, we show that the search for the quantum connection is locally equivalent to the search for a Poincaré–Cartan form. Moreover, the problem of global existence of a quantum connection is solved by a theorem of Kostant–Souriau type (see, for instance, [Kos70, AbMa78]), which states a topological necessary and sufficient condition on the spacetime and the cosymplectic form. Our analysis of Poincaré–Cartan forms plays a key role in the proof of the theorem.

We end the introduction with some mathematical conventions. In this paper, all manifolds and maps between manifolds are C^∞.

As for sheaves, we shall use the definitions and the main results given in [Wel80].

Finally, we recall some basic facts on unit spaces. This theory has been developed in [JaMo93a, JaMo93b] in order to make the independence of classical and quantum mechanics from scales explicit.

A *semi–vector space*, is defined to be a set $\mathbb{U}$ endowed with an abelian semi–group structure and by an outer multiplication by $\mathbb{R}^+$ that fulfill properties analogous to those of vector spaces. A semi–vector space is said to be *positive* if the multiplication cannot be extended to $\mathbb{R}$ or $\mathbb{R}^+ \cup \{0\}$.

Several algebraic constructions of vector spaces can be repeated for semi–vector spaces. In particular, it can be shown that the tensor product (over $\mathbb{R}^+$) of a semi–vector space and a vector space has a natural vector space structure.

A *unit space* is defined to be a one–dimensional positive semi–vector space (over $\mathbb{R}^+$), or a one–dimensional vector space (over $\mathbb{R}$).

Due to the one–dimensional nature of our unit spaces, we will use the following notational conventions. Let $\mathbb{U}$ and $\mathbb{V}$ be unit spaces; if $u \in \mathbb{U}$, $v \in \mathbb{V}$, then we write $uv := u \otimes v$, and if $0 \neq z \in \mathbb{U}$, then we write $\frac{1}{z} := z^*$.

Unit spaces will allow us to take into account at each step of the theory the scales involved; in fact, the basic objects of our theory (metric, electromagnetic field, etc.) are valued into vector bundles multiplied tensorially with such spaces. We will say these tensor fields to be *scaled*.

It is important to remark that the operators like contraction, Lie derivative, exterior derivative, covariant derivative and so on, can be easily extended to scaled tensors.

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1 Lagrangian bicomplex in mechanics

In this section, we sketch the theory of first order Lagrangian bicomplex due to Krupka [Kru90, Kru95], in terms of an intrinsic formalism developed in [Vit95]. The Lagrangian bicomplex arises by a natural geometric construction just from the de Rham sequence and the contact structure of the jet prolongation of the starting fibred manifold.

Thus, our framework is constituted by a fibred manifold

$$t : \boldsymbol{E} \to \boldsymbol{T} \,,$$

with $\dim \boldsymbol{T} = 1$ and $\dim \boldsymbol{E} = m + 1$.

We deal with the tangent bundle $T\boldsymbol{E} \to \boldsymbol{E}$ of $\boldsymbol{E}$, the tangent prolongation $Tt : T\boldsymbol{E} \to T\boldsymbol{T}$ of $t : \boldsymbol{E} \to \boldsymbol{T}$ and the vertical bundle $V\boldsymbol{E} \to \boldsymbol{E}$.

Moreover, for $0 \leq r$, we are concerned with the r–jet space $J_r\boldsymbol{E}$; in particular, we set $J_0\boldsymbol{E} \equiv \boldsymbol{E}$. We recall the natural fibrings $t^r_s : J_r\boldsymbol{E} \to J_s\boldsymbol{E}$, $t^r : J_r\boldsymbol{E} \to \boldsymbol{T}$ and the affine bundle $t^r_{r-1} : J_r\boldsymbol{E} \to J_{r-1}\boldsymbol{E}$ associated with the vector bundle $\odot^r T^*\boldsymbol{T} \otimes V\boldsymbol{E}$, for $0 \leq s \leq r$. A detailed account of the theory of jets can be found in [MaMo83a, Kup80, Sau89].

Charts on $\boldsymbol{E}$ adapted to the fibring are denoted by (x^0, y^i). Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, the index 0 labels the coordinate on $\boldsymbol{T}$; Greek indices $\lambda, \mu, \dots$ run from 0 to m. We denote by (∂_0, ∂_i) and (d^0, d^i), respectively, the local bases of vector fields and 1–forms on $\boldsymbol{E}$ induced by an adapted chart. The check ($\check{}$) denotes vertical restrictions. As an example, $(\check{d}^i)$ denotes the local base of sections of $V^*\boldsymbol{E} \to \boldsymbol{E}$.

We denote multi–indices of dimension n by underlined latin letters such as $\underline{l} = (l_1, \dots, l_n)$, with $0 \leq l_1, \dots, l_n$, by identifying the index i with a multi–index according to

$$i \simeq (l_1, \dots, l_i, \dots, l_n) \equiv (0, \dots, 1, \dots, 0) \,,$$

we can write

$$\underline{l} + i = (l_1, \dots, l_i + 1, \dots, l_n) \,.$$

In this paper we are concerned just with multi–indices of dimension $n = 1$. Clearly, in this case we can write $\underline{l} \in \mathbb{N}, |\underline{l}| = \underline{l}\,,\ \underline{l} + 1 = \underline{(l+1)}$; however, it is still useful to keep the multi–index notation and to distinguish indices and multi–indices.

The charts induced on $J_r\boldsymbol{E}$ are denoted by $(x^0, y^i_{\underline{l}})$, with $0 \leq |\underline{l}| \leq r$; in particular, we set $y^i_{\underline{l}0} \equiv y^i$. For small degrees, $r = 1, 2, 3$, we write indices explicitly, according to: $y^i_{\underline{l}1} = y^i_0$, $y^i_{\underline{l}2} = y^i_{00}$, $y^i_{\underline{l}3} = y^i_{000}$. The local vector fields and forms of $J_r\boldsymbol{E}$ associated with the adapted chart are denoted by $(\partial^{\underline{l}}_i)$ and $(d^i_{\underline{l}})$, $0 \leq |\underline{l}| \leq r, 1 \leq i \leq m$, respectively.

1.1 Contact structure

We start with a brief recall of the contact structure of jets.

Contact maps

A fundamental role will be played in our theory by the "contact maps" on jet spaces (see [MaMo83a]). Namely, for $1 \leq r$, we consider the natural injective fibred morphism over $J_r\boldsymbol{E} \to J_{r-1}\boldsymbol{E}$

$$д_r : J_r\boldsymbol{E} \underset{\boldsymbol{T}}{\times} T\boldsymbol{T} \to TJ_{r-1}\boldsymbol{E}\,,$$

and the complementary surjective fibred morphism

$$\theta_r : J_r\boldsymbol{E} \underset{J_{r-1}\boldsymbol{E}}{\times} TJ_{r-1}\boldsymbol{E} \to VJ_{r-1}\boldsymbol{E}\,,$$

whose coordinate expression are

$$\begin{aligned} д_r &= d^0 \otimes д_{r0} = d^0 \otimes (\partial_0 + y^j_{\underline{l}+1}\partial^{\underline{l}}_j)\,, \qquad 0 \leq |\underline{l}| \leq r-1, \\ \theta_r &= \theta^j_{\underline{l}} \otimes \partial^{\underline{l}}_j = (d^j_{\underline{l}} - y^j_{\underline{l}+1}d^0) \otimes \partial^{\underline{l}}_j\,, \qquad 0 \leq |\underline{l}| \leq r-1\,. \end{aligned}$$

The transpose of the map θ_r is the injective fibred morphism over $J_r\boldsymbol{E} \to J_{r-1}\boldsymbol{E}$

$$\theta^*_r : J_r\boldsymbol{E} \underset{J_{r-1}\boldsymbol{E}}{\times} V^*J_{r-1}\boldsymbol{E} \to T^*J_r\boldsymbol{E}\,.$$

Distinguished sheaves of forms

We are concerned with some distinguished sheaves of forms on jet spaces. Let $0 \leq k$.

For $0 \leq r$, we consider the *standard* sheaf of local *k–forms* on $J_r\boldsymbol{E}$

$$\overset{k}{\Lambda}_r := \{\alpha : J_r\boldsymbol{E} \to \overset{k}{\wedge} T^*J_r\boldsymbol{E}\}\,.$$

1.1 Remark. For $0 \leq s \leq r$, we define the sheaves of *horizontal* forms to be the sheaves of local fibred morphisms over $J_s\boldsymbol{E}$ and $\boldsymbol{T}$

$$\begin{aligned} \overset{k}{\mathcal{H}}_{(r,s)} &:= \{\alpha : J_r\boldsymbol{E} \to \overset{k}{\wedge} T^*J_s\boldsymbol{E}\}\,, \\ \overset{k}{\mathcal{H}}_r &:= \{\beta : J_r\boldsymbol{E} \to \overset{k}{\wedge} T^*\boldsymbol{T}\}\,, \end{aligned}$$

respectively.

Thus, $\alpha \in \overset{k}{\mathcal{H}}_{(r,s)}$ and $\beta \in \overset{1}{\mathcal{H}}_r$ if and only if their coordinate expressions are of the type

$$\begin{aligned} \alpha &= \alpha^{\underline{l}_1\ldots\underline{l}_{k-1}}_{i_1\ldots i_{k-1}0}\, d^{i_1}_{\underline{l}_1} \wedge \ldots \wedge d^{i_{k-1}}_{\underline{l}_{k-1}} \wedge d^0 + \alpha^{\underline{l}_1\ldots\underline{l}_k}_{i_1\ldots i_k}\, d^{i_1}_{\underline{l}_1} \wedge \ldots \wedge d^{i_k}_{\underline{l}_k}\,, \\ \beta &= \beta_0\, d^0 \end{aligned}$$

where $0 \leq |\underline{l}_j| \leq s$, and α's, $\beta_0 \in \overset{0}{\Lambda}_r$.

Clearly, $\overset{k}{\mathcal{H}}_{(r,r)} = \overset{k}{\Lambda}_r$ and, for $k > 1$, $\overset{k}{\mathcal{H}}_r = 0$.

We have the natural injective linear sheaf morphisms, for $0 \leq s \leq r, r \leq r', s \leq s'$,

$$\overset{k}{\Lambda}_s \hookrightarrow \overset{k}{\mathcal{H}}_{(r,s)} \hookrightarrow \overset{k}{\Lambda}_r \,, \qquad \overset{k}{\mathcal{H}}_{(r,s)} \hookrightarrow \overset{k}{\mathcal{H}}_{(r',s')} \,,$$

which will be currently exploited (even without any explicit mention). $\square$

1.2 Remark. For $0 \leq s \leq r$, we define the sheaf of *vertical* forms to be the sheaf of fibred morphisms over $J_s\boldsymbol{E}$

$$\overset{k}{\mathcal{V}}_{(r,s)} := \{\alpha : J_r\boldsymbol{E} \to V^* J_s\boldsymbol{E}\} \,.$$

Thus, $\alpha \in \overset{k}{\mathcal{V}}_{(r,s)}$ if and only if its coordinate expression is of the type

$$\alpha = \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k} \, \check{d}^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \check{d}^{i_k}_{\underline{l}_k} \,, \quad 0 \leq |\underline{l}_j| \leq s \,,$$

with $\alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$.

We have the natural linear sheaf morphisms, for $0 \leq s \leq r, r \leq r', s \leq s'$,

$$\overset{k}{\mathcal{H}}_{(r,s)} \to \overset{k}{\mathcal{V}}_{(r,s)} \,, \qquad \overset{k}{\mathcal{V}}_{(r,s)} \hookrightarrow \overset{k}{\mathcal{V}}_{(r',s')} \,,$$

which will be currently exploited (even without any explicit mention).

For $0 \leq s < r$, the map θ_r induces an injective linear sheaf morphism

$$\overset{k}{\wedge}\theta^*_{s+1} : \overset{k}{\mathcal{V}}_{(r,s)} \hookrightarrow \overset{k}{\mathcal{H}}_{(r,s)} \,,$$

with coordinate expression

$$\overset{k}{\wedge}\theta^*_r(\alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k} \, \check{d}^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \check{d}^{i_k}_{\underline{l}_k}) = \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k} \, \theta^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \theta^{i_k}_{\underline{l}_k} \,. \square$$

1.3 Remark. For $0 \leq s < r$, we define the sheaf of *contact* forms to be the subsheaf

$$\overset{k}{\mathcal{C}}_{(r,s)} := \overset{k}{\wedge}\theta^*_{s+1}(\overset{k}{\mathcal{V}}_{(r,s)}) \subset \overset{k}{\mathcal{H}}_{(r,s)} \,,$$

i.e. the subsheaf of local fibred morphisms over $J_s\boldsymbol{E}$

$$\overset{k}{\mathcal{C}}_{(r,s)} := \{\alpha : J_r\boldsymbol{E} \to \overset{k}{\wedge}(\operatorname{im}\theta^*_{s+1}) \subset \overset{k}{\wedge}T^* J_s\boldsymbol{E}\} \subset \overset{k}{\mathcal{H}}_{(r,s)} \,.$$

Hence, we have the natural linear sheaf isomorphism

$$\overset{k}{\wedge}\theta^*_{s+1} : \overset{k}{\mathcal{V}}_{(r,s)} \hookrightarrow \overset{k}{\mathcal{C}}_{(r,s)} \,.$$

In other words, the sheaf $\overset{k}{\mathcal{C}}_{(r,s)}$ turns out to be the subsheaf of local fibred morphisms $\alpha \in \overset{k}{\mathcal{H}}_{(r,s)}$ which factorise as

$$\alpha = \overset{k}{\wedge}\theta^*_{s+1} \circ \tilde{\alpha}\,,$$

through the composition

$$J_r\boldsymbol{E} \xrightarrow{\tilde{\alpha}} J_{s+1}\boldsymbol{E} \underset{J_s\boldsymbol{E}}{\times} \overset{k}{\wedge} V^* J_s\boldsymbol{E} \xrightarrow{\overset{k}{\wedge}\theta^*_{s+1}} \overset{k}{\wedge} T^* J_s\boldsymbol{E}\,,$$

where $\tilde{\alpha} : J_r\boldsymbol{E} \to \overset{k}{\wedge} T^* J_s\boldsymbol{E}$ is a local fibred morphism over $J_s\boldsymbol{E}$.

Thus, $\alpha \in \overset{k}{\mathcal{C}}_{(r,s)}$ if and only if its coordinate expression is of the type

$$\alpha = \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k}\, \theta^{i_1}_{\underline{l}_1} \wedge \dots \wedge \theta^{i_k}_{\underline{l}_k}\,, \quad 0 \leq |\underline{l}_j| \leq s\,,$$

with $\alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k} \in \overset{0}{\Lambda}_r$. $\square$

1.4 Remark. For $1 \leq r$, we define the sheaf of *affine horizontal* forms to be the subsheaf

$$\overset{1}{\mathcal{H}}{}^A_r \subset \overset{1}{\mathcal{H}}_r$$

of local fibred morphisms $\beta \in \overset{1}{\mathcal{H}}_r$, which are affine fibred morphisms over $J_{r-1}\boldsymbol{E} \to \boldsymbol{T}$.

Thus, in coordinates, $\beta \in \overset{1}{\mathcal{H}}{}^A_r$ if and only if $\beta_0 : J_r\boldsymbol{E} \to I\!R$ is an affine map with respect to the coordinates $y^i_{\underline{l}}$, with $|\underline{l}| = r$. $\square$

1.5 Remark. For $0 \leq s < r$, we define the sheaf of *affine contact* forms to be the subsheaf

$$\overset{k}{\mathcal{C}}{}^A_{(r,s)} \subset \overset{k}{\mathcal{C}}_{(r,s)}$$

of local fibred morphisms $\alpha \in \overset{k}{\mathcal{C}}_{(r,s)}$ such that $\tilde{\alpha}$ is an affine fibred morphism over $J_{r-1}\boldsymbol{E} \to J_s\boldsymbol{E}$.

Thus, in coordinates, $\alpha \in \overset{k}{\mathcal{C}}{}^A_{(r,s)}$ if and only if the components $\alpha's \in \overset{0}{\Lambda}_r$ are affine maps with respect to the coordinates $y^i_{\underline{l}}$, with $|\underline{l}| = r$. $\square$

We have the following important isomorphisms.

1.6 Remark. For $0 \leq s < r$, we have the natural linear sheaf isomorphisms

$$\overset{k}{\mathcal{V}}_{(r,s)} \otimes \overset{1}{\mathcal{H}}_r \to \overset{k}{\mathcal{C}}_{(r,s)} \otimes \overset{1}{\mathcal{H}}_r \to \overset{k}{\mathcal{C}}_{(r,s)} \wedge \overset{1}{\mathcal{H}}_r\,,$$

with coordinate expression

$$\begin{aligned} \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k 0}\, \check{d}^{i_1}_{\underline{l}_1} \wedge \dots \wedge \check{d}^{i_k}_{\underline{l}_k} \otimes d^0 &\mapsto \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k 0}\, \theta^{i_1}_{\underline{l}_1} \wedge \dots \wedge \theta^{i_k}_{\underline{l}_k} \otimes d^0 \\ &\mapsto \alpha^{\underline{l}_1 \dots \underline{l}_k}_{i_1 \dots i_k 0}\, \theta^{i_1}_{\underline{l}_1} \wedge \dots \wedge \theta^{i_k}_{\underline{l}_k} \wedge d^0\,. \end{aligned}$$

We shall currently exploit these isomorphisms (even without explicit mention). $\square$

1.7 Remark. Let us recall the natural linear fibred epimorphism over $J_1\boldsymbol{E}$

$$V^*J_1\boldsymbol{E} \to V^*_{\boldsymbol{E}}J_1\boldsymbol{E} \simeq J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (V^*\boldsymbol{E} \underset{\boldsymbol{T}}{\otimes} T\boldsymbol{T}) .$$

The natural linear fibred morphism over $J_1\boldsymbol{E}$, given by the composition

$$V^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} T^*\boldsymbol{T} \hookrightarrow J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (V^*\boldsymbol{E} \underset{\boldsymbol{T}}{\otimes} T\boldsymbol{T} \underset{\boldsymbol{T}}{\otimes} T^*\boldsymbol{T}) \xrightarrow{\langle,\rangle} J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V^*\boldsymbol{E} \xrightarrow{\theta_1} J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} ,$$

yields the linear sheaf morphism

$$P : \overset{1}{\mathcal{V}}_{(r,1)} \otimes \overset{1}{\mathcal{H}}_r \to \overset{1}{\mathcal{C}}_{(r,0)} ,$$

with coordinate expression

$$P(\alpha_{i0}\theta^i + \alpha^0_{i\,0}\theta^i_0) \otimes d^0 = \alpha^0_{i\,0}\theta^i . \square$$

Main splitting

The contact maps yield a natural linear splitting of the tangent and cotangent exact sequences of the fibring $J_r\boldsymbol{E} \to \boldsymbol{T}$, by pullback over $J_{r+1}\boldsymbol{E}$, for $0 \leq r$.

1.8 Proposition. For $0 \leq r$, we have a natural sheaf splitting

$$\overset{k}{\mathcal{H}}_{(r+1,r)} = \Big(\overset{k-1}{\mathcal{C}}_{(r+1,r)} \wedge \overset{1}{\mathcal{H}}_{r+1} \Big) \oplus \overset{k}{\mathcal{C}}_{(r+1,r)} ,$$

whose first and second projections are respectively denoted by

$$h : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k-1}{\mathcal{C}}_{(r+1,r)} \wedge \overset{1}{\mathcal{H}}_{r+1} , \qquad v : \overset{k}{\mathcal{H}}_{(r+1,r)} \to \overset{k}{\mathcal{C}}_{(r+1,r)} .$$

and have coordinate expressions, for $\alpha \in \overset{k}{\mathcal{H}}_{(r+1,r)}$,

$$\begin{aligned} h(\alpha) = \Big(& \alpha^{\underline{l}_1\ldots\underline{l}_{k-1}}_{i_1\ldots i_{k-1}0}\, \theta^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \theta^{i_{k-1}}_{\underline{l}_{k-1}} + \\ & (-1)^{k-j}\, y^j_{\underline{l}_j}\, \alpha^{\underline{l}_1\ldots\underline{l}_j\ldots\underline{l}_k}_{i_1\ldots j\ldots i_k}\, \theta^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \widehat{\theta}^j_{\underline{l}_j} \wedge \ldots \wedge \theta^{i_k}_{\underline{l}_k} \Big) \wedge d^0 , \\ v(\alpha) = {} & \alpha^{\underline{l}_1\ldots\underline{l}_k}_{i_1\ldots i_k}\, \theta^{i_1}_{\underline{l}_1} \wedge \ldots \wedge \theta^{i_k}_{\underline{l}_k} . \end{aligned}$$

Proof. It follows from the linear splitting over $J_{r+1}\boldsymbol{E}$

$$t^{r+1*}_r(T^*J_r\boldsymbol{E}) = t^{r+1*}(T^*\boldsymbol{T}) \oplus \operatorname{im}\theta^*_{r+1} ,$$

with coordinate expression

$$\alpha_0 d^0 + \alpha^{\underline{l}}_i d^i_{\underline{l}} = (\alpha_0 + y^i_{\underline{l}+1}\alpha^{\underline{l}}_i) d^0 + \alpha^{\underline{l}}_i \theta^i_{\underline{l}} . \square$$

Horizontal and vertical differential

The contact maps $д_r$ and θ_r induce important derivations of the sheaf of graded algebras Λ (see [Sau89, Cos94]).

1.9 Remark. First of all, we have the two derivations of degree 0

$$i_h : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} : \alpha \mapsto i_h \alpha := i(д_{r+1})\alpha \,,$$
$$i_v : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} : \alpha \mapsto i_v \alpha := i(\theta_{r+1})\alpha \,.$$

Then, we obtain the further two derivations of degree 1, namely, the *horizontal* and *vertical differential*

$$d_h := i_h \circ d - d \circ i_h : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,,$$
$$d_v := i_v \circ d - d \circ i_v : \overset{k}{\Lambda}_r \to \overset{k}{\Lambda}_{r+1} \,,$$

which fulfill

$$d_h^2 = 0 \quad d_v^2 = 0 \,, \qquad d_h + d_v = t_r^{r+1*} \circ d \,. \square$$

1.10 Remark. We have

$$d_h \overset{1}{\mathcal{H}}_r = 0 \,, \qquad d_h \overset{k}{\mathcal{C}}_{(r,r-1)} \subset \overset{k}{\mathcal{C}}{}^A_{(r+1,r)} \wedge \overset{1}{\mathcal{H}}_r \,,$$
$$d_v \overset{k}{\mathcal{C}}_{(r,r-1)} \subset \overset{k+1}{\mathcal{C}}{}^A_{(r+1,r)} \,, \qquad d_v \overset{1}{\mathcal{H}}_r = \overset{1}{\mathcal{C}}{}^A_{(r+1,r)} \wedge \overset{1}{\mathcal{H}}_r \,. \square$$

1.2 Lagrangian bicomplex

Next, following Krupka [Kru90, Kru95], we consider the de Rham sequence on the first–order jet space and quotient it by means of a natural subsequence arising from the contact structure. In this way we obtain a diagram whose vertices and arrows describe and organise the main items of Lagrangian calculus.

So, we define, by induction on k, the sheaves

$$\overset{0}{\Theta}_1 := \{0\} \,, \quad \overset{1}{\Theta}_1 := \overset{1}{\mathcal{C}}_{(1,0)} \,, \quad \ldots, \quad \overset{k}{\Theta}_1 := \overset{k}{\mathcal{C}}_{(1,0)} + d \overset{k-1}{\mathcal{C}}_{(1,0)} \,.$$

1.11 Theorem. *We obtain the commutative diagram (see [Kru90])*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \overset{3}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{3}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_1 \arrow[r, "E_0"] \arrow[d] & \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \arrow[r, "E_1"] \arrow[d] & \overset{2}{\Lambda}_1/\overset{2}{\Theta}_1 \arrow[r, "E_2"] \arrow[d] & \overset{3}{\Lambda}_1/\overset{3}{\Theta}_1 \arrow[r, "E_3"] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

in which rows and columns are exact.

1.12 Proposition. [Kru90] The cohomology of the cochain complex of global sections associated with the Lagrangian sequence is naturally isomorphic to the de Rham cohomology of $\boldsymbol{E}$. □

We call the above diagram the first–order *Lagrangian bicomplex* associated with the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$. Moreover, we call the bottom row the *Lagrangian sequence*, the column of 1–forms the *kinematical column*, the column of 2–forms the *dynamical column* and the column of 3–forms the *integrability column*. In fact, we shall see that this bicomplex provides a very useful logical scheme for organising the main objects of Lagrangian formalism and their differential relations. In particular, we shall obtain a table which displays the *Lagrangian* λ, the *momentum* π, the *Poincaré-Cartan form* τ, the *dynamical form* ω, the *Euler-Lagrange form* ϵ, the *Helmholtz form*η, the *Euler-Lagrange operator* E_1 and the *Helmholtz operator* E_2, according to the following scheme

$$\begin{tikzcd}
\pi \in \overset{1}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \overset{3}{\Theta}_1 \arrow[d] \\
\tau \in \overset{1}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \omega \in \overset{2}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{3}{\Lambda}_1 \arrow[d] \\
\lambda \in \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \arrow[r, "E_1"] & \epsilon \in \overset{2}{\Lambda}_1/\overset{2}{\Theta}_1 \arrow[r, "E_2"] & \eta \in \overset{3}{\Lambda}_1/\overset{3}{\Theta}_1
\end{tikzcd}$$

1.3 Kinematical column

Let us examine the kinematical column.

We can interpret the quotient projection in terms of h.

1.13 Proposition. The map given by the composition

$$\overset{1}{\Lambda}_1 \hookrightarrow \overset{1}{\mathcal{H}}_{(2,1)} \xrightarrow{\ h\ } \overset{1}{\mathcal{H}}_2$$

passes to the quotient $\overset{1}{\Lambda}_1/\overset{1}{\Theta}_1$ yielding the linear sheaf isomorphism

$$[h] : \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \to \overset{1}{\mathcal{H}}{}^A_2 : [\alpha] \mapsto h(\alpha)\,.$$

For $\tau \in \overset{1}{\Lambda}_1$, we have the coordinate expression

$$[h] : [\alpha_0 d^0 + \alpha_i \theta^i + \alpha_i^0 d_0^i] \mapsto (\alpha_0 + \alpha_i^0 y_{00}^i) d^0\,.$$

Next, we analyse the possible splittings of the column of 1–forms.

For this purpose we need a reversed arrow. Indeed, the following result shows an important arrow in the opposite direction.

1.14 Definition. We define the *generalised momentum map* as the linear sheaf morphism

$$\Pi : \overset{2}{\Lambda}_1 \to \overset{1}{\mathcal{C}}_{(2,0)}$$

given by the composition

$$\overset{2}{\Lambda}_1 \xrightarrow{\ h\ } \overset{1}{\mathcal{C}}_{(2,1)} \wedge \overset{1}{\mathcal{H}}_2 \longrightarrow \overset{1}{\mathcal{V}}_{(2,1)} \otimes \overset{1}{\mathcal{H}}_2 \xrightarrow{\ P\ } \overset{1}{\mathcal{C}}_{(2,0)}\,,$$

and the *generalised Poincaré–Cartan map* as the linear sheaf morphism

$$\Xi : \overset{1}{\Lambda}_1 \to \overset{1}{\Lambda}_2 : \alpha \mapsto \alpha + \Pi(d\alpha)\,.\,\square$$

1.15 Remark. We have the following coordinate expressions

$$\begin{aligned}\Pi(\alpha) =& \Big((\partial_i^0 \alpha_0 - \partial_0 \alpha_i^0) + (\partial_i^0 \alpha_j - \partial_j \alpha_i^0) y_0^j + \\ & (\partial_i^0 \alpha_j^0 - \partial_j^0 \alpha_i^0) y_{00}^j \Big) \theta^i\end{aligned}$$

and

$$\begin{aligned}\Xi(\alpha) =& (\alpha_0 + \alpha_i y_0^i) d^0 + \\ & \Big(\alpha_i + (\partial_i^0 \alpha_0 - \partial_0 \alpha_i^0) + (\partial_i^0 \alpha_j - \partial_j \alpha_i^0) y_0^j + \\ & (\partial_i^0 \alpha_j^0 - \partial_j^0 \alpha_i^0) y_{00}^j \Big) \theta^i + \alpha_i^0 d_0^i\,.\end{aligned}$$

1.16 Lemma. Let $\alpha \in \overset{1}{\Lambda}_1$. Then, the following conditions are equivalent:

$$\Pi(d\alpha) = -\alpha\,; \tag{1}$$

$$\alpha \in \overset{1}{\mathcal{C}}_{(1,0)}\,. \tag{2}$$

1.17 Proposition. The sheaf morphism Ξ passes to the quotient $\overset{1}{\Lambda}_1/\overset{1}{\Theta}_1$, yielding the injective linear sheaf morphism

$$[\Xi] : \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \to \overset{1}{\Lambda}_2 : [\alpha] \mapsto \alpha + \Pi(d\alpha)\,.$$

PROOF. $[\Xi]$ passes to the quotient because, for each $c \in \overset{1}{\Theta}_1$, we have

$$c + \Pi(dc) = c - c = 0\,.$$

Moreover, $[\Xi]$ is injective because, for each $\alpha \in \overset{1}{\Lambda}_1$,

$$\alpha + \Pi(d\alpha) = 0 \Rightarrow \alpha \in \overset{1}{\Theta}_1\,. \square$$

Let us consider the exact sequence

$$0 \to \overset{1}{\Theta}_1 \to \overset{1}{\Lambda}_1 \to \overset{1}{\mathcal{H}}{}^A_2 \to 0\,.$$

We might hope to split this sequence by means of the maps $[\Xi]$ and v. However, their domains and codomains are too large; on the other hand, we can achieve our goal by a suitable restriction of the sequence.

1.18 Theorem. *The subsequence*

$$0 \to \overset{1}{\Theta}_1 \to \overset{1}{\mathcal{H}}_{(1,0)} \to \overset{1}{\mathcal{H}}_1 \to 0$$

of

$$0 \to \overset{1}{\Theta}_1 \to \overset{1}{\Lambda}_1 \to \overset{1}{\mathcal{H}}{}^A_2 \to 0$$

is the maximal subsequence which splits through v according to

$$0 \to \overset{1}{\mathcal{H}}_1 \longrightarrow \overset{1}{\mathcal{H}}_{(1,0)} \xrightarrow{\;v\;} \overset{1}{\Theta}_1 \to 0$$

and is the maximal subsequence which splits through Ξ according to

$$0 \to \overset{1}{\mathcal{H}}_1 \xrightarrow{\;\Xi\;} \overset{1}{\mathcal{H}}_{(1,0)} \longrightarrow \overset{1}{\Theta}_1 \to 0\,.$$

1.19 Remark. We define the subsheaf

$$\mathcal{P} := \Xi(\overset{1}{\mathcal{H}}_1) \subset \overset{1}{\mathcal{H}}_{(1,0)}\,.$$

The maps

$$h : \mathcal{P} \to \overset{1}{\mathcal{H}}_1\,, \qquad \Xi : \overset{1}{\mathcal{H}}_1 \to \mathcal{P}$$

are mutually inverse sheaf isomorphisms. $\square$

1.4 Dynamical column

Let us examine the dynamical column.

We can interpret the quotient projection in terms of h, by means of the following results.

1.20 Remark. The map given by the composition

$$\overset{2}{\Lambda}_1 \hookrightarrow \overset{2}{\mathcal{H}}_{(2,1)} \xrightarrow{h} \overset{1}{\mathcal{C}}_{(2,1)} \wedge \overset{1}{\mathcal{H}}_2$$

does not pass to the quotient $\overset{2}{\Lambda}_1/\overset{2}{\Theta}_1$ because $\overset{2}{\Theta}_1 \neq \overset{2}{\mathcal{C}}_1$. □

Therefore, it is interesting to find the maximal subsheaf of $\overset{2}{\Lambda}_1$ such that the restriction of h passes to the quotient.

1.21 Theorem. *The maximal subsheaf $\mathcal{D} \subset \overset{2}{\Lambda}_1$, such that the map given by the composition*

$$\mathcal{D} \hookrightarrow \overset{2}{\Lambda}_1 \hookrightarrow \overset{2}{\mathcal{H}}_{(2,1)} \xrightarrow{h} \overset{1}{\mathcal{C}}_{(2,1)} \wedge \overset{1}{\mathcal{H}}_2$$

passes to the quotient $\overset{2}{\Lambda}_1/\overset{2}{\Theta}_1$, is

$$\mathcal{D} = \ker \Pi \,.$$

The coordinate expression of $\omega \in \mathcal{D}$ is of the type

$$\omega = 2(\omega_{i0} + \omega^{0}_{i\,j} y^{j}_{00})\theta^i \wedge d^0 + \omega_{ij}\theta^i \wedge \theta^j + 2\omega^{0}_{i\,j}\theta^i_0 \wedge \theta^j \,,$$

where $\omega_{i0}, \omega^{0}_{i\,j}, \omega_{ij} \in \overset{0}{\Lambda}_1$.

Proof. Let $\omega, \omega' \in \overset{2}{\Lambda}_1$; then, $[\omega] = [\omega']$ if and only if $\omega' = \omega + c + dc'$, with $c \in \overset{2}{\mathcal{C}}_{(1,0)}, c' \in \overset{1}{\mathcal{C}}_{(1,0)}$.

- Let us prove that $\ker \Pi \subset \mathcal{D}$. Let $\Pi(\omega) = \Pi(\omega') = 0$ and $[\omega'] = [\omega]$. Then, $\Pi(dc') = 0$, hence $c' = 0$, hence $h(\omega') = h(\omega)$.

- Let us prove that $\mathcal{D} \subset \ker \Pi$. Let $\omega \in \ker \Pi$, $[\omega'] = [\omega]$ and $h(\omega') = h(\omega)$. Then, $h(dc') = 0$, hence $\Pi(dc') = 0$, hence $\Pi(\omega') = 0$. □

1.22 Corollary. The map given by the composition

$$\mathcal{D} \hookrightarrow \overset{2}{\Lambda}_1 \hookrightarrow \overset{2}{\mathcal{H}}_{(2,1)} \xrightarrow{h} \overset{1}{\mathcal{C}}_{(2,1)} \wedge \overset{1}{\mathcal{H}}_1$$

yields the linear sheaf isomorphism

$$[h] : \mathcal{D}/\overset{2}{\Theta}_1 \to \overset{1}{\mathcal{C}}{}^{A}_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 : [\omega] \mapsto h(\omega) \,.$$

For $\omega \in \mathcal{D}$, we have the coordinate expression

$$[h] : [\omega] \mapsto 2(\omega_{i0} + \omega^{0}_{i\,j} y^{j}_{00})\, \theta^i \wedge d^0 \,.$$

1.5 Kinematical and dynamical columns

The sheaf $\mathcal{D}$ fulfills the following important property.

1.23 Theorem. *Let $\omega \in \mathcal{D}$ be closed and let $\tau \in \overset{1}{\Lambda}_1$ be a local potential of ω. If $\tau \in \overset{1}{\mathcal{H}}_{(1,0)}$, then*

$$\tau \in \mathcal{P} \subset \overset{1}{\mathcal{H}}_{(1,0)} \,.$$

On the other hand,

$$d(\mathcal{P}) \subset \mathcal{D} \,.$$

Proof. Let us prove the first assertion. We have $\tau = h(\tau) + v(\tau)$, and $\Pi(d\tau) = 0$. Hence, $\Pi\Big(d(h(\tau)+v(\tau))\Big) = 0$. But, for a property of Π, $\Pi(d(v(\tau))) = -v(\tau)$. Moreover, $h(d(h(\tau))) = d(h(\tau))$ because of $\dim \boldsymbol{T} = 1$. Hence, we obtain $v(\tau) = \Pi\Big(d(h(\tau))\Big)$. Hence,

$$\tau = h(\tau) + \Pi(d(h(\tau))) \in \mathcal{P} \,.$$

Let us prove the second assertion. We have, for $\lambda \in \overset{1}{\mathcal{H}}_1$,

$$\Pi\left(d\left(\lambda + \Pi(d\lambda)\right)\right) = 0 \,,$$

because $\Pi(\Pi(d\lambda)) = -\Pi(d\lambda)$. $\square$

1.24 Corollary. The following diagrams commute

$$\begin{tikzcd} \mathcal{P} \arrow[r, "d"] \arrow[d, "h"'] & \mathcal{D} \arrow[d, "h"] \\ \overset{1}{\mathcal{H}}_1 \arrow[r, "E_1"'] & \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P} \arrow[r, "d"] & \mathcal{D} \arrow[d, "h"] \\ \overset{1}{\mathcal{H}}_1 \arrow[r, "E_1"'] \arrow[u, "\Xi"] & \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \end{tikzcd}$$

1.25 Corollary. The coordinate expression of $E_1 : \overset{1}{\mathcal{H}}_1 \to \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1$ is

$$E_1(\lambda_0 d^0) = (\partial_i \lambda_0 - (\text{д}_2)_0 . \partial^0_i \lambda_0) \theta^i \wedge d^0 \,.$$

The following important result emphasises the interest of the above subdiagrams of the Lagrangian bicomplex.

1.26 Theorem. *[AnDu80] The sequence*

$$\overset{1}{\mathcal{H}}_1 \xrightarrow{E_1} \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \xrightarrow{E_2} \overset{3}{\Lambda}_3 / \overset{3}{\Theta}_3$$

is exact in $\overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1$.

1.6 Interpretation of the Lagrangian bicomplex

Let us consider the natural linear sheaf isomorphism (arising from the contact structure), for $1 \leq r$,

$$\overset{1}{\mathcal{C}}_{(r,0)} \wedge \overset{1}{\mathcal{H}}_1 \to \overset{1}{\mathcal{V}}_{(r,0)} \otimes \overset{1}{\mathcal{H}}_1 \,.$$

Let $\lambda : J_1\boldsymbol{E} \to T^*\boldsymbol{T}$ be a standard first order Lagrangian of the variational calculus. We can easily verify that the standard second order Euler–Lagrange morphism $\epsilon(\lambda) : J_2\boldsymbol{E} \to V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} T^*\boldsymbol{T}$ associated with this Lagrangian turns out to be just the form $E_1(\lambda)$. Therefore, we are led to interpret $\overset{1}{\mathcal{H}}_1$ as the sheaf of *first order Lagrangians*, $\overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1$ as the sheaf of *Euler–Lagrange forms*, E_1 as the *Euler–Lagrange operator*. Analogously, one could interpret E_2 in terms of the standard third order Helmholtz operator of variational calculus.

Therefore, we are in the position to interpret the other objects arising in the Lagrangian bicomplex.

According to this interpretation, given a Lagrangian $\lambda \in \overset{1}{\mathcal{H}}_1$, we say that $\Pi(d\lambda) \in \overset{1}{\Theta}_1$ is the *momentum form of the Lagrangian* and $\Xi(\lambda) \in \mathcal{P}$ is the *Poincaré–Cartan form of the Lagrangian*.

In the direct approach to Lagrangian formalism one starts with a Lagrangian $\lambda \in \overset{1}{\mathcal{H}}_1$ and fill in the further vertices of the bicomplex (in the direction bottom-up, left-right) by means of the maps Ξ, d, h.

In the inverse approach to Lagrangian formalism one starts with a Euler–Lagrange type morphism $\epsilon \in \overset{1}{\mathcal{V}}_{(2,0)} \otimes \overset{1}{\mathcal{H}}_1$, which fulfills the Helmholtz closure condition, and finds a local Lagrangian, which is defined up to the horizontal differential of a function on $\boldsymbol{E}$. Clearly, this form yields the filling in procedure as in the direct case; but, now, all objects are defined up to a gauge.

In particular, we do not know any natural sheaf morphism $\overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \to \mathcal{D}$, analogous to Ξ, which enables us to move up univocally in the column of 2–forms. In the literature (see, for instance, [Sau89, MaPa94]) such a map can be found under additional hypotheses on the structure of the underlying fibred manifold.

Actually, we shall be involved with the Lagrangian formalism associated with our model of Galilei spacetime. Namely, a metric, a connection and a spacetime 2–form will yield directly a global *dynamical 2–form* $\omega \in \mathcal{D}$ and a global $\epsilon \in \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1$. Indeed, we shall prove that the subsheaves consisting of the above ω and ϵ are naturally isomorphic. Thus, we shall be able to fill in the Lagrangian bicomplex starting equivalently with ω or ϵ. Therefore, the objects recovered on left (Lagrangian, Poincaré–Cartan form and momentum) will be defined only locally and up to a gauge.

By the way, we have illustrated the variational interpretation of the first order Lagrangian bicomplex in the case of standard first order horizontal Lagrangians $\lambda \in \overset{1}{\mathcal{H}}_1$. On the other hand, one could develop a first order variational calculus also starting with a non horizontal first order Lagrangian in $\mu \in \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \simeq \overset{1}{\mathcal{H}}{}^A_2$; Krupka has pointed out that $E_1(\mu)$ is just the standard third order Euler–Lagrange form associated with the affine second order Lagrangian $\mu \in \overset{1}{\mathcal{H}}{}^A_2$.

2 Galilei general relativistic bicomplex

So far, we have analysed the Lagrangian bicomplex associated with a fibred manifold with the only assumption that the base space is one dimensional. Now, we add the hypothesis that the base space is affine and we consider additional structures on the fibred manifold according to the model of general relativistic spacetime developed in [JaMo93b, JaMo93a]. In this richer framework we can develop further results and interpret the general relativistic Galileian Lagrangian mechanics in terms of the bicomplex.

2.1 Spacetime structures

Galilei spacetime

We start with a simple hypothesis on the base space of our fibred manifold and sketch its physical interpretation.

Now on, we assume the base space $\boldsymbol{T}$ to be an affine space associated with an oriented vector space $\mathbb{T}$.

The total space $\boldsymbol{E}$ is said to be the *Galilei spacetime*, $\boldsymbol{T}$ the *absolute time* and t the *time function*.

A *time unit of measurement* is defined to be an oriented basis of $\mathbb{T}$ or its dual

$$u_0 \in \mathbb{T}\,, \qquad u^0 \in \mathbb{T}^*\,.$$

One might choose a time unit of measurement and make the correspondent identification $\mathbb{T} \simeq I\!R$. However, we do not want to make such a choice for physical reasons, in order to make the theory manifestly invariant with respect to units of measurement.

We will use fibred charts (x^0, y^i) on $\boldsymbol{E}$ adapted to a time unit of measurement u_0, according to $Tt \circ \partial_0 = u_0$.

We obtain the scaled form

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}\,,$$

with coordinate expression

$$dt = u_0 \otimes d^0\,.$$

A *motion* is defined as a section $s : \boldsymbol{T} \to \boldsymbol{E}$ and its *velocity* as the jet prolongation $j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$.

An *observer* is defined to be a section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E} \,.$$

The coordinate expression of an observer is $o = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$.

A fibred chart (x^0, y^i) is said to be *adapted* to an observer o if $o^i_0 = 0$. For each observer, many adapted charts exist locally; conversely, each fibred chart determines locally an observer.

An observer o can be regarded as a connection on the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$. Accordingly, we define the translation fibred isomorphism $\nabla[o]$ associated with o

$$\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E} : \sigma \mapsto \nabla[o](\sigma) := \sigma - o(t^0_1(\sigma)) \,.$$

Then, the velocity of a motion s *observed* by the observer o is defined as

$$\nabla[o]s := \nabla[o] \circ j_1 s = j_1 s - o \circ s : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E} \,.$$

Its coordinate expression is

$$\nabla[o]s = (\partial_0 s^i - o^i_0 \circ s) u^0 \otimes \partial_i \,.$$

Spacetime connection

Next, we consider an additional structure on our fibred manifold given by special type of connections.

2.1 Definition. We introduce the following three notions.

A *spacetime connection* is defined to be a torsion free linear connection on $T\boldsymbol{E} \to \boldsymbol{E}$

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \,,$$

such that

$$\nabla dt = 0 \,.$$

A *phase connection* is defined to be a torsion free (with respect to θ_1, see [JaMo93a]) affine connection on $J_1\boldsymbol{E} \to \boldsymbol{E}$

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E} \,.$$

A *dynamical connection* is defined to be a connection on $J_1\boldsymbol{E} \to \boldsymbol{T}$

$$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E} \,,$$

which projects over

$$\text{д}_1 : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$$

and is ‘homogeneous’ in the sense of [KoMo90]. $\square$

A dynamical connection can be regarded as a section

$$\gamma : J_1\boldsymbol{E} \to J_2\boldsymbol{E}\,.$$

2.2 Remark. The coordinate expressions of spacetime connections, phase connections and dynamical connections are, respectively, of the type

$$\begin{aligned}
K &= d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i \dot{\partial}_i)\,, &\quad K_\lambda{}^i &:= K_\lambda{}^i{}_0 \dot{x}^0 + K_\lambda{}^i{}_j \dot{y}^j\,,\\
\Gamma &= d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i \partial_i^0)\,, & \Gamma_\lambda{}^i &:= \Gamma_\lambda{}^i{}_0 + \Gamma_\lambda{}^i{}_j y_0^j\,,\\
\gamma &= u^0 \otimes (\partial_0 + y_0^i \partial_i + \gamma^i \partial_i^0)\,, & \gamma^i &:= \gamma_h{}^i{}_k y_0^h y_0^k + 2\gamma_h{}^i{}_0 y^h + \gamma_0{}^i{}_0\,,
\end{aligned}$$

with

$$K_\lambda{}^i{}_\mu \in C^\infty(\boldsymbol{E})\,, \quad \Gamma_\lambda{}^i{}_\mu \in C^\infty(\boldsymbol{E})\,, \quad \gamma_\alpha{}^i{}_\beta \in C^\infty(\boldsymbol{E})\,.$$

2.3 Remark. A spacetime connection K, a phase connection Γ and a dynamical connection γ are characterised, respectively, by the sections

$$\begin{aligned}
\nu[K] &: T\boldsymbol{E} \to T^*T\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} V\boldsymbol{E}\,,\\
\nu[\Gamma] &: J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T^*J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} V\boldsymbol{E}\,,\\
\nabla[\gamma] &: J_2\boldsymbol{E} \to \mathbb{T}^{*2} \otimes V\boldsymbol{E}\,,
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\nu[K] &= (\dot{d}^i - K_\lambda{}^i d^\lambda) \otimes \partial_i\,,\\
\nu[\Gamma] &= u^0 \otimes (d_0^i - \Gamma_\lambda{}^i d^\lambda) \otimes \partial_i\,,\\
\nabla[\gamma] &= u^0 \otimes u^0 (y_{00}^i - \gamma^i \circ t_1^2)\partial_i\,.\square
\end{aligned}$$

2.4 Theorem. *[JaMo93a, JanMo95] The maps*

$$\nu[K] \mapsto \nu[\Gamma] := \theta_1 \circ \nu[K] \circ T\text{д}_1\,, \qquad \Gamma \mapsto \gamma := \text{д} \lrcorner \Gamma$$

yield a natural bijection

$$K \mapsto \Gamma \mapsto \gamma$$

between spacetime, phase and dynamical connections with coordinate expression

$$\Gamma_\lambda{}^i{}_\mu = K_\lambda{}^i{}_\mu\,, \qquad \gamma_\lambda{}^i{}_\mu = \Gamma_\lambda{}^i{}_\mu\,.$$

Spacelike metric

Eventually, we consider an additional metric structure on our fibred manifold.

A *space–like metric* is defined to be scaled vertical Riemannian metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})\,,$$

where $\mathbb{L}$ is a positive 1–dimensional 'semi–vector space' over $I\!R^+$, called the *space of lengths*.

The coordinate expression of a space–like metric is of the type

$$g = g_{ij} \check{d}^i \otimes \check{d}^j \,, \qquad g_{ij} \in C^\infty(\boldsymbol{E}, \mathbb{L}^2 \otimes I\!R) \,.$$

A space–like metric g yields the linear isomorphisms

$$g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E} \,, \qquad g^\sharp : V^*\boldsymbol{E} \to \mathbb{L}^{*2} \otimes V\boldsymbol{E} \,.$$

One might choose the scale factor of the metric and write $g : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E}$. However, we do not want to make such a choice for physical reasons, in order to make the theory manifestly invariant with respect to units of measurement.

On the other hand the forthcoming geometric constructions will lead to a disturbing scale factor in the objects which are candidate to filling in the bicomplex. In order to remove this factor and allow the physical interpretation, we introduce a further scale space and constants.

So, we consider a *mass* $m \in \mathbb{M}$, where $\mathbb{M}$ is a positive 1–dimensional 'semi–vector space' over $I\!R^+$, called the *space of masses* and the *Planck constant*, i.e. a positively oriented element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We rescale g, by setting

$$G := m/\hbar \; g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}) \,.$$

Spacetime structure

2.5 Definition. A pair (K, g) constituted by a spacetime connection and a spacelike metric is said to be a *spacetime structure*.

We say a spacetime structure (K, g) to be *integrable* if

$$\nabla[K']g = 0 \,, \qquad R[K]^i{}_\lambda{}^h{}_\mu = R[K]^h{}_\mu{}^i{}_\lambda \,,$$

where K' is the restriction of K to the subbundle $V\boldsymbol{E} \subset T\boldsymbol{E}$ and $R[K]$ is the curvature of K. □

We stress that the metric cannot determine the connection fully, because of its degeneracy. On the other hand, we have the following results.

2.6 Remark. Let $o : \boldsymbol{E} \to J_1\boldsymbol{E}$ be an observer and set

$$\sigma[o] := S(\nabla[\Gamma]) : \boldsymbol{E} \to \overset{2}{\odot} T^*\boldsymbol{E} \,,$$
$$\phi[o] := A(\nabla[\Gamma]) : \boldsymbol{E} \to \overset{2}{\wedge} T^*\boldsymbol{E} \,,$$

where S and A are the symmetrization and antisymmetrisation operators associated with G.

We have the coordinate expressions in any chart adapted to o

$$\sigma[o]_{0j} = \phi[o]_{0j} = -m/\hbar\ \Gamma_{0j0}\,,$$
$$\sigma[o]_{ij} = -m/\hbar\ (\Gamma_{ij0} + \Gamma_{ji0})\,, \qquad \phi[o]_{ij} = -m/\hbar\ (\Gamma_{ij0} - \Gamma_{ji0})\,.$$

Also, the connection K is characterised by the triplet $(\check{K}, \check{\sigma}[o], \phi[o])$, where $\check{K}$ is the linear connection induced on the fibres of $t : \boldsymbol{E} \to \boldsymbol{T}$ and $\check{\sigma}[o]$ is the vertical restriction of $\sigma[o]$. $\square$

2.7 Theorem. *[JaMo93b] Let $o : \boldsymbol{E} \to J_1\boldsymbol{E}$ be an observer. Then the space time structure (K, g) is integrable if and only if*

$$\nabla[K']g = 0\,, \qquad \check{\sigma}[o] = G^\flat \circ L_o\bar{g}\,, \qquad d(\phi[o]) = 0\,,$$

where $\bar{g}$ is the contravariant metric.

In other words, the spacetime structure is integrable if and only if the connection induced on the fibres by K is the Riemannian connection associated with g, the symmetric component of K (with respect to o) is the Lie derivative of the metric with respect to o and the antisymmetric component of K (with respect to o) is closed.

2.2 Spacetime structures and Lagrangian bicomplex

Next, we exhibit distinguished objects of the bicomplex arising from spacetime structures.

Dynamical column

We start with objects, arising from spacetime structures, belonging to the dynamical column of the bicomplex. These objects turn out to be global and gauge and observer independent.

2.8 Proposition. A spacetime structure (K, g) yields the 2–form, called the associated *dynamical form*,

$$\omega := \nu_\Gamma \bar{\wedge} \theta : J_1\boldsymbol{E} \to \overset{2}{\wedge} T^* J_1\boldsymbol{E}\,,$$

(the contracted wedge product $\bar{\wedge}$ is taken with respect to G) with coordinate expression

$$\begin{aligned} \omega &= G_{ij} u^0 \otimes (d^i_0 - \Gamma_\lambda{}^i d^\lambda) \wedge \theta^j \\ &= G_{ij} u^0 \otimes (d^i_0 - \gamma^i d^0 - \Gamma_h{}^i \theta^h) \wedge \theta^j\,. \end{aligned}$$

2.9 Remark. The horizontal and vertical projections

$$h(\omega) \in \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_0 \qquad v(\omega) \in \overset{2}{\mathcal{C}}{}^A_2$$

have coordinate expressions

$$\begin{aligned} h(\omega) &= G_{ij} u^0 \otimes (y^i_{00} - \gamma^i)\theta^i \wedge d^0 \\ v(\omega) &= G_{ij} u^0 \otimes (\theta^i_0 - \Gamma_h{}^i \theta^h) \wedge \theta^j\,. \square \end{aligned}$$

2.10 Remark. The dynamical form is non–degenerate, in the sense that

$$dt \wedge \omega \wedge \omega \wedge \omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \overset{7}{\wedge} T^* J_1\boldsymbol{E}$$

is a scaled volume form of $J_1\boldsymbol{E}$. $\square$

2.11 Remark. We have

$$\phi[o] = 2o^*\omega \,. \square$$

It can be proved that ω is the unique non trivial two–form on $J_1\boldsymbol{E}$ induced naturally by (K, g) (see [Jan93]).

The dynamical form associated with (K, g) turns out to be a distinguished *global* element

$$\omega \in \mathcal{D} \,.$$

The local dynamical forms associated with all spacetime structures constitute a distinguished nonlinear subsheaf

$$\mathcal{F} \subset \mathcal{D} \,,$$

on which we will draw our attention.

2.12 Proposition. A spacetime structure (K, g) yields the fibred morphism, called the associated *Euler–Lagrange morphism*,

$$\epsilon := G^\flat \circ \nabla[\gamma] : J_2\boldsymbol{E} \to V^*\boldsymbol{E} \otimes \mathbb{T}^* \,,$$

which can be naturally identified with the form, called the associated *Euler–Lagrange form*,

$$\epsilon := G^\flat \circ \nabla[\gamma] : J_2\boldsymbol{E} \to T^*\boldsymbol{E} \wedge \mathbb{T}^* \,,$$

with coordinate expression

$$\epsilon = G_{ij} u^0 (y^i_{00} - \gamma^i) \check{d}^j \otimes d^0 \simeq G_{ij} u^0 (y^i_{00} - \gamma^i) d^j \wedge d^0 \,.$$

The Euler–Lagrange form associated with (K, g) turns out to be a distinguished *global* element

$$\epsilon \in \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \,.$$

The local Euler–Lagrange forms associated with all spacetime structures constitute a distinguished nonlinear subsheaf

$$\mathcal{E} \subset \overset{1}{\mathcal{C}}{}^A_{(2,0)} \wedge \overset{1}{\mathcal{H}}_1 \,,$$

on which we will draw our attention.

2.13 Proposition. Let us refer to a spacetime structure (K, g). Then, γ is the unique second order connection such that

$$\gamma \lrcorner \omega = 0 \, .$$

Thus, we have a natural nonlinear sheaf isomorphism

$$\mathcal{F} \to \mathcal{E} \, .$$

We can summarise the natural bijections obtained so far as follows

$$\begin{tikzcd}
 & & \omega \arrow[d] \\
K \arrow[r] & \Gamma \arrow[r] \arrow[ur] & \gamma \\
 & & \epsilon \arrow[u]
\end{tikzcd}$$

Kinematical column

Next, we proceed with objects, arising from integrable spacetime structures, belonging to the kinematical column of the bicomplex. These objects are local and gauge dependent, but observer independent.

First of all, we relate the integrability of spacetime structure with the closure of the induced objects in the dynamical column.

2.14 Theorem. *[JaMo93b] Let us consider a spacetime structure (K, g) and the associated dynamical form ω and Euler-Lagrange form ϵ. Then, the following conditions are equivalent.*
i) The spacetime structure (K, g) is integrable.
ii) The dynamical form ω is d–closed

$$d\omega = 0 \, .$$

iii) The Euler–Lagrange form is E_2–closed

$$E_2 \epsilon = 0 \, .$$

PROOF. The equivalence i) $\iff$ ii) has been proved in [JaMo93b].
The equivalence ii) $\iff$ iii) follows from the above equivalence and the commutativity of the Lagrangian bicomplex. □

Thus, in case of integrable spacetime structure, the dynamical form ω turns out to be a cosymplectic form.

So, given an integrable spacetime structure (K, g), we have to analyse the potentials of ω and ϵ.

First of all we introduce the following objects.

2.15 Definition. Let g be a spacelike metric and o an observer. We define the associated *kinetic energy*, *kinetic momentum* and *kinetic momentum form*, respectively, to be the maps

$$\begin{aligned} k[o] &:= 1/2\, G \circ (\nabla[o], \nabla[o]) \in \overset{1}{\mathcal{H}}_1 \subset \overset{1}{\mathcal{H}}_{(1,0)}\,, \\ \check{p}[o] &:= G^{\flat} \circ \nabla[o] \in \overset{1}{\mathcal{V}}_{(1,0)}\,, \\ p[o] &:= \theta^* \lrcorner \check{p}[o] \in \overset{1}{\mathcal{H}}_{(1,0)}\,. \square \end{aligned}$$

We have the coordinate expressions

$$\begin{aligned} k[o] &= 1/2\, u^0 G_{ij} y^i_0 y^j_0 \; d^0\,, \\ \check{p}[o] &= u^0 G_{ij} y^j_0 \; \check{d}^i\,, \\ p[o] &= u^0 (-G_{ij} y^i_0 y^j_0 \; d^0 + G_{ij} y^j_0 \; d^i)\,. \end{aligned}$$

Of course, we obtain

$$o^* k[o] = 0\,, \qquad o^* p[o] = 0\,.$$

2.16 Remark. Let (K, g) be a spacetime structure and o an observer. We denote the local *potentials* of $\phi[o]$ by

$$\alpha[o] : \boldsymbol{E} \to T^*\boldsymbol{E}\,.$$

The local potentials are defined up to a gauge of the type df, with $f \in C^\infty(\boldsymbol{E}, I\!R)$. $\square$

2.17 Definition. We define the sheaf of *spacetime Lagrangians* to be the nonlinear subsheaf

$$\mathcal{L} \subset \overset{1}{\mathcal{H}}_1$$

constituted by local forms $\lambda \in \overset{1}{\mathcal{H}}_1$, which are polynomials of degree 2, with respect to the affine structure of $J_1\boldsymbol{E} \to \boldsymbol{E}$, and whose second fibre derivative is

$$G : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E})\,,$$

where $g := \hbar/m\, G$ is a spacelike metric and $\alpha \in \overset{1}{\Lambda}_0$. $\square$

2.18 Remark. The sheaf of spacetime Lagrangians is constituted by forms of $\overset{1}{\mathcal{H}}_1$ that, for each observer o and chart (x^0, y^i), can be written as

$$\lambda = k[o] + i_h \alpha\,, \quad \text{i.e.} \quad \lambda = (1/2\, u^0 G_{ij} y^i_0 y^j_0 + \alpha_i y^i_0 + \alpha_0) d^0\,,$$

where $g := \hbar/m\, G$ is a spacelike metric and $\alpha \in \overset{1}{\Lambda}_0$. $\square$

2.19 Definition. The sheaf of *spacetime Poincaré–Cartan forms* is defined to be the nonlinear subsheaf

$$\mathcal{M} \subset \overset{1}{\mathcal{H}}_{(1,0)}$$

constituted by local forms $\tau \in \overset{1}{\mathcal{H}}_{(1,0)}$, which are affine, with respect to the affine structure of $J_1\boldsymbol{E} \to \boldsymbol{E}$, and whose fibre derivative is

$$\theta_1 \lrcorner G^\flat : J_1\boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E}) \,,$$

where $g := \hbar/m\, G$ is a spacelike metric and $\alpha \in \overset{1}{\Lambda}_0$. $\square$

2.20 Remark. The sheaf of spacetime Poincaré–Cartan forms is constituted by forms of $\overset{1}{\mathcal{H}}_{(1,0)}$ that, for each observer o and chart (x^0, y^i), can be written as

$$\tau = k[o] + p[o] + \alpha \,,$$

i.e.

$$\tau = -1/2\, u^0 G_{ij} y^i_0 y^j_0 d^0 + u^0 G_{ij} y^j_0 dy^i + \alpha_\lambda d^\lambda \,,$$

where $g := \hbar/m\, G$ is a spacelike metric and $\alpha \in \overset{1}{\Lambda}_0$.

Moreover, we have

$$\Xi(k[o] + i_h\alpha) = k[o] + p[o] + \alpha \,,$$
$$h(k[o] + p[o] + \alpha) = k[o] + i_h\alpha \,.$$

Then, we can write

$$\mathcal{M} = \Xi(\mathcal{L}) \subset \mathcal{P} \,,$$

and the maps Ξ and h are mutually inverse nonlinear sheaf isomorphisms

$$\Xi : \mathcal{L} \to \mathcal{M} \,, \qquad h : \mathcal{M} \to \mathcal{L} \,. \square$$

Now, we are in a position to integrate locally ϵ and ω.

2.21 Theorem. *The subsequence*

$$\mathcal{M} \xrightarrow{\ d\ } \mathcal{F} \xrightarrow{\ d\ } \overset{3}{\Lambda}_1 \qquad \text{of} \qquad \overset{1}{\Lambda}_1 \xrightarrow{\ d\ } \overset{2}{\Lambda}_1 \xrightarrow{\ d\ } \overset{3}{\Lambda}_1$$

is exact.

More precisely, we have the following results.

Let $\omega \in \mathcal{F}$ be the dynamical form associated with an integrable spacetime structure.

i) If o is an observer and $\alpha[o] \in \overset{1}{\Lambda}_0$ a local potential of $\phi[o]$, according to $2d\alpha[o] = \phi[o]$, then the local section

$$\tau := k[o] + p[o] + \alpha[o] \in \mathcal{M}$$

turns out to be a local potential of ω, according to $2d\tau = \omega$.

ii) If ω admits a local potential of the type

$$\tau \in \overset{1}{\mathcal{H}}_{(1,0)} \subset \overset{1}{\Lambda}_1 \,,$$

according to $2d\tau = \omega$, then

$$\tau \in \mathcal{M}$$

and, for each observer o, we can write

$$\tau = k[o] + p[o] + \alpha[o] \,,$$

where

$$\alpha[o] = o^*\tau \in \overset{1}{\Lambda}_0 \,, \qquad d\alpha[o] = o^*\omega \,.$$

Proof. i) We have

$$\begin{aligned} 2d(\tau) &= \\ &= G_{ij} d_0^i \wedge \theta^j - ((\Gamma_{0hj} + \Gamma_{0jh}) y_0^h + \Gamma_{jhk} y_0^h y_0^k) d^0 \wedge d^j + \\ &\quad (\Gamma_{hkj} + \Gamma_{hjk}) y_0^k d^h \wedge d^j + \partial_\mu a_\lambda d^\mu \wedge d^\lambda + \partial_i^0 a_\lambda d_0^i \wedge d^\lambda \\ &= \omega + \partial_\mu a_\lambda d^\mu \wedge d^\lambda + \partial_i^0 a_\lambda d_0^i \wedge d^\lambda + \\ &\quad G_{ij} (\Gamma_0{}^i{}_0 d^0 \wedge d^j + \Gamma_k{}^i{}_0 d^k \wedge d^0) \\ &= \omega \,. \end{aligned}$$

ii) Without losing in generality, we can write locally

$$\tau = -1/2\, u^0 G_{ij} y_0^i y_0^j d^0 + u^0 G_{ij} y_0^i d^j + \alpha_\lambda d^\lambda \,,$$

where $\alpha \in \overset{1}{\mathcal{H}}_{(1,0)}$.

Then, a computation in coordinates shows that the condition $2d\tau = \omega$ is equivalent to the system

$$\partial_i^0 a_\lambda = 0 \,, \qquad 2d\alpha = \phi \,.\, \square$$

Later, we shall realise that this result accounts essentially for the search of a quantum connection.

2.22 Corollary. Let $\tau_1, \tau_2 \in \overset{1}{\mathcal{H}}_{(1,0)}$ be two local potentials of ω with the same domain of definition. Then, we have

$$\tau_1 = \tau_2 + c\,,$$

where

$$c \in \overset{1}{\Lambda}_0 \subset \overset{1}{\mathcal{H}}_{(1,0)}\,, \qquad dc = 0\,.$$

PROOF. Let us set $c := \tau_1 - \tau_2 \in \overset{1}{\mathcal{H}}_{(1,0)}$. For any observer o we can write

$$2\tau_1 = k[o] + p[o] + \alpha_1\,, \qquad 2\tau_2 = k[o] + p[o] + \alpha_2\,,$$

where $\alpha_1, \alpha_2 \in \overset{1}{\Lambda}_0$. Hence, we obtain

$$c = \alpha_1 - \alpha_2 \in \overset{1}{\Lambda}_0\,. \square$$

2.23 Corollary. The subsequence

$$\mathcal{L} \xrightarrow{E_1} \mathcal{E} \xrightarrow{E_2} \overset{3}{\Lambda}_1/\overset{3}{\Theta}_1 \qquad \text{of} \qquad \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \xrightarrow{E_1} \overset{2}{\Lambda}_1/\overset{2}{\Theta}_1 \xrightarrow{E_2} \overset{3}{\Lambda}_1/\overset{3}{\Theta}_1$$

is exact.

More precisely, we have the following results.

Let $\epsilon \in \mathcal{E}$ be the Euler–Lagrange form associated with an integrable spacetime structure.

i) If o is an observer and $\alpha[o] \in \overset{1}{\Lambda}_0$ a local potential of $\phi[o]$, according to $2d\alpha[o] = \phi[o]$ then, the local section

$$\lambda := k[o] + i_h\alpha[o] \in \overset{1}{\mathcal{H}}\,,$$

turns out to be a local potential of ϵ, according to $E_1\lambda = \epsilon$.

ii) If ϵ admits a local potential of the type

$$\lambda \in \overset{1}{\mathcal{H}}_1 \subset \overset{1}{\mathcal{H}}{}^A_2\,,$$

then

$$\lambda \in \mathcal{L}$$

and, for each observer o, we can write

$$\lambda = k[o] + i_h\alpha[o]\,,$$

where

$$\alpha[o] = o^*\Xi[\lambda] \in \overset{1}{\Lambda}_0\,, \qquad d\alpha[o] = o^*\omega\,.$$

PROOF. It follows from the above theorem and the commutativity of the Lagrangian bicomplex. □

Each potential τ and λ of the dynamical form ω and of the Euler-Lagrange form ϵ, associated with an integrable spacetime structure (K, g), are said to be the corresponding *spacetime Poincaré–Cartan form* and *spacetime Lagrangian*, respectively.

Hence, given ω and ϵ, the corresponding Poincaré–Cartan form τ and Lagrangian λ are defined up to a gauge of the type df, with $f \in C^\infty(\boldsymbol{E}, I\!R)$. But, we stress that τ and λ have been defined independently of observers.

2.24 Remark. In the standard direct approach to Lagrangian mechanics one starts with a Lagrangian and derives from it the momentum, the Poincaré–Cartan form, the dynamical form and the Euler–Lagrange morphism.

On the other hand, in our model based on the general Lagrangian bicomplex and on integrable spacetime structures, we follow a different approach. In fact, the distinguished and global objects provided by the structure are only the objects of the dynamical column. So, the objects of the kinematical column are obtained only locally and up to a gauge.

In particular, in this context, the most natural viewpoint for the Poincaré –Cartan form is to regard it as a (gauge dependent) potential of the global dynamical form (instead as an object derived from the gauge dependent and local Lagrangian). □

Eventually, we consider the momentum.

2.25 Remark. Let us consider an integrable spacetime structure (K, g) and the associated dynamical form ω.

We define the *momentum*, associated with the potential $\tau \in \mathcal{M}$ of ω, as the contact form (see [Gar74, MaMo83b])

$$\pi = v(\tau) \in \overset{1}{\Theta}_1 \,,$$

with coordinate expression

$$\pi = (u^0 G_{ij} y^i_0 + \alpha_j) \theta^j \,.$$

We obtain

$$\pi = \Pi\Big(d(h(\tau))\Big) \,.$$

Moreover, for each observer o, we can write

$$\pi = p[o] + v(\alpha[o]) \,,$$

where

$$\alpha[o] = o^* \tau \,, \qquad d\alpha[o] = o^* \omega \,.$$

Hence, given ω, the momentum π is defined up to a gauge of the type $\check{d} f$, with $f \in C^\infty(\boldsymbol{E}, I\!R)$. □

2.3 Galilei structure and Lagrangian bicomplex

So far, we have discussed the general bicomplex associated with a fibred manifold whose base space has dimension 1 and have analysed the objects arising from additional spacetime structures.

Now, we present our model of Galilei spacetime structure, relate it to the spacetime bicomplex and give the physical interpretation.

We assume the following objects:
- a spacelike metric

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E} ,$$

- a spacetime connection, called the *gravitational connection*,

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} ,$$

- a scaled 2–form, called the *electromagnetic field*,

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \overset{2}{\wedge} T^*\boldsymbol{E} ,$$

- a pair

$$(m, q) \in \mathbb{M} \times \mathbb{Q} ,$$

where the oriented 1–dimensional vector space $\mathbb{Q} := \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ is called the *space of charges*.

Moreover, we suppose that
- the spacetime structure (K, g) is integrable,
- the form F is closed.

Next, we couple the gravitational and electromagnetic fields by incorporating them together into the geometric structure of spacetime. So, we obtain *total* objects, which allow us to treat the gravitational and electromagnetic fields as a unique field.

It is convenient to start with ω, because this coupling is immediate.

We define the *total dynamical form* ω as

$$\omega := \omega^\natural + q/2m \; F .$$

2.26 Proposition. [JaMo93b, JanMo95] The bijections between dynamical forms and dynamical connections and between dynamical connections and phase connections yield the *total dynamical connection* and the *total phase connection*, respectively,

$$\Gamma = \Gamma^\natural + \Gamma^e , \qquad \gamma = \gamma^\natural + \gamma^e ,$$

where

$$\gamma^e = -q/m \; g^{\sharp 2} \circ h(F) , \qquad \Gamma^e = -q/2m \; g^{\sharp 2} \circ \Big(F + h(F)\Big) ,$$

(here $g^{\sharp 2}$ denotes the metric isomorphism on the second component after vertical restriction), with coordinate coordinate expressions

$$\gamma^e = -q/m \; (F_0{}^i + F_h{}^i x_0^h) u^0 \otimes \partial_i^0 \,,$$
$$\Gamma^e = -q/2m \; u^0 \Big((F_h{}^i x_0^h + 2F_0{}^i) d^0 + F_j{}^i d^j \Big) \otimes \partial_i^0 \,.$$

The section γ^e turns out to be just the *Lorentz force*.

Of course, the total phase connection yields the total spacetime connection, as well (see [JaMo93b]).

Thus, the gravitational and electromagnetic structures have produced an integrable spacetime structure. Then, we can apply the general machinery and produce distinguished objects of the bicomplex.

In particular, the equation

$$\nabla[\gamma] j_1 s := j_2 s - \gamma \circ j_1 s = 0 \,,$$

associated with the total dynamical connection is said to be the *generalised Newton law of particle motion*.

In coordinates we have

$$\begin{aligned} \partial_{00}^2 s^i - (\Gamma_h{}^i{}_k \circ s) \partial_0 s^h \partial_0 s^k - 2(\Gamma_0{}^i{}_h \circ s) \partial_0 s^h - (\Gamma_0{}^i{}_0 \circ s) = \\ = q/m \; (F^i{}_0 \circ s + F^i{}_h \circ s \partial_0 s^h) \,. \end{aligned}$$

3 Quantum mechanics of a scalar particle

Now we analyse the existence problem for the covariant quantisation of a charged scalar particle according to the scheme presented in [JaMo93a, JaMo93b]. We show the role of the Poincaré–Cartan form in this context.

3.1 Quantum bundle and quantum connection

We start with a brief recall of the covariant quantisation scheme, as formulated in [JaMo93a, JaMo93b].

We refer to a spacetime equipped with an integrable spacetime structure associated with a given gravitational and electromagnetic field and to a charged particle with given mass and charge. In particular, we are concerned with the total cosymplectic form ω introduced in the previous section.

3.1 Definition. We define the *quantum bundle* to be a complex line–bundle $\boldsymbol{Q} \to \boldsymbol{E}$ endowed with a Hermitian fibre metric h. □

Quantum histories are represented by sections of $\boldsymbol{Q} \to \boldsymbol{E}$.

A local trivialisation $(t_0^1)^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathbb{C}$, which maps i and h into the analogous canonical elements of $\mathbb{C}$ and such that $H^2_{\text{de Rham}}(\boldsymbol{U}) = 0$, is said to be *normal*. Normal trivialisations exist locally. We shall always refer to normal trivialisations and to fibred charts adapted to normal trivialisations. A normal trivialisation is called also a *quantum gauge*.

We denote the *Liouville vector field* by

$$\mathbb{I} : \boldsymbol{Q} \to V\boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q)\,.$$

3.2 Definition. A *quantum connection* is defined to be a connection ч on the bundle

$$\boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$$

fulfilling the properties

1. ч is linear Hermitian,
2. ч is *universal* in the sense of [MaMo83a, MaMo83b],
3. the curvature of ч fulfills the equation

$$R[\text{ч}] = i\,\omega \otimes \mathbb{I} : \boldsymbol{Q}^{\uparrow} \to \overset{2}{\wedge} T^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} \boldsymbol{Q}^{\uparrow}\,. \square$$

The requirement of universality of ч is equivalent to the statement that ч can be seen as a *system of connections* on $\boldsymbol{Q} \to \boldsymbol{E}$

$$\xi[\text{ч}] : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$$

i.e. as a family of connections on $\boldsymbol{Q} \to \boldsymbol{E}$ parametrised by observers.

The universality condition for ч is quite important. In fact, it allows us to skip the problems related to polarisations [JaMo93b].

3.3 Lemma. Let ч be a connection on the bundle $\boldsymbol{Q}^{\uparrow}$. Then, the following conditions are equivalent:

1. ч is Hermitian and universal;
2. in the domain of each normal splitting we have the splitting

$$\text{ч} = \text{ч}^{\shortparallel} + i\,\tau \otimes \mathbb{I}\,, \qquad \tau \in \overset{1}{\mathcal{H}}_{(1,0)}\,,$$

   where $\text{ч}^{\shortparallel}$ is the flat connection induced by the normal trivialisation;

3. in the domain of each normal trivialisation we have the coordinate expression

$$\text{ч} = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + i\,\text{ч}_{\lambda} d^{\lambda} \otimes \mathbb{I}\,, \qquad \text{ч}_{\lambda} \in \overset{0}{\Lambda}_1\,.$$

Proof. In fact, Hermitianity is equivalent to $\text{ч}_\lambda, \text{ч}^0_i \in i\overset{0}{\Lambda}_1$ and universality is equivalent to the condition $\text{ч}^0_i = 0$ (see [JaMo93b]). $\square$

We stress that the condition $\text{ч}^0_i = 0$ is intrinsic.

The search for a quantum connection is locally equivalent to the choice of a Poincaré–Cartan form τ associated with ω.

3.4 Theorem. *[JaMo93b] Let* ч *be a connection on the bundle* $\boldsymbol{Q}^\uparrow$*. Then, the following conditions are equivalent:*

1. ч *is a quantum connection;*
2. *in the domain of each normal splitting we have the splitting*

$$\text{ч} = \text{ч}^{\shortparallel} + i\,\tau \otimes \mathbb{I}$$

where $\text{ч}^{\shortparallel}$ *is the flat connection induced by the trivialisation and* τ *is a distinguished local Poincaré–Cartan form associated with* ω *(whose choice is determined just by* ч *and the quantum gauge);*

3. *in the domain of each normal trivialisation we have the coordinate expression*

$$\begin{aligned}\text{ч} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \\ + i\,(-1/2\, G_{ij} y^i_0 y^j_0 d^0 + G_{ij} y^i_0 d^j + \alpha[o]_\lambda d^\lambda) \otimes \mathbb{I}\,,\end{aligned}$$

where $\alpha[o] \in \overset{1}{\Lambda}_0$ *is a distinguished potential of* $\phi[o]$ *(whose choice is determined just by* ч *and the quantum gauge), and* o *is the observer associated with the chart.*

Proof. The proof is essentially that of Theorem 2.21. $\square$

3.5 Corollary. The composition of the dynamical connection γ with the quantum connection ч yields the connection

$$\gamma \lrcorner \text{ч} : \boldsymbol{Q}^\uparrow \to \mathbb{T}^* \otimes T\boldsymbol{Q}^\uparrow$$

on the fibred manifold $\boldsymbol{Q}^\uparrow \to \boldsymbol{T}$.

In the domain of each normal splitting we have the splitting

$$\gamma \lrcorner \text{ч} = \gamma \lrcorner \text{ч}^{\shortparallel} + i\,\lambda \otimes \mathbb{I}\,,$$

where $\text{ч}^{\shortparallel}$ is the flat connection induced by the trivialisation and $\lambda := h(\tau)$ is a distinguished local Lagrangian associated with ω (whose choice is determined just by ч and the quantum gauge).

In the domain of each normal trivialisation we have the coordinate expression

$$\begin{aligned}\gamma \lrcorner \text{ч} =& u^0 \otimes (\partial_0 + y^i_0 \partial_i + \gamma^i \partial^0_i) + \\ &+ i\,(1/2\, u^0 G_{ij} y^i_0 y^j_0 + \alpha_i y^i_0 + \alpha_0) d^0 \otimes \mathbb{I}\,.\end{aligned}$$

3.6 Remark. We stress that the form τ does not depend on the observer, but only on the normal trivialisation.

On the other hand, the potential induced locally by the quantum connection depends on the quantum gauge and on the observer in the following way.

If b, b' are local bases of the quantum bundle associated with two normal trivialisations and o is an observer, then we have

$$\alpha'[o] = \alpha[o] - d\vartheta\,,$$

where $\exp(i\vartheta) := b'/b \in \overset{0}{\Lambda}_0$.

If o, o' are two observers and $o' = o + v$, then we have

$$\alpha[o'] = \alpha[o] - 1/2\, G \circ (v, v) + \nu[o] \lrcorner (G^\flat \circ v)\,,$$

where $v := o' - o$ and $\nu[o] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E}$ is the vertical valued form associated with o (here regarded as a connection on the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$). $\square$

Quantising a classical mechanical system means requiring the existence of a quantum bundle and a quantum connection.

Then, a Lagrangian density on $J_1\boldsymbol{Q}$ can be obtained by means of the given metric structures and the covariant derivative of sections. The corresponding Euler–Lagrange equation turns out to be a generalised Schroedinger equation for a scalar quantum particle in the given classical gravitational and electromagnetic fields.

Moreover, we can exhibit a natural isomorphism between a certain algebra of functions on $J_1\boldsymbol{E}$ and a a distinguished algebra of vector fields on $\boldsymbol{Q}$. Such vector fields yield quantum operators. In this way, we obtain a covariant implementation of the correspondence principle in a curved spacetime with absolute time (see [JaMo93a, JaMo93b] for details).

3.2 Quantisation and Poincaré–Cartan form

In this subsection we give a result on the existence of a quantisation of a classical mechanical systems. More precisely, we give necessary and sufficient conditions to the existence of a quantum bundle and a quantum connection.

This result is a generalisation of the Theorem by Kostant–Soriau to the case of a general relativistic classical mechanical theory on a spacetime with absolute time (see, for example, [Kos70, Gar79]). As one could expect, we will recover a similar result, but involving the topology of spacetime, rather than the topology of the space–like configuration space.

Lets us start with a few cohomological remarks. Let us consider a manifold $\boldsymbol{M}$.

We recall that $\boldsymbol{M}$ admits a covering $\{\boldsymbol{U}_i\}_{i\in I}$, called *good cover*, such that each finite intersection of the open subsets of the covering is contractible [BoTu82]. Moreover, we have a natural isomorphism

$$H^*_{\check{\mathrm{C}}\mathrm{ech}}(\boldsymbol{M}, \mathcal{S}) \simeq H^*_{\check{\mathrm{C}}\mathrm{ech}}(\{\boldsymbol{U}_i\}_{i\in I}, \mathcal{S})\,,$$

for each sheaf $\mathcal{S}$ of abelian groups.

We observe that the natural inclusion $i : \mathbb{Z} \to I\!R$ induces a natural morphism

$$i : H^2_{\check{\mathrm{C}}\mathrm{ech}}(\boldsymbol{M}, \mathbb{Z}) \to H^2_{\check{\mathrm{C}}\mathrm{ech}}(\boldsymbol{M}, I\!R) \,,$$

which is (in general) not injective because of torsion [BoTu82].

We observe that if two line bundles over $\boldsymbol{M}$ are isomorphic then they are isometric (it can be proved easily by taking into account that the base space is 1–dimensional).

We recall that there is a natural bijection between classes of isomorphic hermitian line bundles and cocycles

$$c_{ij} : \boldsymbol{U}_i \cap \boldsymbol{U}_j \to U(1) \,.$$

We shall be concerned with the natural isomorphism between the de Rham and Čech cohomologies of $\boldsymbol{E}$ [BoTu82, Wel80]

$$H^2_{\text{de Rham}}(\boldsymbol{E}) \simeq H^2_{\check{\mathrm{C}}\mathrm{ech}}(\boldsymbol{E}, I\!R) \,.$$

We recall that the de Rham cohomology of $J_1\boldsymbol{E}$ is isomorphic (but not naturally isomorphic) to the de Rham cohomology of $\boldsymbol{E}$.

Now we refer to a Galilei spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ equipped with an integrable gravitational field and a closed electromagnetic field. We shall be concerned with the induced total closed dynamical form $\omega \in \mathcal{F} \subset \overset{2}{\Lambda}_1$.

Clearly, ω induces a cohomology class on $J_1\boldsymbol{E}$. Even more, we can state the following result, which depends on a property of the Poincaré–Cartan form (see Corollary 2.22).

3.7 Lemma. The closed form ω yields naturally a Čech cohomology class of $\boldsymbol{E}$

$$[\omega] \in H^2_{\check{\mathrm{C}}\mathrm{ech}}(\boldsymbol{E}, I\!R) \,.$$

Proof. Let us consider a good cover $\{\boldsymbol{U}_i\}_{i \in I}$ of $\boldsymbol{E}$.

On each tube $(t^1_0)^{-1}(\boldsymbol{U}_i)$ we choose a Poincaré–Cartan form τ_i. In virtue of Corollary 2.22, if $\boldsymbol{U}_i \cap \boldsymbol{U}_j \neq \emptyset$, then $\tau_i - \tau_j$ is a closed form on $\boldsymbol{U}_i \cap \boldsymbol{U}_j$.

Hence, to each pair (i, j) such that $\boldsymbol{U}_i \cap \boldsymbol{U}_j \neq \emptyset$, there exists a function

$$f_{ij} : \boldsymbol{U}_i \cap \boldsymbol{U}_j \to I\!R \,,$$

such that $\tau_i - \tau_j = df_{ij}$.

It is easy to see that, for each triplet (i, j, k) such that $\boldsymbol{U}_i \cap \boldsymbol{U}_j \cap \boldsymbol{U}_k \neq \emptyset$, we have

$$d(f_{ij} + f_{jk} - f_{ik}) = 0$$

hence, the function $f_{ij} + f_{jk} - f_{ik}$ is constant.

This yields the Čech 2-cocycle

$$c := \{f_{ij} + f_{jk} - f_{ik} : \boldsymbol{U}_i \cap \boldsymbol{U}_j \cap \boldsymbol{U}_k \to I\!R\} \,,$$

hence the Čech cohomology class

$$[\omega] := [c] .$$

We can prove that the above class $[\omega]$ does not depend on the chosen gauges of τ_i and f_{ij}. $\square$

3.8 Theorem. *The following two conditions are equivalent.*

1. *There exist a quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$ and a quantum connection* ч.
2. *The closed form ω yields a cohomology class in the subgroup*

$$[\omega] \in i(H^2(\boldsymbol{E}, \mathbb{Z})) \subset H^2(\boldsymbol{E}, I\!R) .$$

Proof. Let us consider a good cover $\{\boldsymbol{U}_i\}_{i\in I}$ of $\boldsymbol{E}$.

- Suppose that the second condition of the statement holds. Then, for all $i, j, k \in I$ such that $U_i \cap U_j \cap U_k \neq \emptyset$, there exist functions f_{ij}, f_{jk}, f_{ik} as in the above Lemma, such that $(f_{ij} + f_{jk} - f_{ik}) \in \mathbb{Z}$. Then, the functions

$$c_{ij} : \boldsymbol{U}_i \cap \boldsymbol{U}_j \to U(1) : x \mapsto \exp i2\pi f_{ij}$$

fulfill $c_{ij} c_{jk} = c_{ik}$, hence yield a Hermitian line bundle cocycle on $\boldsymbol{E}$. Moreover, we obtain

$$\tau_i - \tau_j = df_{ij} = \frac{1}{i2\pi} \frac{dc_{ij}}{c_{ij}} ,$$

hence the forms $i\tau_i \otimes \mathbb{I}$ yield a global quantum connection.

- Suppose that the first condition of the statement holds. On each tube $(t_0^1)^{-1}(\boldsymbol{U}_i)$ refer to the splitting (see Theorem 3.4)

$$ч = ч'' + i\,\tau_i \otimes \mathbb{I} .$$

Then, the constant function $f_{ij} + f_{jk} - f_{ik}$ as in the above Lemma turns out to be valued into $\mathbb{Z}$. $\square$

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