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Variational sequences on finite order jets of submanifolds

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Abstract

The theory of finite order variational sequences has been recently developed trough Krupka's finite order variational bicomplex and Vinogradov's $\mathcal{C}$-spectral sequence. Such theories are formulated on finite order jets of fibred manifolds. In this paper we shall provide a formulation on finite order jets of submanifolds using Vinogradov's $\mathcal{C}$-spectral sequence approach. New insight on the geometry of jets of submanifolds is obtained as a by-product of our research. Intrinsic and coordinate expressions of relevant objects are given.
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Introduction

Spectral sequences are a well-known tool in Algebraic Topology and Homological Algebra (see, for example, [10]). The *$\mathcal{C}$-spectral sequence* was introduced by Vinogradov [12, 13]. It arises from a remarkable filtration of the De Rham complex on jets of fibrings, jets of submanifolds or on differential equations (i. e., submanifolds of a jet space). The filtration is provided by contact forms. These forms are characterised by their vanishing when pulled back via prolonged sections of the given space. The $\mathcal{C}$-spectral sequence yields the *variational sequence* as a by-product. Some terms of this sequence can be identified in a suitable way with objects from the calculus of variations, like Lagrangians, Euler-Lagrange morphisms and Helmholtz morphisms. This is due to the fact that contact forms yield zero contribution to action-like functionals, .

The above formulation has been first carried out in the case of infinite jets, in order to avoid serious technical difficulties due to the computation of jet order. Then, after the partial results by Anderson and Duchamps [1] and Duzhin [4], Krupka provided a finite order formulation on jets of fibrings [8]. After, a finite order formulation of Vinogradov's $\mathcal{C}$-spectral sequence on jets of fibrings appeared in [15, 16].

Here we provide a formulation of Vinogradov's $\mathcal{C}$-spectral sequence in the case of jets of submanifolds of finite order.

In section 1 we recall the basics of jets of order r of submanifolds of dimension n of a given manifold E [2, 5, 6, 9, 14]. Such spaces arise as the equivalence class of (embedded) submanifolds of dimension n of E having a contact of order r at a point. It is easily seen that, locally, jets of submanifolds are diffeomorphic to jets of fibrings. Thus, jets of submanifolds provide a non-trivial generalization of jets of fibrings to situations where a fibering is absent.

We also present new structures that turn out to be essential to the computation of $\mathcal{C}$-spectral sequence. Jets of submanifolds do not have the natural vertical distribution, as jets of fibrings. But they have the Cartan (or contact) distribution. We introduce a new version of this distribution in the higher order jet (*pseudo-horizontal bundle*), inspired by the work by Modugno and Vinogradov [9]. It plays the role of horizontal distribution, and a *pseudo-vertical bundle* can be introduced as a quotient of the tangent space through the pseudo-horizontal bundle. Such bundles *do not yield* a vertical distribution on jets of submanifolds, but provide the same essential information as the vertical distribution on jets of fibrings to the purposes of variational sequence.

In section 2 we introduce Cartan (or contact) forms, i. e., forms annihilating Cartan distribution. Then, we introduce the horizontalization, i. e., the restriction of forms to the pseudo-horizontal bundle. Of course, Cartan forms are characterized as the kernel of the horizontalization. This feature is new in the framework of jets of submanifolds (to our knowledge), and is essential for the computations of $\mathcal{C}$-spectral sequence.

In section 3 we introduce the finite order $\mathcal{C}$-spectral sequence, and compute all of its groups in the case of jets of submanifolds. Results are very close to that of the infinite order case, and the direct limit of the finite order formulation yields the infinite order formulation by Vinogradov. Through the Green-Vinogradov formula for adjoint operators in the infinite order case we are able to compute distinguished representatives for finite order equivalence classes of some of the most important quotient spaces of the $\mathcal{C}$-spectral sequence, like Euler–Lagrange morphisms and Helmholtz morphisms. Locally, these expressions look like the fibred case, but the spaces where they are defined are quite different.

In our paper we show the possibility to compute Vinogradov's $\mathcal{C}$-spectral sequence in the case of finite order jets of submanifolds. We think that it is possible to formulate a finite order $\mathcal{C}$-spectral sequence also in the case of differential equations. These can be regarded as submanifolds of jets of a given order; they act on $\mathcal{C}$-spectral sequence as a constraint. The technical difficulties coming from the absence of a fibring can be solved exactly as we did in this paper. We leave this topic for future research.

1 Jet spaces

In this section we recall basic facts about the geometry of jets of submanifolds (our sources were [2, 5, 6, 9, 14]) together with some new considerations which are suitable to our purposes.

Let E be an $(n+m)$-dimensional manifold. We consider the class $[L]^r_x$ of n–dimensional (embedded) submanifolds $L \subset E$ having a contact of order r at $x \in E$. The set of such classes is said to be the *r–jet of n–dimensional submanifolds of* E, and is denoted by $J^r(E,n)$. We also set $J^0(E,n) = E$. Any submanifold $L \subset E$ can be *prolonged* to $J^r(E,n)$ via the map

$$j_r L\colon L \to J^r(E,n), \quad p \to [L]^r_p. \tag{1}$$

We identify L with its image through $j_r L$, denoted by $L^{(r)}$. The set $J^r(E,n)$ has a natural manifold structure, given as follows. We say a local chart (V,φ) on E to be *fibred* if V is diffeomorphic to $(X \times U) \subset \mathbb{R}^{n+m}$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are open subsets. Obviously, the trivial projection $\pi\colon V \to X$ makes V a fibred manifold on X. In this case, we set $\varphi = (x^\lambda, u^i)$, where (x^λ) are coordinates on X and (u^i) are coordinates on U. Greek indices run from 1 to n and the Latin ones from 1 to m. We say a submanifold $L \subset E$ to be *concordant* with the above chart at $p \in V \cap L$ if L can be (locally) expressed as $u^i = s^i(x^\lambda)$. Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_r)$ with $1 \leq \sigma_i \leq n$ be a multi-index and $|\boldsymbol{\sigma}| \overset{\text{def}}{=} r$. The fibred charts induce the local chart $\big(J^r\pi, (x^\lambda, u^i_{\boldsymbol{\sigma}})\big)$ on $J^r(E,n)$ at $[L]^r_p$, where $J^r\pi$ is the jet of the fibring π. The functions $u^i_{\boldsymbol{\sigma}}$ are determined by $u^i_{\boldsymbol{\sigma}} \circ j_r L = \partial^{|\boldsymbol{\sigma}|} s^i / \partial x^{\boldsymbol{\sigma}}$, where $\partial x^{\boldsymbol{\sigma}}$ stands for $\partial x^{\sigma_1} ... \partial x^{\sigma_r}$.

The natural projections $\pi_{r,h} : J^r(E,n) \longrightarrow J^h(E,n)$, $r \geq h$, have a fibre bundle structure. In particular, it is known [2, 6] that $\pi_{r+1,r}$ are affine bundles for $r \geq 1$. Of course, if $r = 0$ then $\pi_{1,0}$ is the Grassmann bundle of n–dimensional subspaces in TE. The infinite order jet $J^\infty(E,n)$ is just the inverse limit of the projections $\pi_{r+1,r}$.

Let us consider the following bundles over $J^{r+1}(E,n)$: the pull-back bundle

$$T^{r+1,r} \stackrel{\text{def}}{=} J^{r+1}(E,n) \underset{J^r(E,n)}{\times} TJ^r(E,n)$$

the subbundle $H^{r+1,r} \stackrel{\text{def}}{=} \{([L]^{r+1}_p, \upsilon) \in T^{r+1,r} \mid \upsilon \in T_{[L]^r_p} L^{(r)}\}$ and the quotient bundle $V^{r+1,r} \stackrel{\text{def}}{=} T^{r+1,r}/H^{r+1,r}$.

Definition 1. We say $H^{r+1,r}$ and $V^{r+1,r}$ to be, respectively, the *pseudo-horizontal* and the *pseudo-vertical* bundle of $J^r(E,n)$.

The most important property of the bundles $T^{r+1,r}$, $H^{r+1,r}$ and $V^{r+1,r}$ is the following *contact exact sequence*

$$0 \longrightarrow H^{r+1,r} \overset{D^{r+1}}{\hookrightarrow} T^{r+1,r} \overset{\omega^{r+1}}{\longrightarrow} V^{r+1,r} \longrightarrow 0 \tag{2}$$

and the corresponding sequence of dual morphisms, yielding

$$0 \longleftarrow \wedge^k (H^{r+1,r})^* \overset{\wedge^k (D^{r+1})^*}{\longleftarrow} \wedge^k (T^{r+1,r})^* \overset{(\omega^{r+1})^* \wedge \,\mathrm{id}}{\hookleftarrow} (V^{r+1,r})^* \wedge \wedge^{k-1} (T^{r+1,r})^* \longleftarrow 0 \tag{3}$$

We observe that $\wedge^k (T^{r+1,r})^*$ is a vector subbundle of $\wedge^k (T^* J^{r+1}(E,n))$.

Now we provide coordinate expressions of main objects. A local basis of the set of sections of the bundle $H^{r+1,r}$ is

$$D_\lambda = \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma},\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}},$$

where the index $\boldsymbol{\sigma}, \lambda$ stands for $(\sigma_1, \ldots, \sigma_r, \lambda)$; D_λ is said to be *total derivative* with respect to x^λ. A local chart of $H^{r+1,r}$ associated with the above basis is $(x^\lambda, u^i_{\boldsymbol{\tau}}, z^\mu)$, $|\boldsymbol{\tau}| \leq r+1$, $z^\mu([L]^{r+1}_p, \upsilon) = \upsilon^\mu$. A local basis of $(H^{r+1,r})^*$ dual to (D_λ) is given by the restriction of the 1-forms dx^λ to $H^{r+1,r}$, and is denoted by $\overline{dx}^\lambda$. The local expression of D^{r+1} turns out to be

$$D^{r+1} = \overline{dx}^\lambda \otimes D_\lambda = \overline{dx}^\lambda \otimes \left(\frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma},\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \right)$$

A local basis of the set of the sections of the bundle $V^{r+1,r}$ is $B^j_{\boldsymbol{\sigma}} \stackrel{\text{def}}{=} [\partial/\partial u^i_{\boldsymbol{\sigma}}] \in V^{r+1,r}$, $|\boldsymbol{\sigma}| \leq r$. A local chart of $V^{r+1,r}$ associated with the above basis is $(x^\lambda, u^i_{\boldsymbol{\sigma}}, z^j_{\boldsymbol{\sigma}})$, $z^j_{\boldsymbol{\sigma}}([\upsilon]) = \upsilon^j_{\boldsymbol{\sigma}} - \upsilon^\lambda u^j_{\boldsymbol{\sigma},\lambda}$. The local expression of ω^{r+1} is

$$\omega^{r+1} = \omega^j_{\boldsymbol{\sigma}} \otimes B^j_{\boldsymbol{\sigma}} = \left(du^j_{\boldsymbol{\sigma}} - u^j_{\boldsymbol{\sigma},\lambda} dx^\lambda \right) \otimes B^j_{\boldsymbol{\sigma}}.$$

Finally, we have a natural distribution C^r on $J^r(E,n)$ generated by the tangent spaces $TL^{(r)}$ for any n-dimensional submanifold $L \subset E$, namely the *Cartan Distribution* (see, for example, [3]). It is generated by the vector fields D_λ and $\partial/\partial u^i_{\boldsymbol{\sigma}}$, with $|\boldsymbol{\sigma}| = r$. This distribution has not to be confused with $H^{r,r-1}$, which is a subbundle of a different bundle and is generated by D_λ.

When E is endowed with a fibering $\pi\colon E \to M$, the space $J^r\pi$ of r-th jets of sections $s\colon M \to E$ of π is an open dense subset of $J^r(E,n)$ [9], and the vertical bundle $V\pi \overset{\text{def}}{=} \ker T\pi_{r+1,r}$ yields a splitting of the contact sequence (2).

2 Forms on jets

Here we introduce distinguished spaces of forms on jet spaces. Then, we give an isomorphism between spaces of forms and spaces of differential operators. This allows us to 'import' the theory of adjoint operators and Green's formula in our setting.

We denote by $\mathcal{F}_r$ the algebra $\mathcal{C}^\infty(J^r(E,n))$. For $k \geq 0$ we consider the standard $\mathcal{F}_r$-module Λ^k_r of k-forms on $J^r(E,n)$. We set also $\Lambda^*_r = \oplus_k \Lambda^k_r$. We introduce the submodule of Λ^k_r of the *contact forms*

$$\mathcal{C}^1\Lambda^k_r \overset{\text{def}}{=} \{\alpha \in \Lambda^k_r \mid (j_rL)^*\alpha = 0 \text{ for each submanifold } L \subset E\}.$$

Clearly, contact forms annihilate Cartan distribution. We set $\mathcal{C}^1\Lambda^*_r = \oplus_k \mathcal{C}^1\Lambda^k_r$. Moreover, we define $\mathcal{C}^p\Lambda^*_r$ as the p-th exterior power of $\mathcal{C}^1\Lambda^*_r$. Next we introduce the $\mathcal{F}_{r+1}$-module $\Lambda^k_{r+1,r}$ of the k-forms along $\pi_{r+1,r}$, i. e. k-forms on $J^r(E,n)$ with coefficients in $\mathcal{F}_{r+1}$. Obviusly, $\Lambda^k_r \subset \Lambda^k_{r+1,r} \subset \Lambda^k_{r+1}$.

We also consider the $\mathcal{F}_{r+1}$-module $\mathcal{H}^k_{r+1,r}$ of *pseudo-horizontal* k-forms, i. e., sections $\alpha\colon J^{r+1}(E,n) \to \wedge^k(H^{r+1,r})^*$. It is a submodule of $\Lambda^k_{r+1,r}$.

Definition 2. We define the *horizontalization* to be the map

$$h\colon \Lambda^k_r \to \mathcal{H}^k_{r+1,r}, \quad \alpha \mapsto (\wedge^k(D^{r+1})^*) \circ \pi^*_{r+1,r}(\alpha)$$

where $\wedge^k(D^{r+1})^*$ is the map of equation (3).

If $\alpha \in \Lambda^k_r$, then we have the coordinate expression

$$\alpha = \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_h}_{i_1\ldots i_h}{}_{\lambda_{h+1}\ldots\lambda_k} du^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge du^{i_h}_{\boldsymbol{\sigma}_h} \wedge dx^{\lambda_{h+1}} \wedge \cdots \wedge dx^{\lambda_k},$$

where $0 \leq h \leq k$. Hence

$$h(\alpha) = u^{i_1}_{\boldsymbol{\sigma}_1,\lambda_1} \cdots u^{i_h}_{\boldsymbol{\sigma}_h,\lambda_h} \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_h}_{i_1\ldots i_h}{}_{\lambda_{h+1}\ldots\lambda_k} \overline{dx}^{\lambda_1} \wedge \cdots \wedge \overline{dx}^{\lambda_k}.$$

Let us introduce the $\mathcal{F}_r$-module $\overline{\Lambda}^q_r \overset{\text{def}}{=} \operatorname{im} h$. It is easy to realize from the above coordinate expressions that, if $r \geq 1$, then $\overline{\Lambda}^q_r$ is made by sections of $\wedge^q(H^{r+1,r})^* \to$

$J^{r+1}(E,n)$ whose coefficients are polynomial of degree $\leq q$ in the variables $u^i_{\boldsymbol{\sigma}}$, with $|\boldsymbol{\sigma}| = r+1$. Note that coefficients *are not* generic polynomials: they have a distinguished antisymmetry property on indexes [16]. The case $r = 0$ needs special attention. In fact, $\pi_{1,0}$ has no affine structure; but the local expression of the horizontalization in a fibred chart is again polynomial. So, $\overline{\Lambda}^q_0$ is a subspace of the space of *locally* polynomial forms of degree $\leq q$ in u^i_λ.

Finally, we study the relationship of contact forms with horizontalization.

Proposition 1. *We have*

$$\mathcal{C}^1\Lambda^k_r = \ker h \quad \textit{if} \quad 0 \leq k \leq n, \qquad \mathcal{C}^1\Lambda^k_r = \Lambda^k_r \quad \textit{if} \quad k > n.$$

Proof. Let $\alpha \in \Lambda^k_r$. Then, $h(\alpha) = 0$ if and only if $\wedge^k(D^{r+1})^*(\alpha) = 0$. But $\ker \wedge^k(D^{r+1})^* = \mathrm{im}((\omega^{r+1})^* \wedge \mathrm{id})$. The second assertion is trivial. □

Let $p \leq k$ and $\alpha \in \mathcal{C}^p\Lambda^k_r$. From the above proposition we have the coordinate expression

$$\pi^*_{r+1,r}(\alpha) = \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \wedge \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_p}_{i_1\ldots i_p}, \qquad \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_p}_{i_1\ldots i_p} \in \Lambda^{k-p}_r, \tag{4}$$

where $|\boldsymbol{\sigma}_l| \leq r$ for $l = 1, \ldots, p$. Note that derivatives of order $r+1$ appear in the above expression in the forms $\omega^{i_l}_{\boldsymbol{\sigma}_l}$ with $|\boldsymbol{\sigma}_l| = r$. It is possible to obtain an expression containing just r-th order derivatives by using contact forms of the type $d\omega^{i_l}_{\boldsymbol{\sigma}_l}$ with $|\boldsymbol{\sigma}_l| = r-1$; see [8].

In view of the above considerations, we introduce the subspace $\mathcal{C}^p\Lambda^p_{r,r+1} \subset \mathcal{C}^p\Lambda^p_{r+1}$ of contact forms of order $r+1$ with coefficients in $\mathcal{F}_r$.

Now, we establish a correspondence between forms and differential operators.

Let P, Q be projective modules over an $\mathbb{R}$-algebra A . We recall [2] that a *linear differential operator* of order k is defined to be an $\mathbb{R}$-linear map $\Delta : P \to Q$ such that

$$[\delta_{a_0}, [\ldots, [\delta_{a_k}, \Delta] \ldots]] = 0$$

for all $a_0, \ldots, a_k \in A$. Here, square brackets stand for commutators and δ_{a_i} is the multiplication morphism by a_i. Of course, linear differential operators of order zero are just morphisms of modules. The A-module of differential operators of order k from P to Q is denoted by $\mathrm{Diff}_k(P,Q)$. The A-module of differential operators of any order from P to Q is denoted by $\mathrm{Diff}(P,Q)$. This definition can be generalised to maps with l arguments in P. The corresponding space is denoted by $\mathrm{Diff}_{(l)}(P,Q)$.

Let $r \leq s$, P be a $\mathcal{F}_r$-module and Q be a $\mathcal{F}_s$-module. We consider $\mathcal{C}$*-differential* operators [3] from P to Q, i. e. differential operators whose expression contains total derivatives instead of standard ones. In local coordinates, $\mathcal{C}$-differential operators have the form $(a^{\boldsymbol{\sigma}}_{ij} D_{\boldsymbol{\sigma}})$, where $a^{\boldsymbol{\sigma}}_{ij} \in \mathcal{F}_s$, $D_{\boldsymbol{\sigma}} = D_{\sigma_1} \circ \cdots \circ D_{\sigma_k}$. We denote the $\mathcal{F}_s$-module of $\mathcal{C}$-differential operators of order k from P to Q by $\mathcal{C}\mathrm{Diff}_k(P,Q)$. We also introduce the $\mathcal{F}_s$-module of differential operators from P to Q of any order $\mathcal{C}\mathrm{Diff}(P,Q)$. We generalize the definition to maps with l arguments in P.

Finally, for $r \geq 1$ we denote the $\mathcal{F}_r$-module of bundle morphisms $\varphi\colon J^r(E,n) \to V^{1,0}$ over $\mathrm{id}_{J^1(E,n)}$ by $\varkappa_r$. We define also $\varkappa_0 \subset \varkappa_1$ to be the subset of morphisms of the type $\varphi' = \omega^1(\varphi \circ \pi_{1,0})$, with $\varphi\colon E \to TE$. In coordinates, $\varphi \in \varkappa_r$ has the local expression $\varphi = \varphi^i[\partial/\partial u^i]$, where $\varphi^i \in \mathcal{F}_r$. We have $\varkappa_r \subset \varkappa_{r+1}$, so we introduce the direct limit $\varkappa$ of morphisms on $J^\infty(E,n)$. Any $\varphi \in \varkappa_r$ can be uniquely prolonged to a bundle morphism $\ni_\varphi\colon J^{r+s}(E,n) \to V^{s+1,s}$., which is said to be *evolutionary morphism*. In coordinates, $\ni_\varphi = D_{\boldsymbol{\sigma}}\varphi^i[\partial/\partial u^i_\sigma]$.

Proposition 2. *We have the natural isomorphism*

$$\mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^q_r), \qquad \alpha \to \nabla_\alpha$$

where $\nabla_\alpha(\varphi_1, \ldots, \varphi_p) = \ni_{\varphi_1} \lrcorner(\ldots \lrcorner(\ni_{\varphi_p} \lrcorner\, \alpha)\ldots)$.

The above proposition can be proved by analogy with the infinite order case (see [3]). Just recall that the isomorphism is realized due to the fact that to any pseudovertical tangent vector to $J^r(E,n)$ there exists an evolutionary morphism passing through it.

3 Spectral sequence

The $\mathcal{C}$-spectral sequence was introduced by Vinogradov in the late Seventies [12, 13]. Here, we present a new finite order approach to $\mathcal{C}$-spectral sequence on the jets of submanifolds of order r.

The module Λ^k_r is filtered by the submodules $\mathcal{C}^p\Lambda^k_r$; namely, we have the obvious *finite* chain of inclusions

$$\Lambda^k_r \overset{\mathrm{def}}{=} \mathcal{C}^0\Lambda^k_r \supset \mathcal{C}^1\Lambda^k_r \supset \cdots \supset \mathcal{C}^p\Lambda^k_r \supset \cdots \supset \mathcal{C}^I\Lambda^k_r \supset \mathcal{C}^{I+1}\Lambda^k_r = \{0\},$$

where I is the dimension of the contact distribution (see [3]). We say the above graded filtration of Λ^k_r to be the *$\mathcal{C}$-filtration* on the jet space of order r.

The $\mathcal{C}$-filtration gives rise to a spectral sequence $(E^{p,q}_N, e_N)_{N,p,q\in\mathbb{N}}$ in a standard way [10]. We say it to be Vinogradov's *$\mathcal{C}$-spectral sequence of (finite) order* r on E.

Our goal is to describe all terms in the $\mathcal{C}$-spectral sequence.

We recall that $E^{p,q}_0 \equiv \mathcal{C}^p\Lambda^{p+q}_r/\mathcal{C}^{p+1}\Lambda^{p+q}_r$. The spaces $\mathcal{C}^p\Lambda^k_r$ admit a coordinate description trough pull-back (proposition 1).

Generalizing h, we introduce the map $h^p\colon \Lambda^{p+q}_r \to \Lambda^p_{r+1} \otimes \overline{\Lambda}^q_r$ such that

$$(5)\quad \alpha_1 \wedge \ldots \wedge \alpha_{p+q} \mapsto \frac{1}{p!\,q!} \sum_{\sigma\in S_{p+q}} |\sigma|\, \alpha_{\sigma(1)} \wedge \ldots \wedge \alpha_{\sigma(p)} \otimes h(\alpha_{\sigma(p+1)} \wedge \ldots \wedge \alpha_{\sigma(p+q)})$$

where S_{p+q} is the set of permutations of $p+q$ elements.

The following proposition is an immediate consequence of proposition 1 and equation (4).

Theorem 1 (Computation of E_0)**.** *The restriction of h^p to $C^p\Lambda_r^{p+q}$ yields the isomorphism*

$$E_0^{p,q} = \frac{C^p\Lambda_r^{p+q}}{C^{p+1}\Lambda_r^{p+q}} \to C^p\Lambda^p_{r,r+1} \otimes \overline{\Lambda}^q_r, \quad [\alpha] \mapsto h^p(\alpha).$$

We denote the differential e_0 (which is the quotient of d) by $\bar{d} \overset{\mathrm{def}}{=} e_0$. Obviously we have

$$\bar{d}\colon E_0^{p,q} \to E_0^{p,q+1},\ h^{p+1}(\alpha) \to h^{p+2}(d\alpha) \tag{6}$$

Hence, the bigraded complex (E_0, e_0) is isomorphic to the sequence of complexes

$$\begin{tikzcd}
0 & 0 & 0 & & 0 \\
\bar{\Lambda}^n_r \arrow[u, "\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r) \arrow[u, "-\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r}(\varkappa_0, \bar{\Lambda}^n_r) \arrow[u, "\bar{d}"'] & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r}(\varkappa_0, \bar{\Lambda}^n_r) \arrow[u, "(-1)^I\bar{d}"'] \\
\dots \arrow[u] & \dots \arrow[u] & \dots \arrow[u] & \dots & \dots \arrow[u] \\
\bar{\Lambda}^1_r \arrow[u, "\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^1_r) \arrow[u, "-\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r}(\varkappa_0, \bar{\Lambda}^1_r) \arrow[u, "\bar{d}"'] & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r}(\varkappa_0, \bar{\Lambda}^1_r) \arrow[u, "(-1)^I\bar{d}"'] \\
\Lambda^0_r \arrow[u, "\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \mathcal{F}_r) \arrow[u, "-\bar{d}"'] & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(2)\,r}(\varkappa_0, \mathcal{F}_r) \arrow[u, "\bar{d}"'] & \dots & \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(I)\,r}(\varkappa_0, \mathcal{F}_r) \arrow[u, "(-1)^I\bar{d}"'] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \dots & 0 \arrow[u]
\end{tikzcd}$$

The sequence becomes trivial after the I-th column, and the minus signs are put in order to agree with an analogous convention on infinite order variational bicomplexes.

Now we study the term E_1. We recall that $E_1 = H(E_0)$, where the homology is taken with respect to $\bar{d}$. We need the following technical result. The proof is analogous to the case of jets of fibring [16].

Lemma 1. *The sequence*

$$0 \longrightarrow \mathcal{C}^p\Lambda^p_r \xrightarrow{d} \mathcal{C}^p\Lambda^{p+1}_r \xrightarrow{d} \dots \xrightarrow{d} \mathcal{C}^p\Lambda^{p+n-1}_r \xrightarrow{d} \dots$$

is exact up to the term $\mathcal{C}^p\Lambda^{p+n-1}_r$.

Then, we observe that the $\mathcal{C}$-spectral sequence of order r converges to the de Rham cohomology of $J^r(E,n)$. So, according to the definition of convergence [10], there exists $n_0 \in \mathbb{N}$ such that $E_{n_0} = E_s$ for $s > n_0$, and E_{n_0} is isomorphic to the quotient vector spaces $i^p H^*(\mathcal{C}^p\Lambda^*)/i^{p+1}H^*(\mathcal{C}^{p+1}\Lambda^*)$ of the filtration

$$H^*(\Lambda^*) \supset iH^*(\mathcal{C}^1\Lambda^*) \supset i^2H^*(\mathcal{C}^2\Lambda^*) \supset \cdots \supset i^I H^*(\mathcal{C}^I\Lambda^*) \supset 0,$$

of the de Rham cohomology of $J^r(E,n)$ (i is the natural inclusion). This is due to the fact that the $\mathcal{C}$-spectral sequence is a first quadrant spectral sequence [10]. Moreover, the de Rham cohomology of $J^r(E,n)$ is equal to $H^*(J^1(E,n))$ because $J^r(E,n)$ has topologically trivial fibre over $J^1(E,n)$.

Theorem 2 (Computation of $E_1^{*,n}$)**.** *The term $(E_1^{*,n}, e_1)$ is isomorphic to the (short) complex*

$$0 \longrightarrow \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r) \xrightarrow{e_1} \dots \xrightarrow{e_1} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(E_0^{p,n-1}) \xrightarrow{e_1} \dots$$

where $E_1^{p,n} = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(E_0^{p,n-1})$, $e_1^{p,n}([h^{p+1}(\alpha)]) = [h^{p+2}(d\alpha)]$.

Proof. The identifications of spaces come directly from the definition of E_1. As for the last statement, we have by definition (see, *e. g.*, [10]) $e_1 = \pi \circ \delta$, where δ is the Bockstein operator induced by the exact sequence and π is the cohomology map induced by the corresponding map π of the exact sequence. The proof is completed by expanding into the above equation the definition of δ. □

Theorem 3 (Computation of E_1 and E_2)**.** *We have*

$$E_1^{0,q} = H^q(J^1(E,n)), \quad q \neq n; \qquad E_1^{p,q} = 0, \quad q \neq n,\ p \neq 0;$$
$$E_2^{p,n} = H^p(J^1(E,n)), \quad p \geq 1.$$

Proof. The first statement follows from the fact that $E_0^{0,q}$ is the quotient of the de Rham sequence with an exact sequence (see lemma 1), and analogously for the second statement. □

We can combine the complex $E_0^{0,*}$ with the complex $E_1^{*,n}$, producing the complex

$$\dots \xrightarrow{\bar{d}} \bar{\Lambda}^n_r \xrightarrow{\tilde{e}_1} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(E_0^{1,n-1}) \xrightarrow{e_1} \dots,$$

where $\tilde{e}_1$ is the composition of the quotient projection with e_1. Due to the above theorem, the cohomology of the above complex is equal to the de Rham cohomology of $J^1(E,n)$.

Definition 3. We say the above complex to be the *finite order variational sequence associated with the $\mathcal{C}$-spectral sequence of order r on $J^r(E,n)$.*

The word 'variational' comes from the fact that we can identify the objects of the space $\bar{\Lambda}^n_r$ with $(r+1)$-*st order Lagrangians* [16]. Moreover, next two spaces in the sequence can be identified with a space of (finite order) *Euler–Lagrange morphisms* and a space of (finite order) *Helmholtz morphism*, and the differential e_1 is the operator sending Lagrangians into corresponding Euler–Lagrange morphism and Euler–Lagrange type morphisms into Helmholtz morphisms.

Now we show that each equivalence class in the quotient spaces of the variational sequence can be represented by a distinguished form. To this aim, we observe that pull-back includes the $\mathcal{C}$-spectral sequence of order r into the $\mathcal{C}$-spectral sequence of order $r+1$. We can evaluate the direct limit of the finite order $\mathcal{C}$-spectral sequences: this is equivalent to the $\mathcal{C}$-spectral sequence of infinite order (as formulated by Vinogradov). Hence, we have the embedding

$$(7)\qquad \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0,\bar{\Lambda}^n_r)/\bar{d}(E_0^{p,n-1}) \hookrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa,\bar{\Lambda}^n)/\widehat{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa,\bar{\Lambda}^{n-1})),$$

where $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}$ is the space of operators of any order, $\bar{\Lambda}^k$ is the space of pseudo-horizontal forms on the jet space of any order and $\widehat{d}$ is the direct limit of $\bar{d}$ (which coincides, in a local chart, with the horizontal differential, see [11]).

We recall Vinogradov's geometric formulation of Green's formula for adjoint operators. Let $\mathcal{F}$ be the space of functions on jet spaces of any order, and set $\widehat{\varkappa}\stackrel{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{F}}(\varkappa,\bar{\Lambda}^n)$. If $\Delta\colon P\to Q$ is a $\mathcal{C}$-differential operator, then [3] there exists an operator $\Delta^*\colon \widehat{Q}\to\widehat{P}$ fulfilling $\widehat{q}(\Delta(p))-(\Delta^*(\widehat{q}))(p)=\widehat{d}\omega_{p,\widehat{q}}(\Delta)$. In coordinates, if $\Delta=\Delta^{\boldsymbol{\sigma}}_{ij}D_{\boldsymbol{\sigma}}$, then $\Delta^*=(-1)^{|\boldsymbol{\sigma}|}D_{\boldsymbol{\sigma}}\circ\Delta^{\boldsymbol{\sigma}}_{ji}$.

Now, the quotient spaces (7) are isomorphic to $K_p(\varkappa)$ ([12, 13]; see also [3, p. 192]) which is the subspace of $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa,\widehat{\varkappa})$ whose elements are skew-adjoint in each argument, i. e. $(\nabla(\varphi_1,\dots,\varphi_{p-2}))^*=-\nabla(\varphi_1,\dots,\varphi_{p-2})$ for all $\varphi_1,\dots,\varphi_{p-2}\in\varkappa$. Note that, if $p=1$, then the isomorphism reads as the evaluation of the adjoint of the given operator at the constant function 1 [3].

The above considerations show that the equivalence class $[\alpha]$ in the quotient space with contact degree p is represented, through the embedding (7) and the characterization of $K_p(\varkappa)$, as the operator ∇_α obtained after skew-adjoining α in its first $(p-1)$-arguments and adjoining it in its p-th argument.

Locally, $\alpha\in\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0,\bar{\Lambda}^n_r)$ is of the form $\alpha^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_{p-1}\boldsymbol{\tau}}_{i_1\dots i_{p-1}j}\,\omega^{i_1}_{\boldsymbol{\sigma}_1}\wedge\cdots\wedge\omega^{i_{p-1}}_{\boldsymbol{\sigma}_{p-1}}\wedge\omega^j_{\boldsymbol{\tau}}\wedge\overline{dx}$, where $\overline{dx}\stackrel{\mathrm{def}}{=}\overline{dx}^1\wedge\cdots\wedge\overline{dx}^n$. Hence, if $p=1$, then $\nabla_\alpha=(-1)^{|\boldsymbol{\sigma}|}D_{\boldsymbol{\sigma}}\alpha^{\boldsymbol{\sigma}}_i\omega^i\wedge\overline{dx}$. This clearly shows that the first quotient space in the variational sequence is the space of Euler–Lagrange type operators. If $p=2$, then

$$(8)\qquad \nabla_\alpha=\frac{1}{2}\left(\tilde{\alpha}^{\boldsymbol{\sigma}}_{ij}-\sum_{|\boldsymbol{\rho}|=0}^{s-|\boldsymbol{\sigma}|}(-1)^{|(\boldsymbol{\sigma},\boldsymbol{\rho})|}\binom{|(\boldsymbol{\sigma},\boldsymbol{\rho})|}{|\boldsymbol{\rho}|}D_{\boldsymbol{\rho}}\tilde{\alpha}^{(\boldsymbol{\sigma},\boldsymbol{\rho})}_{ji}\right)\omega^i_{\boldsymbol{\sigma}}\wedge\omega^j\wedge\overline{dx},$$

where $(\boldsymbol{\sigma}, \boldsymbol{\rho})$ denotes the union of the multiindexes $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, s is the jet order of $\tilde{\alpha}^{\boldsymbol{\sigma}}_{ij}$, and the factor $1/2$ comes from skew-symmetrization. Note that we also used the Leibniz rule for total derivatives [11] to derive the expression of the $\mathcal{C}$-differential operator $(-1)^{\boldsymbol{\sigma}} D_{\boldsymbol{\sigma}} \circ \tilde{\alpha}^{\boldsymbol{\sigma}}_{ji}$, where $\tilde{\alpha}^{\boldsymbol{\sigma}}_{ji}$ are the coefficients of $\nabla^*_\alpha(1)$. This clearly shows that the second quotient space in the variational sequence is the space of Helmholtz-type operators.

Through the above expressions it is possible to derive a representation formula for any p, together with a representation of the differential e_1.

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